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Two Permutation Classes Enumerated by the Central Binomial Coefficients

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Abstract

We define a map between the set of permutations that avoid either the four patterns $3214, 3241, 4213, 4231$ or $3124, 3142, 4123, 4132$, and the set of Dyck prefixes. This map, when restricted to either of the two classes, turns out to be a bijection that allows us to determine some notable features of these permutations, such as the distribution of the statistics "number of ascents", "number of left-to-right maxima", "first element", and "position of the maximum element".

1 Introduction

The well-known pattern containment order over the set of permutations is defined as follows: a permutation σ *contains* a permutation τ if there exists a subsequence of σ that has the same relative order as τ, and in this case τ is said to be a *pattern* occurring in σ. Otherwise, σ is said to *avoid the pattern* τ.

A *class* of permutations is a downset in the permutation pattern containment order. Every class can be defined by its basis B, namely, the set of minimal permutations that are not contained in it; the class is denoted by $\mathrm{Av}(B)$. We let $S_n(B)$ denote the set $\mathrm{Av}(B) \cap S_n$, where S_n is the set of all permutations of length n.

The classes of permutations that avoid one or more patterns of length 3 have been exhaustively studied since the seminal paper [8]. In many cases, the properties of these permutations have been determined by establishing suitable bijections with lattice paths (see the paper by Claesson and Kitaev [2] for a survey).

The case of patterns of length 4 still seems to be incomplete, even with regard to the mere enumeration in the case of multiple avoidance (for a comprehensive overview on the subject see the book by Kitaev [5, Chapter 6.1]. See also the paper by Mansour and Vainshtein [7] for a case of multiple avoidance concerning the pattern 132 and other patterns of length 4 or more). Guibert [4] dealt with some enumerative problems concerning multiple avoidance in his Ph.D. thesis. In particular, [4, Theorem 4.6] exhibits 12 different classes of permutations avoiding four patterns of length 4, each one enumerated by the sequence of central binomial coefficients.

On the other hand, it is well known that the central binomial coefficient $\binom{2n}{n}$ (see A000984 in [9]) enumerates the set of Dyck prefixes of length $2n$, namely, lattice paths in the integer lattice $\mathbb{N} \times \mathbb{N}$ starting from the origin, consisting of up steps $U = (1,1)$ and down steps $D = (1,-1)$, and never passing below the x-axis.

In this paper we consider two of Guibert's classes, namely,

$$\mathrm{Av}(T_1) = \mathrm{Av}(3214, 3241, 4213, 4231) \quad \text{and} \quad \mathrm{Av}(T_2) = \mathrm{Av}(3124, 3142, 4123, 4132),$$

and introduce a map Φ between the union of these classes and the set of Dyck prefixes. The restrictions of this map to the sets $\mathrm{Av}(T_1)$ and $\mathrm{Av}(T_2)$ turn out to be bijections. The key tool in determining this map is Theorem 1, which describes the structure of permutations of both classes. If we consider the decomposition $\sigma = \alpha\, n\, \beta$, where n is the maximum symbol in σ and α and β are possibly empty words, then if the permutation σ avoids the four patterns in T_1, the prefix α avoids 321 and the suffix β avoids both 231 and 213, while if σ avoids T_2, the prefix α avoids 312 and the suffix β avoids both 123 and 132, where α and β satisfy some additional constraints described in the next section. In both cases, the lattice path $\Phi(\sigma)$ is obtained by first associating with α a Dyck prefix by a procedure similar to the one used by Krattenthaler [6], and then appending to this prefix a sequence of up steps and down steps that depends on the suffix β.

The map Φ allows us to relate some properties of a permutation σ with some particular features of the corresponding Dyck prefix P. For example, the Dyck prefix P does not touch the x-axis (except for the origin) whenever σ is connected (see Section 3), while P is a Dyck path whenever σ ends with the maximum symbol. Moreover, if $\sigma \in \mathrm{Av}(T_1)$, the y-coordinate of the last point of P gives information about the maximum length of a decreasing subsequence in σ.

The map Φ allows us also to prove that each one of the three statistics "number of left-to-right maxima", "position of the maximum element", and "first element" are equidistributed over the two classes, and we determine the generating function of these statistics. Finally, in the last section we determine the distribution of the statistic "number of ascents", which is not equidistributed over the two classes.

2 The classes $\mathrm{Av}(3214, 3241, 4213, 4231)$ and $\mathrm{Av}(3124, 3142, 4123, 4132)$

In this paper we are interested in the two classes $\mathrm{Av}(T_1)$ and $\mathrm{Av}(T_2)$, where $T_1 = \{3214, 3241, 4213, 4231\}$ and $T_2 = \{3124, 3142, 4123, 4132\}$. First of all, we characterize the permutations in these classes by means of their left-to-right-maxima decomposition.

Recall that a permutation σ has a *left-to-right maximum* at position i if $\sigma(i) \geq \sigma(j)$ for every $j \leq i$, and that every permutation σ can be decomposed as

$$\sigma = M_1\, w_1\, M_2\, w_2\, \cdots\, M_k\, w_k,$$

where $M_1, \dots, M_k$ are the left-to-right maxima of σ and $w_1, \dots, w_k$ are (possibly empty) words. A characterization of permutations in $\mathrm{Av}(T_1)$ and $\mathrm{Av}(T_2)$ is easily deduced as follows:

Theorem 1. Let $\sigma \in S_n$. Consider the decomposition

$$\sigma = M_1\, w_1\, M_2\, w_2\, \cdots\, M_k\, w_k,$$

where $M_k = n$. Let l_i denote the length of the word w_i. Then

a. σ belongs to $\mathrm{Av}(T_1)$ if and only if

- the reduced form of w_k is a permutation in $\mathrm{Av}(231, 213)$, and
- if $k > 1$, the juxtaposition of the words $w_1, \dots, w_{k-1}$ consists of the smallest $l_1 + \cdots + l_{k-1}$ symbols in the set $[n-1] \setminus \{M_1, \dots, M_{k-1}\}$, listed in increasing order. In particular, the permutation $\alpha = M_1\, w_1\, M_2\, w_2\, \cdots\, M_{k-1}\, w_{k-1}$, after reduced form, avoids 321.

b. σ belongs to $\mathrm{Av}(T_2)$ if and only if

- the reduced form of w_k is a permutation in $\mathrm{Av}(123, 132)$, and
- if $k > 1$, every word w_i, $i \leq k-1$, consists of the $l_i + 1$ greatest unused symbols among those that are less than M_i, listed in decreasing order. In particular the reduced form of $\alpha = M_1\, w_1\, M_2\, w_2\, \cdots\, M_{k-1}\, w_{k-1}$ avoids 312.

