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On the Unavoidability of k-Abelian Squares in Pure Morphic Words

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Abstract

We consider a recently defined notion of *k-abelian equivalence* of words by concentrating on avoidability problems. The equivalence class of a word depends on the number of occurrences of different factors of length k for a fixed natural number k and the prefix of the word. We show that over a ternary alphabet, k-abelian squares cannot be avoided in pure morphic words for any natural number k. Nevertheless, computational experiments support the conjecture that even 3-abelian squares can be avoided over a ternary alphabet. This illustrates that the simple but widely used method to produce infinite words by iterating a single morphism is not powerful enough with k-abelian avoidability questions.

1 Introduction

The theory of avoidability is one of the oldest and most studied topics in combinatorics on words. The first results were obtained by Axel Thue already at the beginning of the 20th century [18, 19]. He showed, among other things, the existence of an infinite binary word that does not contain any factor three times consecutively, i.e., that avoids cubes (A010060). Similarly, Thue showed that squares can be avoided in infinite ternary words.

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Since the late 1960's *abelian*, i.e., commutative variants of the above problems were studied. The first nontrivial results were obtained by Evdokimov [8], who showed that commutative squares can be avoided in infinite words over a 25-letter alphabet. The size of the alphabet was reduced to five by Pleasant [17], until the optimal value, four, was found by Keränen [13]. Dekking [7] had already managed to prove that the optimal value for the size of the alphabet on which abelian cubes are avoidable is three.

In this paper we concentrate on a new variant of the square-freeness by defining repetitions via equivalence relations which lie properly in between equality and abelian equality. For this relation we use the notion of k-*abelian equivalence*, where $k \geq 1$ is a natural number. We have computational results that support the conjecture of the existence of an infinite ternary 3-abelian square-free word. On the other hand, we will show that an infinite ternary k-abelian square-free word cannot be obtained by iterating a morphism for any $k \geq 1$. This illustrates the intricacy of our concept, and hints that a simple iteration of a morphism alone might not be a useful method in search for avoidability results on k-abelian repetitions. Whether the use of a combination of morphisms, e.g., more general morphic words, might work here remains unknown. As is clarified in the next section there exist other cases where the avoidability can be established, but not in pure morphic words.

2 Preliminaries

For basic terminology of words, as well as avoidability, we refer to [14, 4]. The basic notion in this paper, k-*abelian equivalence* of words, is defined as follows.

Definition 1. Let $k \geq 1$ be a natural number and Σ an alphabet. We say that words u and v in Σ^+ are k-*abelian equivalent*, in symbols $u \equiv_{a,k} v$, if

1. $\mathrm{pref}_{k-1}(u) = \mathrm{pref}_{k-1}(v)$ and $\mathrm{suf}_{k-1}(u) = \mathrm{suf}_{k-1}(v)$,
2. for all $w \in \Sigma^k$, the number of occurrences of w in u and v coincide, i.e., $|u|_w = |v|_w$, and
3. different words of length at most $k-1$ are inequivalent.

Here pref_{k-1} (resp., suf_{k-1}) is used to denote the prefix (resp., suffix) of length $k-1$. It is easy to see that $\equiv_{a,k}$ is an equivalence relation and the first condition makes the notion a sharpening of abelian equality and even a congruence. In fact, it is enough to require the words to have either a common prefix of length $k-1$ or a common suffix of length $k-1$.

Another definition for k-abelian equivalent words with length at least k is the following:

Definition 2. Let $k \geq 1$ be a natural number and Σ an alphabet. Words u and v in Σ^+ are k-abelian equivalent if for all $w \in \Sigma^{k-1} \cup \Sigma^k$, the number of occurrences of w in u and v coincide, i.e., $|u|_w = |v|_w$.

It is easy to see that the definitions 1 and 2 yields the same equivalence. The condition of Definition 2 follows directly from Definition 1. In addition, $|u|_w = |v|_w$ for all $w \in \Sigma^k$ is required in both definitions. To prove the rest we may assume to the contrary that for words $u, v \in \Sigma^+$ the condition of second definition holds but $\mathrm{pref}_{k-1}(u) = x \neq \mathrm{pref}_{k-1}(v)$. Let $|u|_x = r$ so $|v|_x = r$ and $\sum_{a\in\Sigma}(|u|_{ax}) = r - 1$. Now from $\mathrm{pref}_{k-1}(v) \neq x$ and $|v|_x = r$ follows that $\sum_{a\in\Sigma}(|u|_{ax}) = r$ giving a contradiction.

