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A Simplified Binet Formula for k-Generalized Fibonacci Numbers

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Abstract

In this paper, we present a Binet-style formula that can be used to produce the k-generalized Fibonacci numbers (that is, the Tribonaccis, Tetranaccis, etc.). Furthermore, we show that in fact one needs only take the integer closest to the first term of this Binet-style formula in order to generate the desired sequence.

1 Introduction

Let $k \geq 2$ and define $F_n^{(k)}$, the n^{th} k-generalized Fibonacci number, as follows:

$$F_n^{(k)} = \begin{cases} 0, & \text{if } n < 1; \\ 1, & \text{if } n = 1; \\ F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)}, & \text{if } n > 1 \end{cases}$$

These numbers are also called generalized Fibonacci numbers of order k, Fibonacci k-step numbers, Fibonacci k-sequences, or k-bonacci numbers. Note that for $k = 2$, we have $F_n^{(2)} = F_n$, our familiar Fibonacci numbers. For $k = 3$ we have the so-called Tribonaccis (sequence number A000073 in Sloane's *Encyclopedia of Integer Sequences*), followed by the Tetranaccis (A000078) for $k = 4$, and so on. According to Kessler and Schiff [6], these numbers also appear in probability theory and in certain sorting algorithms. We present here a chart of these numbers for the first few values of k:

k	name	first few non-zero terms
2	Fibonacci	$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$
3	Tribonacci	$1, 1, 2, 4, 7, 13, 24, 44, 81, \ldots$
4	Tetranacci	$1, 1, 2, 4, 8, 15, 29, 56, 108, \ldots$
5	Pentanacci	$1, 1, 2, 4, 8, 16, 31, 61, 120, \ldots$

We remind the reader of the famous Binet formula (also known as the de Moivre formula) that can be used to calculate F_n, the Fibonacci numbers:

$$\begin{aligned} F_n &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] \\ &= \frac{\alpha^n - \beta^n}{\alpha - \beta} \end{aligned}$$

for $\alpha > \beta$ the two roots of $x^2 - x - 1 = 0$. For our purposes, it is convenient (and not particularly difficult) to rewrite this formula as follows:

$$F_n = \frac{\alpha - 1}{2 + 3(\alpha - 2)}\alpha^{n-1} + \frac{\beta - 1}{2 + 3(\beta - 2)}\beta^{n-1} \tag{1}$$

We leave the details to the reader.

Our first (and very minor) result is the following representation of $F_n^{(k)}$:

Theorem 1. *For $F_n^{(k)}$ the n^{th} k-generalized Fibonacci number, then*

$$F_n^{(k)} = \sum_{i=1}^{k} \frac{\alpha_i - 1}{2 + (k+1)(\alpha_i - 2)}\alpha_i^{n-1} \tag{2}$$

for $\alpha_1, \ldots, \alpha_k$ the roots of $x^k - x^{k-1} - \cdots - 1 = 0$.

This is a new presentation, but hardly a new result. There are many other ways of representing these k-generalized Fibonacci numbers, as seen in the articles [2, 3, 4, 5, 7, 8, 9]. Our Eq. (2) of Theorem 1 is perhaps slightly easier to understand, and it also allows us to do some analysis (as seen below). We point out that for $k = 2$, Eq. (2) reduces to the variant of the Binet formula (for the standard Fibonacci numbers) from Eq. (1).

As shown in three distinct proofs [9, 10, 13], the equation $x^k - x^{k-1} - \cdots - 1 = 0$ from Theorem 1 has just one root α such that $|\alpha| > 1$, and the other roots are strictly inside the unit circle. We can conclude that the contribution of the other roots in Eq. 2 will quickly become trivial, and thus:

$$F_n^{(k)} \approx \frac{\alpha - 1}{2 + (k+1)(\alpha - 2)} \alpha^{n-1} \qquad \text{for } n \text{ sufficiently large.} \tag{3}$$