This result implies that a permutation $\sigma \in S_n(T_1)$ can be decomposed into

$$\sigma = \alpha\, n\, \beta,$$

where α avoids 321 and β avoids both 231 and 213. Similarly, a permutation $\sigma' \in S_n(T_2)$ can be decomposed into

$$\sigma' = \alpha'\, n\, \beta',$$

where α' avoids 312 and β' avoids both 123 and 132. We note that, in both cases, this property is not a characterization, since the permutations α and α' can not be chosen arbitrarily, according to Theorem 1: for example, the permutation $3\,2\,4\,1$, that belongs to T_1, has precisely the described structure.

It is easy to see that a permutation $\tau = x_1\, x_2\, \ldots\, x_j$ belongs to $\mathrm{Av}(231, 213)$ whenever, for every $i \leq j$, the integer x_i is either the minimum or the maximum of the set $\{x_i, x_{i+1}, \ldots, x_j\}$. Analogously, τ belongs to $\mathrm{Av}(123, 132)$ whenever, for every $i \leq j$, the integer x_i is either the greatest or the second greatest element of the set $\{x_i, x_{i+1}, \ldots, x_j\}$ (see, e.g., [8]).

For example, if we consider the permutation in $S_{10}(T_1)$

$$\sigma = 4\,1\,2\,6\,7\,3\,10\,5\,9\,8,$$

we have

$$\alpha = 4\,1\,2\,6\,7\,3,$$

with

$$M_1 = 4 \quad M_2 = 6 \quad M_3 = 7 \quad M_4 = 10,$$

and

$$\beta = 5\,9\,8.$$

Analogously, if we consider the permutation in $S_{10}(T_2)$

$$\tau = 4\,3\,2\,6\,7\,5\,10\,8\,9\,1,$$

we have

$$\alpha = 4\,3\,2\,6\,7\,5,$$

with

$$M_1 = 4 \quad M_2 = 6 \quad M_3 = 7 \quad M_4 = 10,$$

and

$$\beta = 8\,9\,1.$$

The preceding considerations provide a characterization of the permutations in the two classes that end with the maximum symbol. Let B_n denote the set of permutations in S_n ending by n, by $B_n^{(1)} = B_n \cap \mathrm{Av}(T_1)$, and by $B_n^{(2)} = B_n \cap \mathrm{Av}(T_2)$. Theorem 1 yields immediately the following result:

Corollary 2. The function $\psi_n : B_n \to S_{n-1}$ that maps a permutation σ into the permutation in S_{n-1} obtained by deleting the last symbol in σ yields a bijection between

- $B_n^{(1)}$ and $S_{n-1}(321)$;
- $B_n^{(2)}$ and $S_{n-1}(312)$.

3 Bijections with Dyck prefixes

A *Dyck prefix* is a lattice path in the integer lattice $\mathbb{N} \times \mathbb{N}$ starting from the origin, consisting of up steps $U = (1, 1)$ and down steps $D = (1, -1)$, and never passing below the x-axis. Obviously, a Dyck prefix can be also seen as a word W in the alphabet $\{U, D\}$ such that every initial subword of W contains at least as many symbols U as symbols D.

It is well known (see, e.g., [10]) that the number of Dyck prefixes of length n is $\binom{n}{\lfloor \frac{n}{2} \rfloor}$.

A Dyck prefix ending at ground level is a *Dyck path*.

We now define a map $\Phi : \mathrm{Av}(T_1) \cup \mathrm{Av}(T_2) \to \mathscr{P}$, where $\mathscr{P}$ is the set of Dyck prefixes of even length. First of all, associate the permutation $\sigma = 1$ with the empty path. Then, for every $n \geq 1$, associate a permutation $\sigma \in S_{n+1}(T_1) \cup S_{n+1}(T_2)$ with a Dyck prefix of length $2n$, as follows. Set $\sigma = M_1\, w_1\, M_2\, w_2\, \cdots\, M_k\, w_k$ as above and let l_i be the length of the word w_i.

- If w_k is empty, then

$$\Phi(\sigma) = U^{M_1} D^{l_1+1} U^{M_2-M_1} D^{l_2+1} \cdots U^{M_{k-1}-M_{k-2}} D^{l_{k-1}+1};$$

- If $w_k = x_1 \cdots x_{l_k}$ is not empty, then

$$\Phi(\sigma) = U^{M_1} D^{l_1+1} U^{M_2-M_1} D^{l_2+1} \cdots U^{M_k-M_{k-1}} Q,$$

  where Q is the sequence $Q_1 \cdots Q_{l_k-1}$ of $l_k - 1$ steps such that, for every j, Q_j is an up step if $x_j = \max\{x_j, x_{j+1}, \ldots, x_{l_k}\}$, a down step otherwise.

We point out that, in both cases, the last element of σ is not processed. It is easy to check that the word $\Phi(\sigma)$ is a Dyck prefix of length $2n$.

For example, consider the permutation in $S_{12}(T_1)$

$$\sigma = 6\ 1\ 2\ 9\ 3\ 4\ 5\ 11\ 12\ 7\ 10\ 8.$$

We have $M_1 = 6$, $w_1 = 1\ 2$, $M_2 = 9$, $w_2 = 3\ 4\ 5$, $M_3 = 11$, w_3 is empty, $M_4 = 12$, and $w_4 = 7\ 10\ 8$. The Dyck prefix $\Phi(\sigma)$ is shown in Figure 1.

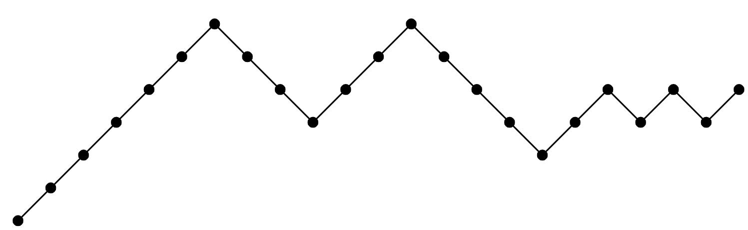

Figure 1: The Dyck prefix $\Phi(6\ 1\ 2\ 9\ 3\ 4\ 5\ 11\ 12\ 7\ 10\ 8)$.

Note that, given a permutation $\sigma \in S_{n+1}(T_1) \cup S_{n+1}(T_2)$, $\sigma = M_1\, w_1 \cdots M_k\, w_k$, the position of the symbol $n + 1$ (that is related to the existence and position of the $(n + 1)$-th up step in the associated Dyck prefix) plays an important role in the definition of the map Φ. For this reason, the $(n+1)$-th up step in $\Phi(\sigma)$, if any, will be called the *cut step* of the path. Needless to say, $\Phi(\sigma)$ contains a cut step if and only if it is not a Dyck path. In fact, if w_k is empty, the path $\Phi(\sigma)$ contains $M_{k-1} = n$ up steps, hence it is a Dyck path. On the other hand, if w_k is not empty, $\Phi(\sigma)$ contains at least $M_k = n+1$ up steps, therefore it does not end at the ground level. The preceding considerations can be summarized as follows:

Proposition 3. Consider a permutation $\sigma \in S_{n+1}(T_1) \cup S_{n+1}(T_2)$. Then, $\Phi(\sigma)$ is a Dyck path if and only if $\sigma(n+1) = n+1$.