Now, notions like *k-abelian repetitions* are naturally defined. For instance, $w = uv$ is a *k-abelian square* if and only if $u \equiv_{a,k} v$. We have the following simple observation:

Lemma 3. [9] *If an infinite word w contains a k-abelian m-power then w contains a k'-abelian m-power for each $1 \leq k' \leq k$.*

For some earlier results about k-abelian equivalence we refer to [11, 10, 12]. We have shown with Saarela and Saari [11] that the longest ternary word avoiding 2-abelian squares has 537 letters. In addition, we have the following result [9] concerning such prefix-preserving morphisms over an alphabet Σ that they map each letter to a word of length at least two:

Theorem 4. [9] *Let h be a morphism over Σ such that* $\min\{|h(a)| : a \in \Sigma\} > 1$. *Then the following two conditions are equivalent:*

1. *The infinite word $h^\infty(a)$ contains a k-abelian m-power for some $k \geq 2$.*

2. *The infinite word $h^\infty(a)$ contains a k-abelian m-power for each $k \geq 1$.*

A prefix-preserving (or *prolongable*) *morphism* is a morphism $h : \Sigma^* \to \Sigma^*$ for which there exists a letter $a \in \Sigma$ and a word $\alpha \in \Sigma^*$ such that $h(a) = a\alpha$ and $h^n(\alpha) \neq \epsilon$ for every $n \geq 0$. We call an infinite word *a pure morphic word* if it is obtained by iterating a prefix-preserving morphism. *A morphic word* is obtained from a pure morphic word by taking an image of it by a morphism or equivalently under a coding, see [1].

We remark that Theorem 4 also holds in the case where the prefix-preserving morphism h over Σ has the following property:

$$\forall a \in \Sigma\ \exists n > 0 : |h^n(a)| > 1. \tag{1}$$

Theorem 4 can be extended in this way because if (1) holds we can choose $n' > 0$ such that $\min\left\{|h^{n'}(a)| : a \in \Sigma\right\} > 1$ and we can consider the morphism $\hat{h} = h^{n'}$. If $a \in \Sigma$ is a letter for which h is prolongable then $\hat{h}^\infty(a) = h^\infty(a)$. Let us denote by H the set of prefix-preserving morphisms having the property (1).

Thue already showed that there exists an infinite pure morphic square-free word over ternary alphabet, see [19]. The question whether pure morphic words can avoid k-abelian squares over ternary alphabets is reasonable because we have a strong evidence that an infinite ternary 3-abelian square-free word might exist.

Example 5. By producing in a lexicographic order longer and longer 3-abelian square-free ternary words we have obtained words containing more than 100,000 letters. The number of ordinary ternary square-free words of length n is listed as a sequence (A006156) and in our case the numbers of 3-abelian square-free words over a ternary alphabet seems to grow exponentially, at least for small values of n, see figure 1. These computer based analyzes suggest that there might exist an infinite 3-abelian square-free word over a ternary alphabet.

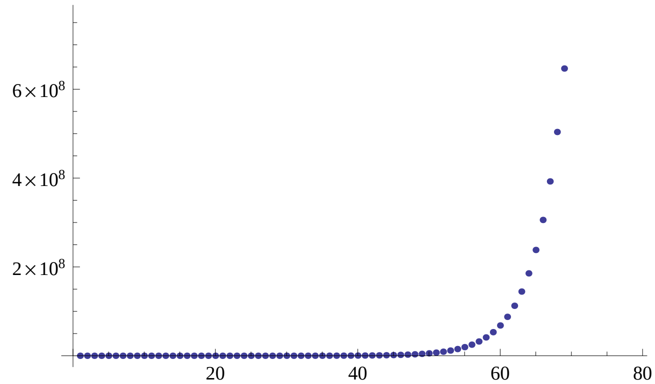

Figure 1: The numbers of 3-abelian square-free words of length n.

In contrary, each infinite word over a three-letter alphabet contains 2-abelian squares, see [11], so it follows from Theorem 4 that, for each $h \in H$ over a ternary alphabet, $h^\infty(a)$ contains a k-abelian square for all $k \geq 1$. In this paper we will show that, a bit surprisingly, we cannot obtain an infinite ternary k-abelian square-free word, for any $k \geq 1$, by simply iterating any prefix-preserving morphism.