It's well known that for the Fibonacci sequence $F_n^{(2)} = F_n$, the "sufficiently large" n in Eq. (3) is $n = 0$, as shown here:

n	0	1	2	3	4	5	6
F_n	0	1	1	2	3	5	8
$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$	0.447	0.724	1.171	1.894	3.065	4.960	8.025
$\lvert$error$\rvert$	.447	.277	.171	.106	.065	.040	.025

It is perhaps surprising to discover that a similar statement holds for all the k-generalized Fibonacci numbers. Let's first define $\operatorname{rnd}(x)$ to be the the value of x rounded to the nearest integer: $\operatorname{rnd}(x) = \lfloor x + \frac{1}{2} \rfloor$. Then, our main result is the following:

Theorem 2. *For $F_n^{(k)}$ the n^{th} k-generalized Fibonacci number, then*

$$F_n^{(k)} = \operatorname{rnd}\left(\frac{\alpha - 1}{2 + (k+1)(\alpha - 2)} \alpha^{n-1}\right)$$

for all $n \geq 2 - k$ and for α the unique positive root of $x^k - x^{k-1} - \cdots - 1 = 0$.

We point out that this theorem is not as trivial as one might think. Note the error term for the generalized Fibonacci numbers of order $k = 6$, as seen in the following chart; it is not monotone decreasing in absolute value.

n	0	1	2	3	4	5	6	7
$F_n^{(6)}$	0	1	1	2	4	8	16	32
$\frac{\alpha-1}{2+7(\alpha-2)}\alpha^5$	0.263	0.522	1.035	2.053	4.072	8.078	16.023	31.782
$\lvert$error$\rvert$	.263	.478	.035	.053	.072	.078	.023	.218

We also point out that not every recurrence sequence admits such a simple formula as seen in Theorem 2. Consider, for example, the scaled Fibonacci sequence $10, 10, 20, 30, 50, 80, \ldots$, which has Binet formula:

$$\frac{10}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{10}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

This can be written as $\operatorname{rnd}\left(\frac{10}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n\right)$, but only for $n \geq 5$. As another example, the sequence $1, 2, 8, 24, 80, \ldots$ (defined by $G_n = 2G_{n-1} + 4G_{n-2}$) can be written as

$$G_n = \frac{(1+\sqrt{5})^n}{2\sqrt{5}} - \frac{(1-\sqrt{5})^n}{2\sqrt{5}},$$

but because both $1+\sqrt{5}$ and $1-\sqrt{5}$ have absolute value greater than 1, then it would be impossible to express G_n in terms of just one of these two numbers.

2 Previous Results

We point out that for $k = 3$ (the Tribonacci numbers), our Theorem 2 was found earlier by Spickerman [11]. His formula (modified slightly to match our notation) reads as follows, where α is the real root, and σ and $\overline{\sigma}$ are the two complex roots, of $x^3 - x^2 - x - 1 = 0$:

$$F_n^{(3)} = \operatorname{rnd}\left(\frac{\alpha^2}{(\alpha-\sigma)(\alpha-\overline{\sigma})}\alpha^{n-1}\right) \tag{4}$$

It is not hard to show that for $k = 3$, our coefficient $\frac{\alpha-1}{2+(k+1)(\alpha-2)}$ from Theorem 2 is equal to Spickerman's coefficient $\frac{\alpha^2}{(\alpha-\sigma)(\alpha-\overline{\sigma})}$. We leave the details to the reader.

In a subsequent article [12], Spickerman and Joyner developed a more complex version of our Theorem 1 to represent the generalized Fibonacci numbers. Using our notation, and with $\{\alpha_i\}$ the set of roots of $x^k - x^{k-1} - \cdots - 1 = 0$, their formula reads

$$F_n^{(k)} = \sum_{i=1}^{k} \frac{\alpha_i^{k+1} - \alpha_i^k}{2\alpha_i^k - (k+1)}\alpha_i^{n-1} \tag{5}$$

It is surprising that even after calculating out the appropriate constants in their Eq. (5) for $2 \leq k \leq 10$, neither Spickerman nor Joyner noted that they could have simply taken the first term in Eq. (5) for all $n \geq 0$, as Spickerman did in Eq. (4) for $k = 3$.