Now let Φ_1 and Φ_2 denote the two restrictions of Φ to the sets $\mathrm{Av}(T_1)$ and $\mathrm{Av}(T_2)$. Our next goal is to prove that the two restrictions Φ_1 and Φ_2 are indeed bijections, by defining their inverses as follows. Both Φ_1^{-1} and Φ_2^{-1} associate the empty path with the permutation 1. Consider now a Dyck prefix $P = U^{h_1}D^{s_1}U^{h_2}D^{s_2}\cdots U^{h_r}D^{s_r}$ of length $2n$, $n \geq 1$. The permutation $\sigma = \Phi_1^{-1}(P)$ is defined as follows:

- If P is a Dyck path, namely, $h_1 + \cdots + h_r = n = s_1 + \cdots + s_r$, set

  $$\begin{aligned}
  &\sigma(1) = h_1,\\
  &\sigma(s_1+1) = h_1 + h_2,\\
  &\vdots\\
  &\sigma(s_1 + \cdots + s_{r-1} + 1) = n,\\
  &\sigma(n+1) = n+1.
  \end{aligned}$$

  Then, place the remaining symbols in increasing order in the unassigned positions.

- If P is not a Dyck path, let t denote the index of the ascending run U^{h_t} containing the cut step.

  - if $t = 1$, P decomposes into
    $$P = U^{n+1}Q,$$
    where Q is a lattice path. In this case, set
    $$\sigma(1) = n+1;$$

  - if $t > 1$, P decomposes into
    $$P = U^{h_1}D^{s_1}\cdots U^{h_t}D^{s_t}U^{n+1-h_1-\cdots-h_t}Q.$$

    Set

    $$\begin{aligned}
    &\sigma(1) = h_1,\\
    &\sigma(s_1+1) = h_1 + h_2,\\
    &\vdots\\
    &\sigma(s_1 + \cdots + s_{t-1} + 1) = n+1.
    \end{aligned}$$

In both cases, set $i = s_1 + \cdots + s_{t-1} + 1$ (or $i = 1$ if $t = 1$). Fill the unassigned positions less than i with the smallest remaining symbols placed in increasing order. Then, for every $j = 1, \ldots, n-i$, set either

$\sigma(i+j) = \min[n+1] \setminus \{\sigma(1), \sigma(2), \ldots, \sigma(i+j-1)\}$ if the j-th step of the path Q is a down step, or

$\sigma(i+j) = \max[n+1] \setminus \{\sigma(1), \sigma(2), \ldots, \sigma(i+j-1)\}$ if the j-th step of the path Q is an up step.

Finally, $\sigma(n+1)$ equals the last unassigned symbol.

The permutation $\tau = \Phi_2^{-1}(P)$ can be defined similarly:

- If P is a Dyck path set

$$\begin{aligned}
&\tau(1) = h_1;\\
&\tau(s_1+1) = h_1 + h_2,\\
&\vdots\\
&\tau(s_1 + \cdots + s_{r-1} + 1) = n,\\
&\tau(n+1) = n+1.
\end{aligned}$$

  Then, scan the unassigned positions from left to right and fill them with the greatest unused symbol among those that are less then the closest preceding left-to-right maximum.

- If P is not a Dyck path, let t denote the index of the ascending run U^{h_t} containing the cut step.

  - if $t = 1$, P decomposes into
$$P = U^{n+1}Q,$$
    where Q is a lattice path. In this case, set
$$\tau(1) = n+1;$$

  - if $t > 1$, P decomposes into
$$P = U^{h_1}D^{s_1} \cdots U^{h_t}D^{s_t}U^{n+1-h_1-\cdots-h_t}Q.$$
    Set
$$\begin{aligned}
&\tau(1) = h_1,\\
&\tau(s_1+1) = h_1 + h_2,\\
&\vdots\\
&\tau(s_1 + \cdots + s_{t-1} + 1) = n+1.
\end{aligned}$$

In both cases, set $i = s_1 + \cdots + s_{t-1} + 1$ (or $i = 1$ if $t = 1$). Then, scan the unassigned positions less then i from left to right and fill them with the greatest unused symbol among those that are less than the closest preceding left-to-right maximum. Then, for every $j = 1, \ldots, n-i$, set either

$\tau(i+j) = \max([n+1] \setminus \{\tau(1), \tau(2), \ldots, \tau(i+j-1)\})$ if the j-th step of the path Q is an up step, or

$\tau(i+j) =$ the second greatest element in the set $[n+1] \setminus \{\tau(1), \tau(2), \ldots, \tau(i+j-1)\}$ if the j-th step of the path Q is a down step.

Finally, $\tau(n+1)$ equals the last unassigned symbol.

Theorem 1 ensures that σ belongs to $S_{n+1}(T_1)$, while τ belongs to $S_{n+1}(T_2)$. Moreover, it is easily seen that $\Phi_1^{-1}(\Phi_1(\sigma)) = \sigma$ and $\Phi_2^{-1}(\Phi_2(\tau)) = \tau$. As an immediate consequence, we have the following result:

Theorem 4. The two maps Φ_1 and Φ_2 are bijections. Hence, the cardinality of both $S_{n+1}(T_1)$ and $S_{n+1}(T_2)$ is the central binomial coefficient $\binom{2n}{n}$.

We observe that the enumerative result contained in this theorem can be also deduced from the results by Guibert [4, Theorem 4.6].

In the following, we show that some properties of the permutations in $\mathrm{Av}(T_1)$ and $\mathrm{Av}(T_2)$ can be deduced from certain features of the corresponding Dyck prefix.

First of all, a permutation $\sigma \in S_n$ is *connected* if it does not have a proper prefix of length $l < n$ that is a permutation in S_l. Connected permutations appear in the literature also as *irreducible permutations*.

On the other hand, recall that a *return* of a Dyck prefix is a down step ending on the x-axis. A Dyck prefix P can be uniquely decomposed into $P = P' P''$, where P' is a Dyck path and P'' is a floating Dyck prefix, namely, a Dyck prefix with no return (*last return decomposition*). The last return decomposition of the path $\Phi(\sigma)$ gives information about the connected components of σ. More precisely, we have the following result:

Proposition 5. Let σ be a permutation in $\mathrm{Av}(T_1) \cup \mathrm{Av}(T_2)$. The following are equivalent:

a) the Dyck prefix $\Phi(\sigma)$ can be decomposed as $\Phi(\sigma) = P' P''$, where P' is a Dyck path of length $2l$, and P'' is a (possibly empty) Dyck prefix,

b) σ is the juxtaposition $\sigma' \sigma''$, where σ' a permutation of the set $\{1, \ldots, l\}$, and σ'' is a non-empty permutation.