Although, iterated morphisms constitute a common tool in avoidability questions, in the ordinary word case there also exist patterns that can be avoided in binary words, but not in words produced by only iterating a morphism. Cassaigne gave a classification of binary patterns according to avoidability in binary words, in binary pure morphic words and in ternary pure morphic words, [3]. The patterns $\alpha^2\beta^2\alpha$, $\alpha\beta\alpha^2\beta$ and $\alpha\beta\alpha^2\beta\alpha$ are such that they can be avoided over a binary alphabet, but not in infinite binary pure morphic words. Similarly, it seems that 3-abelian squares can be avoided over a ternary alphabet but not in infinite ternary pure morphic words. A related well known example is given by the famous (cube-free) Kolakoski word (A000002): it is not pure morphic [5], but it is unknown whether it is morphic. Indeed, it is not known whether its subword complexity is quadratic, as would be in the case of a morphic word. On the other hand Currie has conjectured (see [6] and [15, Problem 3.1.5, p. 132]), that if a pattern p is avoidable on alphabet A, there exist an alphabet B, two morphisms $f : B^* \to A^*$ and $g : B^* \to B^*$ and a letter $a \in B$ such that the infinite word $f(g^\infty(a))$ avoids p, that is, p is avoidable in morphic words. Based on our results and intuition we do not dare to make a related conjecture for k-abelian repetitions, even in the case where the pattern is an integer power.

Rather little is known about avoiding k-abelian repetitions in general or even in morphic words. Results in [10] and [16] are examples of the case where the k-abelian cube-freeness over a binary alphabet is obtained in morphic words. The results cover the cases $k \geq 5$.

3 Unavoidability of 3-abelian squares in pure morphic words

In this section we consider k-abelian square-freeness and nonerasing morphisms that do not belong to the set H. We start by stating few lemmas and conclude our avoidability result for the case $k = 3$ from these lemmas. Let h be such a prefix-preserving morphism over $\Sigma = \{a, b, c\}$ that is prolongable for a and let $h^\infty(a) = w$. If w is k-abelian square-free then at least one letter has to map to a letter as a consequence of Theorem 4. On the other hand, Lemma 7 shows that at most one letter may map to a letter. We continue by remarking that an infinite ternary k-abelian square-free word, in fact a word avoiding ordinary word squares, has to contain each possible factor of length two except aa, bb and cc.

Lemma 6. *Each word of length ≥ 14 over an alphabet $\{a, b, c\}$ contains a square if some of the factors ab, ac, ba, bc, ca or cb do not belong to the set of the factors of w, i.e., to the set $F(w)$.*

Proof. Assume that $ab \notin F(w)$. The other cases are symmetric. It is easy to check that $bcbacbcacbaca$ is the longest word avoiding ab and squares. □

By using Lemma 6 we may prove the following:

Lemma 7. *Consider the morphism*

$$h : \begin{cases} a \mapsto a\alpha \\ b \mapsto \beta \\ c \mapsto \gamma \end{cases}, \text{ where } \alpha \in \Sigma^+ \text{ and } \beta, \gamma \in \Sigma.$$

Now $h^\infty(a) = w$ contains a square.

Proof. If $h(b) = a$ then $h(ba) = aa\alpha$ and by Lemma 6 ba as well as $h(ba)$ are factors of w. Similarly, the case $h(c) = a$ gives a square.

If $h(b) = b = h(c)$ then the image of the factor bc gives a square $h(bc) = bb$ and by Lemma 6 $bc, h(bc) \in F(w)$. The case $h(b) = c = h(c)$ is similar.