The Spickerman-Joyner Eq. (5) was extended by Wolfram [13] to the case with arbitrary starting conditions (rather than the initial sequence $0, 0, \ldots, 0, 1$). In the next section we will show that our Eq. (2) in Theorem 1 is equivalent to the Spickerman-Joyner formula given above (and thus is a special case of Wolfram's formula).

Finally, we note that the polynomials $x^k - x^{k-1} - \cdots - 1$ in Theorem 1 have been studied rather extensively. They are irreducible polynomials with just one zero outside the unit circle. That single zero is located between $2(1-2^{-k})$ and 2 (as seen in Wolfram's article [13]; Miles [9] gave earlier and less precise results). It is also known [13, Lemma 3.11] that the polynomials have Galois group S_k for $k \leq 11$; in particular, their zeros can not be expressed in radicals for $5 \leq k \leq 11$. Wolfram conjectured that the Galois group is always S_k. Cipu and Luca [1] were able to show that the Galois group is not contained in the alternating group A_k, and for $k \geq 3$ it is not 2-nilpotent. They point out that this means the zeros of the polynomials $x^k - x^{k-1} - \cdots - 1$ for $k \geq 3$ can not be constructed by ruler and compass, but the question of whether they are expressible using radicals remains open for $k \geq 12$.

3 Preliminary Lemmas

First, a few statements about the the number α.

Lemma 3. *Let $\alpha > 1$ be the real positive root of $x^k - x^{k-1} - \cdots - x - 1 = 0$. Then,*

$$2 - \frac{1}{k} < \alpha < 2 \tag{6}$$

In addition,

$$2 - \frac{1}{3k} < \alpha < 2 \qquad \text{for } k \geq 4. \tag{7}$$

Proof. We begin by computing the following chart for $k \leq 5$:

k	$2 - \frac{1}{k}$	$2 - \frac{1}{3k}$	α
2	1.5	1.833...	1.618...
3	1.666...	1.889...	1.839...
4	1.75	1.916...	1.928...
5	1.8	1.933...	1.966...

It's clear that $2-\frac{1}{k} < \alpha < 2$ for $2 \leq k \leq 5$ and that $2-\frac{1}{3k} < \alpha < 2$ for $4 \leq k \leq 5$. We now focus on $k \geq 6$. At this point, we could finish the proof by appealing to $2(1 - 2^{-k}) < \alpha < 2$ as seen in the article [13, Lemma 3.6], but here we present a simpler proof.

Let $f(x) = (x-1)(x^k - x^{k-1} - \cdots - x - 1) = x^{k+1} - 2x^k + 1$. We know from our earlier discussion that $f(x)$ has one real zero $\alpha > 1$. Writing $f(x)$ as $x^k(x-2)+1$, we have

$$f\left(2 - \frac{1}{3k}\right) = \left(2 - \frac{1}{3k}\right)^k \left(\frac{-1}{3k}\right) + 1 \tag{8}$$

For $k \geq 6$, it's easy to show

$$3k < \left(\frac{5}{3}\right)^k = \left(2 - \frac{1}{3}\right)^k < \left(2 - \frac{1}{3k}\right)^k$$

Substituting this inequality into the right-hand side of (8), we can re-write (8) as

$$f\left(2 - \frac{1}{3k}\right) < (3k) \cdot \left(\frac{-1}{3k}\right) + 1 = 0.$$

Finally, we note that

$$f(2) = 2^{k+1} - 2 \cdot 2^k + 1 = 1 > 0,$$

so we can conclude that our root α is within the desired bounds of $2 - 1/3k$ and 2 for $k \geq 6$. ☐

We now have a lemma about the coefficients of α^{n-1} in Theorems 1 and 2.