In this case, letting τ be the permutation in $S_{l+1}(T_1) \cup S_{l+1}(T_2)$ obtained by placing the symbol $l+1$ at the end of σ', and ρ be the reduced form of σ'', we have $P' = \Phi(\tau)$ and $P'' = \Phi(\rho)$.

Proof. We prove the assertion for permutations in $\mathrm{Av}(T_1)$, the other case being analogous. Suppose that $\Phi(\sigma)$ can be decomposed as follows:

$$\Phi(\sigma) = U^{h_1} D^{s_1} \cdots U^{h_p} D^{s_p} P'',$$

where $h_1 + \cdots + h_p = s_1 + \cdots + s_p$, and P'' is a Dyck prefix. Set $l = s_1 + \cdots + s_p$. By the definition of Φ_1^{-1}:

$\sigma(1) = h_1,$

$\sigma(s_1 + 1) = h_1 + h_2,$

$\vdots$

$\sigma(s_1 + \cdots + s_{p-1} + 1) = l.$

Now, we must fill the unassigned positions from 2 to l with the $l - p$ smallest integers different from h_1, $h_1 + h_2$, …, l. This implies that $\sigma(1) \ldots \sigma(l)$ is a permutation of the set $\{1, \ldots, l\}$.

On the other hand, suppose $\sigma = \sigma' \sigma''$, where σ' is a permutation of the set $\{1, \ldots, l\}$, and σ'' is a non-empty permutation. In this case,

$$\sigma = M_1\, w_1\, M_2\, w_2 \cdots M_r\, w_r\, \sigma'',$$

where $M_r = l$. Note that the maximum symbol of σ appears in σ''. This implies that the portion of $\Phi(\sigma)$ that corresponds to the entries in σ' consists of l up steps and l down steps, hence, it is a Dyck path. □

For example, the path $P = P'\, P''$ in Figure 2 corresponds to the permutation $\sigma = \sigma'\, \sigma''$, where $\sigma' = 2\,4\,1\,3$ and $\sigma'' = 7\,5\,9\,6\,8$. Moreover, we have $\tau = \Phi_1^{-1}(P') = 2\,4\,1\,3\,5$ and $\rho = \Phi_1^{-1}(P'') = 3\,1\,5\,2\,4$.

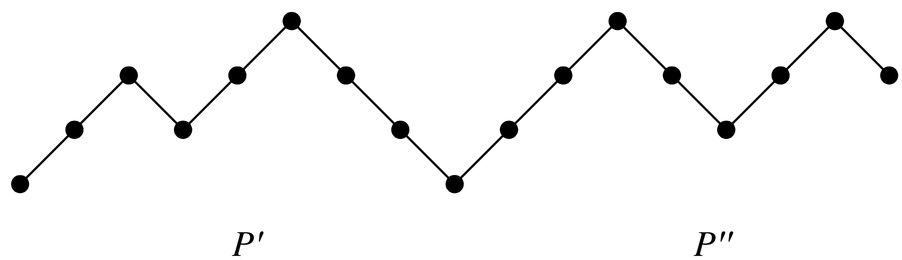

Figure 2: The Dyck prefix $\Phi(2\,4\,1\,3\,7\,5\,9\,6\,8)$.

Proposition 5 implies immediately the following result:

Corollary 6. Connected permutations in $\mathrm{Av}(T_1)$ (resp. $\mathrm{Av}(T_2)$) are in bijection with floating Dyck prefixes. Hence, for every n, there are as many connected permutations in $S_n(T_1)$ (resp. $S_n(T_2)$) as non-connected permutations.

Proof. By Proposition 5, we immediately deduce that connected permutations in $S_n(T_1)$ correspond bijectively to floating Dyck prefixes of length $2n - 2$. These paths are in turn in bijection with Dyck prefixes of length $2n - 3$ (one simply erases the first up step). Hence, the number of connected permutations in $S_n(T_1)$ is

$$\binom{2n-3}{n-2} = \frac{|S_n(T_1)|}{2}.$$

□

Finally, we consider the class $\mathrm{Av}(T_1) \cap \mathrm{Av}(T_2) = \mathrm{Av}(T_1 \cup T_2)$. We have the following result:

Theorem 7. $|S_n(T_1 \cup T_2)| = n \cdot 2^{n-2}$.

Proof. First of all, observe that a connected permutation $\sigma = \alpha\, n\, \beta$ belongs to $S_n(T_1 \cup T_2)$ if and only if α is an arbitrary increasing sequence not containing the symbol 1 (since it must avoid both 321 and 312 and it must be connected) and β is non-empty and either decreasing or order isomorphic to $j\, j-1\, \cdots\, 1\, 2$ (since it must avoid 123, 132, 213 et 231). Let k denote the length of α. Then, the number of connected permutations in $S_n(T_1 \cup T_2)$ is

$$\left(\sum_{k=0}^{n-2} 2\binom{n-2}{k}\right) - 1 = 2^{n-1} - 1.$$

Now, consider a permutation τ in $S_n(T_1 \cup T_2)$ and decompose it as $\tau = \tau'\, \tau''$, where τ'' is its longest connected suffix. Since τ'' contains the symbol n, then τ' must avoid 321 and 312. We distinguish the following cases:

- τ' is empty, hence τ is connected. We have $2^{n-1} - 1$ permutations of this kind.
- τ' is non-empty and τ'' contains at least two elements. If k denotes the length of τ', we have $2^{k-1}(2^{n-1-k} - 1)$ permutations of this kind.
- $\tau'' = n$. In this case τ avoids 321 and 312. We have 2^{n-2} permutations of this kind.

This means that

$$|S_n(T_1 \cup T_2)| = 2^{n-1} - 1 + \left(\sum_{k=1}^{n-2} 2^{k-1}(2^{n-1-k} - 1)\right) + 2^{n-2} = n \cdot 2^{n-2}.$$

□

4 Some statistics over the classes $\mathrm{Av}(T_1)$ and $\mathrm{Av}(T_2)$

The definition of the map Φ suggests that some permutation statistics can be studied simultaneously on the two sets $\mathrm{Av}(T_1)$ and $\mathrm{Av}(T_2)$:

Proposition 8. The three statistics "first element", "position of maximum symbol", and "number of left-to-right maxima" are equidistributed over the classes $\mathrm{Av}(T_1)$ and $\mathrm{Av}(T_2)$.

Proof. Consider a Dyck prefix P of length $2n - 2$ and the two permutations $\sigma = \Phi_1^{-1}(P)$ and $\tau = \Phi_2^{-1}(P)$. Then, we can easily deduce the following:

- Let q denote the length of the first ascending run in P, namely the first maximal sequence of up steps. If $q \geq n$, then $\sigma(1) = \tau(1) = n$. Otherwise, $\sigma(1) = \tau(1) = q$;
- The position of n in both σ and τ equals the number of down steps preceding the cut step, plus one;

- The left-to-right maxima different from n in both σ and τ correspond bijectively to peaks preceding the cut step.