Now there are two cases left. First, if $h(b) = c$ and $h(c) = b$ then $h^2(b) = b$ and $h^2(c) = c$ so without loss of generality it is enough to consider the case $h(b) = b$ and $h(c) = c$. If $a^{-1}h(a) \in \{b, c\}^+$ then $a^{-1}h^\infty(a) \in \{b, c\}^\omega$ and the binary infinite word cannot be square-free. Thus we have to check if $h(a) = a\alpha_1 a\alpha_2$, where $\alpha_1 \in \{b, c\}^+$ and $\alpha_2 \in \{a, b, c\}^*$. Now

$$a \overset{h}{\mapsto} a\alpha_1 a\alpha_2 \overset{h}{\mapsto} a\alpha_1 a\alpha_2 \alpha_1 a\alpha_1 a\alpha_2 h(\alpha_2),$$

because $h(\alpha_1) = \alpha_1$ and thus $h^2(a)$ contains a square $\alpha_1 a\alpha_1 a$. This completes the proof. □

Thus, if there exists a morphism h over a three-letter alphabet Σ that generates an infinite k-abelian square-free word it maps exactly one letter to a letter and, in fact, it has to map the letter to itself. Otherwise $|h^2(a)| > 1$ for all $a \in \Sigma$. Without loss of generality, we may assume that $h(b) = b$. By Lemma 6 we may consider the images of all the factors of length two except aa, bb and cc to further restrict the form of the morphism.

Lemma 8. *If h is a prefix-preserving morphism over the alphabet $\Sigma = \{a, b, c\}$ that generates an infinite k-abelian square-free word, it has to be one of the following forms to avoid usual word squares:*

$$h : \begin{cases} a \mapsto a\alpha a \\ b \mapsto b \\ c \mapsto c\gamma c \end{cases} \quad \textit{or} \quad h : \begin{cases} a \mapsto a\alpha c \\ b \mapsto b \\ c \mapsto a\gamma c \end{cases}, \quad \textit{where } \alpha, \gamma \in \Sigma^+.$$

Proof. We already have that

$$h : \begin{cases} a \mapsto a\alpha' \\ b \mapsto b \\ c \mapsto \gamma' \end{cases}, \text{ where } \alpha', \gamma' \in \Sigma^+ \text{ and } |\gamma'| \geq 2.$$

Thus $h(ab) = a\alpha' b$, from which it follows that b cannot be a suffix of α'. From $h(cb) = \gamma' b$ and $h(ca) = \gamma' a\alpha'$ it follows that c has to be a suffix of γ'. In addition, b cannot be a prefix of γ' because $h(bc) = b\gamma'$. Now we have that $h(a) = a\alpha a$ or $h(a) = a\alpha c$ and $h(c) = a\gamma c$ or $h(c) = c\gamma c$, where $\alpha, \gamma \in \Sigma^*$. By considering $h(ac)$ we conclude that

$$h : \begin{cases} a \mapsto a\alpha a \\ b \mapsto b \\ c \mapsto c\gamma c \end{cases} \quad \text{or} \quad h : \begin{cases} a \mapsto a\alpha c \\ b \mapsto b \\ c \mapsto a\gamma c \end{cases}, \quad \text{where } \alpha, \gamma \in \Sigma^*.$$

If $\alpha = \epsilon$ (and similarly for γ) then $h(a) = aa$ or $h(a) = ac$, and both cases lead to a square. In the latter case, $ca \mapsto a\gamma cac \mapsto ach(\gamma)a\gamma caca\gamma c$. Thus neither α nor γ can be the empty word, which completes the proof. □

Next we restrict our consderation to k-abelian square-freeness with $k = 3$.

Lemma 9. *Consider the morphism*

$$h : \begin{cases} a \mapsto a\alpha c \\ b \mapsto b \\ c \mapsto a\gamma c \end{cases}, \quad \textit{where } \alpha, \gamma \in \Sigma^+.$$

Now $h^\infty(a)$ has 3-abelian square.

Proof. Let $h^\infty(a) = w$, and let us assume that it is square-free. As mentioned, each infinite ternary word contains a 2-abelian square. In particular, $u_1 u_2 \in F(w)$ where u_1 and u_2 are 2-abelian equivalent. We have that $u_1 \neq u_2$ and thus $|u_1| > 3$. Each factor of length three of the word $h(u_1)$ (resp., for $h(u_2)$) is a factor of $h(x_1 x_2 x_3)$, where $x_1 x_2 x_3$ is a factor of u_1 and $x_1, x_2, x_3 \in \Sigma$. In fact, the only case where the image of two consecutive letters is not enough is the case where $x_2 = b$ and we take the factor $\mathrm{suf}_1(h(x_1))h(x_2)\mathrm{pref}_1(h(x_3)) = cba$. Thus the factors of length three of the word $h(u_1)$ are determined by the factors of length two of the word u_1 and the number of factors $x_1 b x_3$ in u_1 where $x_1, x_3 \in \{a, c\}$. Because u_1 and u_2 are 2-abelian equivalent, the words $h(u_1)$ and $h(u_2)$ have the same occurrences of the factors of length three.