Lemma 4. *Let $k \geq 2$ be an integer, and let $m^{(k)}(x) = \dfrac{x-1}{2+(k+1)(x-2)}$. Then,*

1. $m^{(k)}(2-1/k) = 1$.
2. $m^{(k)}(2) = \frac{1}{2}$.
3. $m^{(k)}(x)$ *is continuous and decreasing on the interval* $[2-1/k, \infty)$.
4. $m^{(k)}(x) > \frac{1}{x}$ *on the interval* $(2-1/k, 2)$.

Proof. Parts 1 and 2 are immediate. As for 3, note that we can rewrite $m^{(k)}(x)$ as

$$m^{(k)}(x) = \frac{1}{k+1}\left(1 + \frac{1-\frac{2}{k+1}}{x-(2-\frac{2}{k+1})}\right)$$

which is simply a scaled translation of the map $y = 1/x$. In particular, since this $m^{(k)}(x)$ has a vertical asymptote at $x = 2-\frac{2}{k+1}$, then by parts 1 and 2 we can conclude that $m^{(k)}(x)$ is indeed continuous and decreasing on the desired interval.

To show part 4, we first note that in solving $\frac{1}{x} = m^{(k)}(x)$, we obtain a quadratic equation with the two intersection points $x = 2$ and $x = k$. It's easy to show that $\frac{1}{x} < m^{(k)}(x)$ at $x = 2-1/k$, and since both functions $\frac{1}{x}$ and $m^{(k)}(x)$ are continuous on the interval $[2-1/k, \infty)$ and intersect only at $x = 2$ and $x = k \geq 2$, we can conclude that $\frac{1}{x} < m^{(k)}(x)$ on the desired interval. □

Lemma 5. *For a fixed value of $k \geq 2$ and for $n \geq 2-k$, define E_n to be the error in our Binet approximation of Theorem 2, as follows:*

$$\begin{aligned} E_n &= F_n^{(k)} - \frac{\alpha-1}{2+(k+1)(\alpha-2)} \cdot \alpha^{n-1} \\ &= F_n^{(k)} - m^{(k)}(\alpha) \cdot \alpha^{n-1}, \end{aligned}$$

for α the positive real root of $x^k - x^{k-1} - \cdots - x - 1 = 0$ and $m^{(k)}$ as defined in Lemma 4. Then, E_n satisfies the same recurrence relation as $F_n^{(k)}$:

$$E_n = E_{n-1} + E_{n-2} + \cdots + E_{n-k} \qquad (\textit{for } n \geq 2).$$

Proof. By definition, we know that $F_n^{(k)}$ satisfies the recurrence relation:

$$F_n^{(k)} = F_{n-1}^{(k)} + \cdots + F_{n-k}^{(k)} \tag{9}$$

As for the term $m^{(k)}(\alpha) \cdot \alpha^{n-1}$, note that α is a root of $x^k - x^{k-1} - \cdots - 1 = 0$, which means that $\alpha^k = \alpha^{k-1} + \cdots + 1$, which implies

$$m^{(k)}(\alpha) \cdot \alpha^{n-1} = m^{(k)}(\alpha)\alpha^{n-2} + \cdots + m^{(k)}(\alpha)\alpha^{n-(k+1)} \tag{10}$$

We combine Equations (9) and (10) to obtain the desired result. □

4 Proof of Theorem 1

As mentioned above, Spickerman and Joyner [12] proved the following formula for the k-generalized Fibonacci numbers:

$$F_n^{(k)} = \sum_{i=1}^{k} \frac{\alpha_i^{k+1} - \alpha_i^k}{2\alpha_i^k - (k+1)} \alpha_i^{n-1} \tag{11}$$

Recall that the set $\{\alpha_i\}$ is the set of roots of $x^k - x^{k-1} - \cdots - 1 = 0$. We now show that this formula is equivalent to our Eq. (2) in Theorem 1:

$$F_n^{(k)} = \sum_{i=1}^{k} \frac{\alpha_i - 1}{2 + (k+1)(\alpha_i - 2)} \alpha_i^{n-1} \tag{12}$$