$\square$

Now we study the joint distribution of the two statistics "position of maximum symbol" and "number of left-to-right maxima" over the set $\mathrm{Av}(T_1)$ (bearing in mind that this joint distribution is the same over $\mathrm{Av}(T_2)$). More precisely, we determine the following generating function:

$$J(x, y, w) = \sum_{n \geq 1} \sum_{\sigma \in S_n(T_1)} x^n y^{\mathrm{pos}(\sigma)} w^{\mathrm{lmax}(\sigma)},$$

where $\mathrm{lmax}(\sigma)$ denotes the number of left-to-right maxima in σ, and $\mathrm{pos}(\sigma)$ denotes the position of the maximum symbol in σ.

In the study of permutation statistics over the two considered classes we exploit the last return decomposition of a Dyck prefix described in the previous section. The next Proposition analyzes the behavior of the statistics $\mathrm{pos}(\sigma)$ and $\mathrm{lmax}(\sigma)$ with respect to this decomposition:

Proposition 9. Consider a non connected permutation $\sigma \in \mathrm{Av}(T_1)$. Consider the last return decomposition of $\Phi_1(\sigma)$

$$\Phi_1(\sigma) = P' \, P'',$$

where P' is a non-empty Dyck path and P'' is a floating Dyck prefix. Set $\tau = \Phi_1^{-1}(P')$ and $\rho = \Phi_1^{-1}(P'')$. Then

$$\mathrm{lmax}(\sigma) = \mathrm{lmax}(\tau) + \mathrm{lmax}(\rho) - 1,$$

$$\mathrm{pos}(\sigma) = |\tau| + \mathrm{pos}(\rho) - 1.$$

Proof. Proposition 5 implies that in this case $\sigma = \sigma' \, \sigma''$, where σ' is obtained from τ by deleting its last entry (which is a left-to-right maximum), while σ'' is order isomorphic to ρ. For this reason, $\mathrm{lmax}(\tau) = \mathrm{lmax}(\sigma') + 1$ and $\mathrm{lmax}(\rho) = \mathrm{lmax}(\sigma'')$. Since the symbols appearing in σ'' are greater than those appearing in σ', we get the first assertion. The second assertion is straightforward. $\square$

For example, consider the path P in Figure 2 and the permutation $\sigma = \Phi_1^{-1}(P) = 2\,4\,1\,3\,7\,5\,9\,6\,8$. In this case, $\tau = 2\,4\,1\,3\,5$ and $\rho = 3\,1\,5\,2\,4$, and

$$\mathrm{lmax}(\tau) + \mathrm{lmax}(\rho) - 1 = 4 = \mathrm{lmax}(\sigma),$$

$$|\tau| + \mathrm{pos}(\rho) - 1 = 7 = \mathrm{pos}(\sigma).$$

The above result suggests to determine the joint distribution of the two considered statistics over the set $B_n^{(1)}$ of permutations in $S_n(T_1)$ ending with the maximum symbol, hence corresponding to Dyck paths, and over the set C_n of connected permutations in $S_n(T_1)$, hence corresponding to floating Dyck prefixes.

Set

$$B(x, y, w) = \sum_{n \geq 1} \sum_{\sigma \in B_n^{(1)}} x^n y^{\mathrm{pos}(\sigma)} w^{\mathrm{lmax}(\sigma)}.$$

As shown in the proof of Proposition 8, given a permutation $\sigma \in B_n^{(1)}$, we have $\mathrm{pos}(\sigma) = |\sigma| = n$. Moreover, the number of left-to-right maxima in σ equals the number of peaks in $\Phi_1(\sigma)$, plus one. Hence, if we let $N(x,z)$ denote the Narayana function (A001263 in [9])), namely, the generating function of Dyck paths according to semi-length and number of peaks, then

$$B(x,y,w) = xywN(xy,w).$$

Exploiting the well-known expression of the Narayana function

$$N(x,z) = 1 + \frac{1 - x(1+z) - \sqrt{(1-x(1+z))^2 - 4x^2z}}{2x}$$

(for more detailed information see, e.g., [3]), we get the following identity:

$$B(x,y,w) = w\frac{1 + xy(1-w) - \sqrt{(1-xy(1+w))^2 - 4x^2y^2w}}{2}. \tag{1}$$

Let now

$$C(x,y,w) = \sum_{n\geq 2} \sum_{\sigma \in C_n} x^n y^{\mathrm{pos}(\sigma)} w^{\mathrm{lmax}(\sigma)}$$

be the generating function of the joint distribution of pos and lmax over C. Note that the summation above does not include the case $n = 1$, since the image under Φ of the unique permutation of length 1 is the empty path, that is considered as a Dyck path.

Proposition 9 yields the following functional equation involving the generating functions $J(x,y,w)$, $B(x,y,w)$, and $C(x,y,w)$:

$$J(x,y,w) = B(x,y,w) + \frac{B(x,y,w)C(x,y,w)}{xyw}. \tag{2}$$

Finally, we express the generating function $C(x,y,w)$ in terms of $J(x,y,w)$ and $B(x,y,w)$. To this aim, we describe a relation between the set of floating Dyck prefixes of length $2n$ and the set of all Dyck prefixes of length $2n-2$.

Given a floating Dyck prefix Q, the lattice path obtained from Q by dropping its first and last step is a Dyck prefix. On the other hand, given any Dyck prefix P, we can prepend to P an up step and append either an up or a down step, hence obtaining two Dyck prefixes P_U and P_D, respectively. The prefix P_U is always floating, while P_D is floating if and only if the prefix P is not a Dyck path.

Now let σ denote the permutation in $\mathrm{Av}(T_1)$ corresponding to a given Dyck prefix P and suppose that $\mathrm{pos}(\sigma) = h$ and $\mathrm{lmax}(\sigma) = k$.