In addition, if $\mathrm{pref}_1(u_1) = b$ then $\mathrm{pref}_2(h(u_1)) = ba$ because bb cannot be a prefix of u_1 and in other cases $|h(\mathrm{pref}_1(u_1))| \geq 3$. Because $\mathrm{pref}_1(u_1) = \mathrm{pref}_1(u_2)$ we have that $\mathrm{pref}_2(h(u_1)) = \mathrm{pref}_2(h(u_2))$. Correspondingly, $\mathrm{suf}_2(h(u_1)) = \mathrm{suf}_2(h(u_2))$. Now $h(u_1u_2) \in F(w)$ and $h(u_1)$ and $h(u_2)$ are 3-abelian equivalent. $\square$

From the previous lemmas we have that if there exists a prefix-preserving morphism over a three-letter alphabet generating an infinite 3-abelian square-free word, it has to be of the following form:

$$h : \begin{cases} a \mapsto a\alpha a \\ b \mapsto b \\ c \mapsto c\gamma c \end{cases}, \text{ where } \alpha, \gamma \in \Sigma^+.$$

With the following three lemmas we will show that the morphism above always leads to a word containing 3-abelian square. For the lemmas 10, 11 and 12 let us denote $h^\infty(a) = w$.

Lemma 10. *If $aba, cbc \in F(w)$ then w contains a square.*

Proof. Here $h(aba) = a\alpha aba\alpha a$ and $h(cbc) = c\gamma cbc\gamma c$. This means that the only nontrivial case is $h(a) = a\alpha' ca$ and $h(c) = ca\gamma' c$ where $\alpha', \gamma' \in \Sigma^*$. Now $h(ac) = a\alpha' caca\gamma' c$ gives a square $caca$. $\square$

Thus we may restrict the word w not to contain the factor aba or cbc. These cases are symmetric and it is enough to discuss the other one.

Lemma 11. *If $aba \in F(w)$ and $cbc \notin F(w)$ then w contains a square.*

Proof. Now $h(aba) = a\alpha aba\alpha a$. By avoiding trivial squares this gives $h(a) = aca$ or $h(a) = ac\alpha_1 ca$ where $\alpha_1 \in \Sigma^+$. If $h(a) = aca$ then $h(ac) = acac\gamma c$ gives a square $acac$. Thus $h(ac) = ac\alpha_1 cac\gamma c$ and we may conclude that $h(a) = ac\alpha_2 bca$ and $h(c) = cb\gamma_1 c$ where $\alpha_2, \gamma_1 \in \Sigma^*$. The cases $h(a) = acbca$ and $h(c) = cbc$ are not possible because $cbc \notin F(w)$. Further, $h(ca)$ gives $h(a) = acb\alpha_3 bca$ and $h(c) = cb\gamma_2 bc$ where $\alpha_3, \gamma_2 \in \Sigma^+$. The assumption $cbc \notin F(w)$ also gives that $h(a) = acba\alpha_4 bca$ where $\alpha_4 \in \Sigma^*$. Now $cba \in F(h(a)) \subset F(w)$ and $h(cba) = cb\gamma_2 bcbacba\alpha_4 bca$ which contains the square $cbacba$. $\square$

Now we know that w cannot have either aba or cbc as a factor.

Lemma 12. *If $aba, cbc \notin F(w)$ then w contains a 3-abelian square.*

Proof. By [18, 19, 2], the only infinite pure morphic square-free word over a ternary alphabet avoiding aba and cbc is obtained by iterating a morphism

$$g : \begin{cases} a \mapsto abc \\ b \mapsto ac \\ c \mapsto b \end{cases}.$$

Now $g \in H$ and thus $g^\infty(a)$ contains a 3-abelian square. In fact, already $g^5(a)$ contains a 3-abelian square: $g^5(a) = ab\underbrace{cacbabcbacab}\underbrace{cacbacabcbab}\ldots$. $\square$

Now consider an erasing morphism (the one with $c \mapsto \epsilon$ behaves similarly)

$$e : \begin{cases} a \mapsto a\alpha \\ b \mapsto \epsilon \\ c \mapsto \gamma \end{cases} .$$

The word $e^{\infty}(a)$ cannot contain aba as well as cbc because $e(aba) = a\alpha a\alpha$ and $e(cbc) = \gamma\gamma$. Thus this case returns to the proof of Lemma 12 and, in fact, the same argument can be used for general k-abelian case, too.