Since $\alpha_i^k - \alpha_i^{k-1} - \cdots - 1 = 0$, we can multiply by $\alpha_i - 1$ to get $\alpha_i^{k+1} - 2\alpha_i^k = -1$, which implies $(\alpha_i - 2) = -1 \cdot \alpha_i^{-k}$. We use this last equation to transform (12) as follows:

$$\frac{\alpha_i - 1}{2 + (k+1)(\alpha_i - 2)} = \frac{\alpha_i - 1}{2 + (k+1)(-\alpha_i^{-k})} = \frac{\alpha_i^{k+1} - \alpha_i^k}{2\alpha_i^k - (k+1)}$$

This establishes the equivalence of the two formulas (11) and (12), as desired. □

5 Proof of Theorem 2

Let E_n be as defined in Lemma 5. We wish to show that $|E_n| < \frac{1}{2}$ for all $n \geq 2 - k$. We proceed by first showing that $|E_n| < \frac{1}{2}$ for $n = 0$, then for $n = -1, -2, -3, \ldots, 2 - k$, then for $n = 1$, and finally that this implies $|E_n| < \frac{1}{2}$ for all $n \geq 2 - k$.

To begin, we note that since our initial conditions give us that $F_n^{(k)} = 0$ for $n = 0, -1, -2, \ldots, 2 - k$, then we need only show $|m^{(k)}(\alpha) \cdot \alpha^{n-1}| < 1/2$ for those values of n. Starting with $n = 0$, it's easy to check by hand that $m^{(k)}(\alpha) \cdot \alpha^{-1} < 1/2$ for $k = 2$ and 3, and as for $k \geq 4$, we have the following inequality from Lemma 3:

$$2 - \frac{1}{3k} < \alpha,$$

which implies

$$\alpha^{-1} < \frac{3k}{6k-1}.$$

Also, by Lemma 4,

$$m^{(k)}(\alpha) < m^{(k)}(2 - 1/3k) = \frac{3k-1}{5k-1},$$

so thus:

$$m^{(k)}(\alpha) \cdot \alpha^{-1} < \frac{3k-1}{5k-1} \cdot \frac{3k}{6k-1} < \frac{(3k) \cdot 1}{(5k-1) \cdot 2} < \frac{1}{2},$$

as desired. Thus, $0 < |m^{(k)}(\alpha) \cdot \alpha^{-1}| < 1/2$ for all k, as desired.

Since $\alpha^{-1} < 1$, we can conclude that for $n = -1, -2, \ldots, 2-k$, then $|E_n| = m^{(k)}(\alpha) \cdot \alpha^{n-1} < 1/2$.

Turning our attention now to E_1, we note that $F_1^{(k)} = 1$ (again by definition of our initial conditions) and that

$$\frac{1}{2} = m(2) < m(\alpha) < m(2 - 1/k) = 1$$

which immediately gives us $|E_1| < 1/2$.

As for E_n with $n \geq 2$, we know from Lemma 5 that

$$E_n = E_{n-1} + E_{n-2} + \cdots + E_{n-k} \qquad \text{(for } n \geq 2)$$

Suppose for some $n \geq 2$ that $|E_n| \geq 1/2$. Let n_0 be the smallest positive such n. Now, subtracting the following two equations:

$$\begin{aligned} E_{n_0+1} &= E_{n_0} + E_{n_0-1} + \cdots + E_{n_0-(k-1)} \\ E_{n_0} &= E_{n_0-1} + E_{n_0-2} + \cdots + E_{n_0-k} \end{aligned}$$

gives us:

$$E_{n_0+1} = 2E_{n_0} - E_{n_0-k}$$

Since $|E_{n_0}| \geq |E_{n_0-k}|$ (the first, by assumption, being larger than, and the second smaller than, 1/2), we can conclude that $|E_{n_0+1}| > |E_{n_0}|$. In fact, we can apply this argument repeatedly to show that $|E_{n_0+i}| > \cdots > |E_{n_0+1}| > |E_{n_0}|$. However, this contradicts the observation from Eq. (3) that the error must eventually go to 0. We conclude that $|E_n| < 1/2$ for all $n \geq 2$, and thus for all $n \geq 2-k$. □

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