- If P is floating, then both P_U and P_D are floating. The definition of P_U and P_D implies that the cut steps in both P_U and P_D correspond to the cut step in P. Set $\sigma_U = \Phi_1^{-1}(P_U)$ and $\sigma_D = \Phi_1^{-1}(P_D)$. Then

$$\mathrm{pos}(\sigma_U) = h = \mathrm{pos}(\sigma_D)$$

$$\mathrm{lmax}(\sigma_U) = k = \mathrm{lmax}(\sigma_D);$$

- If P is a Dyck path, only the path P_U is floating. The cut step in P_U is obviously the last one. Then

$$\mathrm{pos}(\sigma_U) = h,$$
$$\mathrm{lmax}(\sigma_U) = k.$$

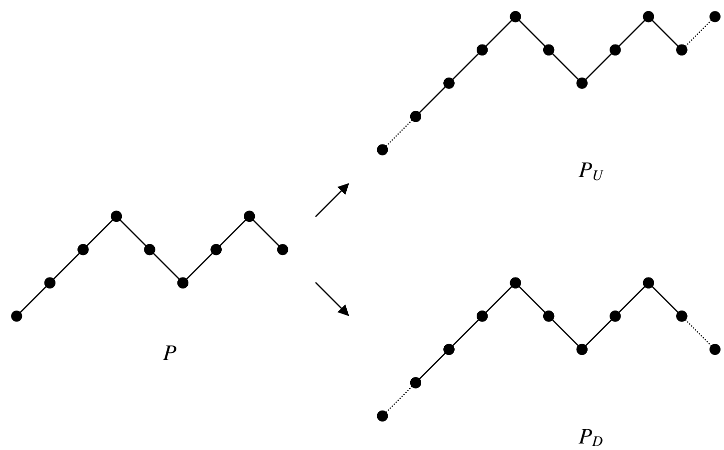

Figure 3: The three Dyck prefixes P, P_U, and P_D

For example, consider the permutations $\sigma = \Phi_1^{-1}(P) = 3\,1\,5\,2\,4$, $\sigma_U = \Phi_1^{-1}(P_U) = 4\,1\,6\,2\,5\,3$, and $\sigma_D = \Phi_1^{-1}(P_D) = 4\,1\,6\,2\,3\,5$, where P, P_U, and P_D are the Dyck prefixes in Figure 3. We have $\mathrm{pos}(\sigma) = \mathrm{pos}(\sigma_U) = \mathrm{pos}(\sigma_D) = 3$, and $\mathrm{lmax}(\sigma) = \mathrm{lmax}(\sigma_U) = \mathrm{lmax}(\sigma_D) = 2$.

Then

$$C(x, y, w) = 2xJ(x, y, w) - xB(x, y, w). \tag{3}$$

Now, exploiting Identities (2) and (3), we get the following expression of $J(x, y, w)$ in terms of $B(x, y, w)$:

Theorem 10. We have

$$J(x, y, w) = \frac{B(x, y, w)(B(x, y, w) - yw)}{2B(x, y, w) - yw}. \tag{4}$$

An explicit expression for $J(x, y, w)$ can be obtained by combining Identities (1) and (4).

Let's now turn our attention to the statistic "first element". Given a permutation σ, we define $\mathrm{head}(\sigma) = \sigma(1)$. We determine the generating function

$$H(x, y) = \sum_{n \geq 1} \sum_{\sigma \in S_n(T_1)} x^n y^{\mathrm{head}(\sigma)} = \sum_{n,k \geq 1} h_{n,k} x^n y^k,$$

where $h_{n,k}$ denotes the number of permutations $\sigma \in S_n(T_1)$ such that $\mathrm{head}(\sigma) = k$ (by Proposition 8, this is also the generating function of the same distribution on $S_n(T_2)$). First of all, given a permutation $\sigma \in S_n(T_1)$, if $\sigma(1) = k$, then the Dyck prefix $\Phi(\sigma)$ starts with

- k up steps followed by a down step, if $k < n$;
- n up steps, if $k = n$.

Hence, in the case $k < n$, if we delete from $\Phi_1(\sigma)$ the first peak we obtain a Dyck prefix whose first ascending run contains at least $k-1$ up steps. It is easy to see that this gives a bijection between the set of Dyck prefixes of length $2n-2$ starting with U^kD and the set of Dyck prefixes of length $2n-4$ starting with U^t, $t \geq k-1$. These arguments imply that, if $n \geq 2$ and $k < n$:

$$h_{n,k} = \sum_{j=k-1}^{n-1} h_{n-1,j}. \tag{5}$$

For $k > 1$, this is equivalent to:

$$h_{n,k} = h_{n,k-1} - h_{n-1,k-2}, \tag{6}$$

with the convention $h_{s,0} = 0$ for every integer s.

The special cases $k = 1$ and $k = n$ can be easily handled as follows. First of all, the permutations in $S_n(T_1)$ such that $\sigma(1) = 1$ correspond to the Dyck prefixes of length $2n-2$ starting with UD, which are in one-to-one correspondence with Dyck prefixes of length $2n-4$. Hence,

$$h_{n,1} = \binom{2n-4}{n-2}.$$

On the other hand, we observe that, given a Dyck prefix of length $2n-2$ starting with U^n, we can change the n−th up step into a down step, obtaining a new lattice path that is still a Dyck prefix. This implies that there are as many Dyck prefixes of length $2n-2$ starting with U^n as those starting with $U^{n-1}D$, namely, $h_{n,n-1} = h_{n,n}$. Recall that permutations $\sigma \in S_n(T_1)$ such that $\sigma(1) = n$ are in bijection with permutations in $S_{n-1}(213, 231)$. It is well known that the number of such permutations is 2^{n-2} (see [8]). Hence,

$$h_{n,n-1} = h_{n,n} = 2^{n-2}.$$

Theorem 11. We have

$$H(x,y) = \frac{xy\left[(xy-1)^2(1-y)\sqrt{1-4x} + x(1-2xy)\right]}{(1-y+xy^2)(1-2xy)\sqrt{1-4x}}.$$

Proof. Formula (6) gives a recurrence for the integers $h_{n,k}$ for every $n \geq 3$ and $2 \leq k \leq n-1$. This fact suggests to consider first the generating function

$$G(x,y) = \sum_{n\geq 2}\sum_{k=1}^{n-1} h_{n,k}x^ny^k.$$

Formula (5) yields

$$G(x,y) = \sum_{n\geq 3}\sum_{k=2}^{n-1} h_{n,k-1}x^ny^k - \sum_{n\geq 3}\sum_{k=2}^{n-1} h_{n-1,k-2}x^ny^k + \sum_{n\geq 2} h_{n,1}x^ny =$$

$$= y\left(G(x,y) - x^2y - \sum_{n\geq 3} h_{n,n-1}x^n y^{n-1}\right) - xy^2\left(G(x,y) - \sum_{n\geq 2} h_{n,n-1}x^n y^{n-1}\right) +$$

$$+ \sum_{n\geq 2} h_{n,1}x^n y.$$

The previous considerations allow us to deduce

$$(1 - y + xy^2)G(x,y) = \frac{x^3y^3 - x^2y^2}{1 - 2xy} + \frac{x^2y}{\sqrt{1-4x}}.$$

Now, $H(x,y)$ can be obtained from $G(x,y)$ as follows:

$$H(x,y) = G(x,y) + xy + \sum_{n\geq 2} 2^{n-2}x^n y^n = G(x,y) + \frac{xy - x^2y^2}{1 - 2xy}.$$

□

5 Other statistics over $\mathrm{Av}(T_1)$ and $\mathrm{Av}(T_2)$

This section is devoted to the study of some permutation statistics that are not equidistributed over the two classes. In both cases, we will translate occurrences of permutation statistics into configurations of the corresponding path.