Now we are ready to state the first part of the main result.

Theorem 13. *Every ternary infinite pure morphic word contains a 3-abelian square.*

Proof. The proof is clear from the lemmas above. With Lemmas 7, 8, and 9 we have restricted the form of the morphism that could produce an infinite ternary 3-abelian square-free word and with Lemmas 10, 11, and 12 we have shown that the word obtained by iterating that type of morphism always contains a 3-abelian square. □

4 Unavoidability of k-abelian squares in pure morphic words

In this section we extend the result of Theorem 13 from 3-abelian squares to arbitrary k-abelian squares. We start with a lemma.

Lemma 14. *If a ternary infinite pure morphic word contains a $(k-1)$-abelian square for $k > 3$, then it contains a k-abelian square.*

Proof. Consider a ternary prefix-preserving morphism h and assume that $w = h^{\infty}(a)$ contains a $(k-1)$-abelian square, i.e., there exist $(k-1)$-abelian equivalent words u_1 and u_2 such that $u_1u_2 \in F(w)$. Assume, contrary to what we want to prove, that w is k-abelian square-free. From Lemma 8 it follows that we may assume $|h(a)|, |h(c)| \geq 3$ and $h(b) = b$.

Now every factor of length k in the word $h(u_1)$ (resp., $h(u_2)$) is a factor of $h(v_1)$ (resp., $h(v_2)$) where $v_1 \in F(u_1)$ and $|v_1| \leq \left\lfloor \frac{k-3}{2} \right\rfloor + 3$. This is because if the word contains factors of which at most every other has length one and all the others have length at least three we can reach at most $\left\lfloor \frac{k-3}{2} \right\rfloor + 3$ factors with a subword of length k. For example, with a subsword of length 11 we can reach at most 7 factors as depicted below.

$$\ldots - * - - \underbrace{- * - - - * - - - * -}_{} - - * - \ldots$$

Thus the factors of $h(u_1)$ (resp., $h(u_2)$) of length k are determined by factors of u_1 (resp., u_2) of length at most $k-1$ because $k - 1 \geq \left\lfloor \frac{k-3}{2} \right\rfloor + 3$ for $k > 3$. We know that u_1 and u_2 are $(k-1)$-abelian equivalent thus $h(u_1)$ and $h(u_2)$ have the same number of occurrences of different factors of length k.

In addition, $\mathrm{pref}_{k-2}(u_1) = \mathrm{pref}_{k-2}(u_2)$ implying $\mathrm{pref}_{k-1}(h(u_1)) = \mathrm{pref}_{k-1}(h(u_2))$. The suffixes of length $k-1$ of the words $h(u_1)$ and $h(u_2)$ coincide, too. This shows that $h(u_1)$ and $h(u_2)$ are k-abelian equivalent and clearly $h(u_1)h(u_2) \in F(w)$. □

Now we are ready to state the second part of the main result generalizing Theorem 13.

Theorem 15. *Every ternary infinite pure morphic word contains a k-abelian square for any $k \geq 1$.*

Proof. Theorem 13 states the result for $k = 3$ and by lemma 3 claim holds also for $1 \leq k \leq 3$. Now we can use Lemma 14 for the case $k = 4$ and inductively prove the claim for $k \geq 4$. □

5 Conclusion

We know from Thue that an infinite square-free word over a ternary alphabet can be obtained by iterating a morphism. In this paper we have shown that infinite pure morphic words over ternary alphabets do not avoid k-abelian squares for any $k \geq 1$. Thus if an infinite ternary k-abelian square-free word exists for some $k \geq 1$, an iterated morphism is not the way to produce it. This differs from the situation of ordinary square-freeness and abelian square-freeness and emphasizes the intricate nature of the k-abelian equivalence with respect to morphism iteration.

As mentioned, we have been able to obtain ternary 3-abelian square-free words longer than 100,000 letters by a computer. This suggests that there might exist an infinite 3-abelian square-free word over a ternary alphabet, but it remains open how to produce it. For example, it is not known whether a morphic word could give an infinite 3-abelian square-free word.[2]

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