First of all, we recall that a permutation σ has an *ascent* at position i whenever $\sigma(i) > \sigma(i+1)$, and let $\mathrm{asc}(\sigma)$ denote the number of ascents of σ.

5.1 The class $\mathrm{Av}(T_1)$

We consider the generating function of the distribution of ascents over $\mathrm{Av}(T_1)$:

$$F(x,y) = \sum_{n\geq 1} \sum_{\sigma\in S_n(T_1)} x^n y^{\mathrm{asc}(\sigma)}.$$

The ascents of $\sigma \in \mathrm{Av}(T_1)$ can be recovered from the Dyck prefix $\Phi_1(\sigma)$ as follows:

Proposition 12. The number of ascents of a permutation $\sigma \in \mathrm{Av}(T_1)$ is the sum of:

- The number of valleys and the number of triple descents (i.e., occurrences of DDD) preceding the cut step in $\Phi_1(\sigma)$ (if $\Phi_1(\sigma)$ is a Dyck path, its final down step counts as a valley); and
- The number of down steps following the cut step in $\Phi_1(\sigma)$.

Proof. Decompose σ as

$$\sigma = M_1\, w_1\, M_2\, w_2\, \cdots\, M_k\, w_k,$$

where $M_1, \ldots, M_k$ are the left-to-right maxima of σ. Theorem 1 implies that an ascent can occur in σ only in one of the following positions:

- between two consecutive symbols in w_i, $i \leq k-1$. These two symbols correspond to two consecutive down steps in $\Phi_1(\sigma)$ coming before the cut step. By the definition of Φ_1 these two down steps are necessarily preceded by a further down step;
- before every left-to-right maximum M_i, except for the first one. These positions correspond exactly to the valleys of $\Phi_1(\sigma)$ coming before the cut step. In the special case when σ ends with its maximum symbol, the final ascent of σ corresponds to the final down step of the Dyck path $\Phi_1(\sigma)$;
- in w_k, every time that the minimum unassigned symbol is chosen. These ascents are of course in bijection with the down steps following the cut step.

□

Also in this case, we study the distribution of ascents on the set $B_n^{(1)}$ of permutations in $\mathrm{Av}(T_1)$ such that $\Phi_1(\sigma)$ is a Dyck path, and on the set C_n of connected permutations in $\mathrm{Av}(T_1)$ that correspond to floating Dyck prefixes. Afterwards, we study the behavior of the ascent distribution with respect to the last return decomposition of the corresponding path. Arguments similar to those used in the proof of Proposition 9 lead to

Proposition 13. Consider a permutation $\sigma \in \mathrm{Av}(T_1)$. Suppose that $\Phi_1(\sigma)$ can be decomposed into

$$\Phi_1(\sigma) = P'\, P'',$$

where P' is a non-empty Dyck path and P'' is any Dyck prefix. Set $\tau = \Phi_1^{-1}(P')$ and $\rho = \Phi_1^{-1}(P'')$. Then

$$\mathrm{asc}(\sigma) = \mathrm{asc}(\tau) + \mathrm{asc}(\rho).$$

Consider the generating function of the ascent distribution over the set $B_n^{(1)}$:

$$E(x,y) = \sum_{n\geq 1} \sum_{\sigma \in B_n^{(1)}} x^n y^{\mathrm{asc}(\sigma)}.$$

The present authors [1] determined the generating function $A(x,y,z)$ of the joint distribution of valleys and triple descents over the set of Dyck paths, namely,

$$\begin{aligned} A(x,y,z) &= \sum_{n\geq 0} \sum_{P \in \mathcal{P}_n} x^n y^{v(P)} z^{td(P)} = \\ &= \frac{1}{2xy(xyz - z - xy)} \big(-1 + xy + 2x^2 y \\ &\quad -2x^2y^2 + xz - 2xyz - 2x^2yz + 2x^2y^2z \\ &\quad + \sqrt{1 - 2xy - 4x^2y + x^2y^2 - 2xz + 2x^2yz + x^2z^2}\big) \end{aligned} \tag{7}$$

where $\mathcal{P}_n$ is the set of Dyck paths of semilenght n, $v(P)$ denotes the number of valleys of the path P, and $td(P)$ is the number of occurrences of DDD in P.

We infer

$$E(x,y) = xy(A(x,y,y) - 1) + x. \tag{8}$$

The last summand in Formula (8) takes into account the permutation 1. This implies that:

Proposition 14. We have

$$E(x,y) = \frac{\sqrt{1-4xy+4x^2y(y-1)}-1}{2y(x(y-1)-1)}. \tag{9}$$

Consider now the generating function of the ascent distribution over the set C_n of connected permutations in $\mathrm{Av}(T_1)$.

$$V(x,y) = \sum_{n\geq 2}\sum_{\sigma\in C_n} x^n y^{\mathrm{asc}(\sigma)}.$$

Proposition 13 yields the following functional equation involving the generating functions $F(x,y)$, $E(x,y)$, and $V(x,y)$:

$$F(x,y) = E(x,y) + \frac{E(x,y)V(x,y)}{x}. \tag{10}$$

Finally, we express the generating function $V(x,y)$ in terms of $F(x,y)$ and $E(x,y)$. Given any Dyck prefix P, we can obtain two Dyck prefixes P_U and P_D by prepending to P an up step and appending either an up or a down step, as explained in the previous section and shown in Figure 3. In this case, we have

- If P is floating, then both P_U and P_D are floating, and

$$\mathrm{asc}(\sigma_U) = h, \qquad \mathrm{asc}(\sigma_D) = h+1;$$

- If P is a Dyck path, only the path P_U is floating, and

$$\mathrm{asc}(\sigma_U) = h.$$

Then, we have

$$V(x,y) = (x+xy)F(x,y) - xyE(x,y). \tag{11}$$

Now, exploiting Identities (10) and (11), we get the following expression of $F(x,y)$ in terms of $E(x,y)$:

Theorem 15. We have

$$F(x,y) = \frac{E(x,y)(1-yE(x,y))}{1-E(x,y)-yE(x,y)}. \tag{12}$$

An explicit expression for $F(x,y)$ can be obtained by combining Identities (9) and (12).

In the remaining of this subsection, we characterize the permutations in $\mathrm{Av}(T_1)$ according to the height of the last point of the path $\Phi_1(\sigma)$. Proposition 3 characterizes permutations in $\mathrm{Av}(T_1)$ whose associated prefix ends at the ground level. Now we characterize permutations $\sigma \in S_n(T_1)$ whose corresponding path $\Phi_1(\sigma)$ ends at $(2n-2, 2h)$, $h>0$.

First of all, it is well known that the number of Dyck prefixes of length $2n-2$ ending at $(2n-2, 2h)$ is

$$\binom{2n-3}{n-1-h} - \binom{2n-3}{n-3-h}, \tag{13}$$

(see [10] and A039599 in [9]).

Theorem 16. Let σ be a permutation in $\mathrm{Av}(T_1)$ not ending with the maximum symbol. If the y-coordinate of the last point of the Dyck prefix $\Phi_1(\sigma)$ is $2k-2$, then the longest decreasing subsequence of σ has cardinality k.

Proof. Recall that every permutation $\sigma \in S_n(T_1)$ can be decomposed as follows:

$$\sigma = \alpha\, n\, \beta,$$

where α avoids 321 and β avoids 213 and 231. Moreover, $\beta = x_1\, x_2\, \cdots\, x_j$ is such that the integer x_i is either the minimum or the maximum of the set $\{x_i, x_{i+1}, \ldots, x_j\}$. Let $x_{i_1}, \ldots, x_{i_q}$ denote the subsequence of β consisting of the integers x_i $(1 \le i \le j-1)$ such that x_i is the maximum of the set $\{x_i, x_{i+1}, \ldots, x_j\}$. By definition of the bijection Φ, it is immediately seen that the y-coordinate of the last point of $\Phi_1(\sigma)$ is

$$j + 1 + q - (j - 1 - q) = 2q + 2.$$

It is easy to check that the sequence

$$n\, x_{i_1}\, \cdots\, x_{i_q}\, x_j,$$

of length $k = q + 2$, is the longest decreasing subsequence in σ. This ends the proof. $\square$

The preceding result allows us to characterize the set of Dyck prefixes of length $2n-2$ corresponding via Φ_1 to permutations in $S_n(T_1)$ that avoid also the pattern $k\ k-1\ \cdots\ 2\,1$:

Theorem 17. We have

$$|S_n(T_1, k\ k-1\ \cdots\ 2\,1)| = \binom{2n-2}{n-1} - \binom{2n-2}{n-k}$$

Proof. The preceding results yield immediately:

$$|S_n(T_1, k\ k-1\ \cdots\ 2\,1)| = \sum_{i=0}^{k-2} \binom{2n-3}{n-1-i} - \binom{2n-3}{n-3-i} =$$

$$= \binom{2n-2}{n-1} - \binom{2n-2}{n-k}.$$

$\square$

In particular, consider the case $k = 3$. Of course, we have $S_n(T_1, 321) = S_n(321)$. The set of Dyck prefixes of length $2n-2$ corresponding via Φ_1 to permutations in $S_n(321)$ can be partitioned into two subsets:

a) the set of Dyck paths;

b) the set of Dyck prefixes ending at $(2n-2, 2)$.

It is well known that the set $S_n(321)$ is enumerated by n-th Catalan number. Many bijections between permutations in $S_n(321)$ and Dyck paths of semilength n appear in the literature, notably the bijection defined by Krattenthaler [6]. If σ is a permutation in $S_n(321)$, the relation between the Dyck prefix $\Phi_1(\sigma)$ and the Dyck path $K(\sigma)$ associated with σ by Krattenthaler's bijection can be described as follows:

- If $\Phi_1(\sigma)$ is a Dyck path, $K(\sigma) = \Phi_1(\sigma)UD$;
- If the last point of $\Phi_1(\sigma)$ has coordinates $(2n-2, 2)$, $K(\sigma) = \Phi_1(\sigma)DD$.

6 The class $\mathrm{Av}(T_2)$

In this last section, we study the generating function of the ascent distribution over $\mathrm{Av}(T_2)$

$$M(x,y) = \sum_{n\geq 1} \sum_{\sigma \in S_n(T_2)} x^n y^{\mathrm{asc}(\sigma)}.$$

Proposition 18. The number of ascents in a permutation $\sigma \in \mathrm{Av}(T_2)$ is the number of peaks in the Dyck prefix $\Phi_2(\sigma)$.

Proof. Decompose σ as

$$\sigma = M_1\, w_1\, M_2\, w_2\, \cdots\, M_k\, w_k,$$

where $M_1, \ldots, M_k$ are the left-to-right maxima of σ. Theorem 1 implies that an ascent can occur in σ only in one of the following positions:

- Before every left-to-right maximum M_i, except for the first one. These positions correspond exactly to the peaks of $\Phi_2(\sigma)$ coming before the cut step.
- In w_k, every time that the second greatest unassigned symbol x_j is chosen either immediately after the cut step or immediately after the choice of the maximum unassigned element x_{j-1}. This means that there is an element p such that $x_j < p < n$ or $x_j < p < x_{j-1}$, respectively. In both cases, when p will be placed, it will give rise to an ascent in σ. These ascents are easily seen to be in bijection with peaks following (or involving) the cut step.

□

Hence, we study the distribution of peaks on the set $\mathscr{P}$ of Dyck prefixes, namely, the generating function

$$S(x,y) = \sum_{n\geq 0} \sum_{P \in \mathcal{P}_n} x^n y^{\mathrm{peak}(P)},$$

where $\mathrm{peak}(P)$ denotes the number of peaks in the prefix P.

Let $R(x,y)$ and $\hat{R}(x,y)$, respectively, denote the generating functions of the same distribution on the set of floating Dyck prefixes and on the set of Dyck prefixes ending with U. We have

Proposition 19. The two generating functions $R(x,y)$ and $\hat{R}(x,y)$ coincide.

Proof. Consider a floating Dyck prefix P. We can obviously write $P = U\,P'$, where P' is still a Dyck prefix. Consider now the path $Q = P'\,U$, ending with U. Then, the map $P \mapsto Q$ is a size-preserving bijection between the set of floating Dyck prefixes and the set of Dyck prefixes ending by U, such that $\mathrm{peak}(P) = \mathrm{peak}(Q)$. $\square$

Consider now a Dyck prefix Q ending with U. Then, according to the last return decomposition, we can decompose Q into $Q = P\,U\,P'\,U$, where P is a Dyck path and P' is any Dyck prefix. We deduce that:

$$\hat{R}(x, y) = xN(x, y)S(x, y) = R(x, y), \tag{14}$$

where $N(x, y)$ is the Narayana generating function.

Afterwards, exploiting once again the last return decomposition, we have

$$S(x, y) = N(x, y)\left(R(x, y) + 1\right). \tag{15}$$

Then, combining Identities (14) and (15), we obtain

$$S(x, y) = \frac{N(x, y)}{1 - xN(x, y)^2}.$$

Consider now the generating function $M(x, y)$ of the ascent distribution over $\mathrm{Av}(T_2)$. The definition of the map Φ_2 yields immediately that $M(x, y) = xS(x, y)$. Hence

Theorem 20. We have

$$M(x, y) = \frac{N(x, y)}{1 - xN(x, y)^2}.$$

An expression for $M(x, y)$ can be found by replacing $N(x, y)$ by its explicit formula.

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