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# The Generalized Stirling and Bell Numbers Revisited

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**Abstract**

The generalized Stirling numbers $\mathfrak{S}_{s;h}(n,k)$ introduced recently by the authors are shown to be a special case of the three parameter family of generalized Stirling numbers $S(n,k;\alpha,\beta,r)$ considered by Hsu and Shiue. From this relation, several properties of $\mathfrak{S}_{s;h}(n,k)$ and the associated Bell numbers $\mathfrak{B}_{s;h}(n)$ and Bell polynomials $\mathfrak{B}_{s;h|n}(x)$ are derived. The particular case $s=2$ and $h=-1$ corresponding to the meromorphic Weyl algebra is treated explicitly and its connection to Bessel numbers and Bessel polynomials is shown. The dual case $s = -1$ and $h = 1$ is connected to Hermite polynomials. For the general case, a close connection to the Touchard polynomials of higher order recently introduced by Dattoli et al. is established, and Touchard polynomials of negative order are introduced and studied. Finally, a $q$-analogue $\mathfrak{S}_{s;h}(n,k|q)$ is introduced and first properties are established, e.g., the recursion relation and an explicit expression. It is shown that the $q$-deformed numbers $\mathfrak{S}_{s;h}(n,k|q)$ are special cases of the type-II $p,q$-analogue of generalized Stirling numbers introduced by Remmel and Wachs, providing the analogue to the undeformed case ($q = 1$). Furthermore, several special cases are discussed explicitly, in particular, the case $s = 2$ and $h = -1$ corresponding to the $q$-meromorphic Weyl algebra considered by Diaz and Pariguan.

# 1 Introduction

The Stirling numbers (of the first and second kind) are certainly among the most important combinatorial numbers as can be seen from their occurrence in many varied contexts, see, e.g., [16, 64, 66, 75] and the references given therein. One of these interpretations is in terms of *normal ordering* special words in the *Weyl algebra* generated by the variables $U, V$ satisfying

$$UV - VU = 1, \tag{1}$$

where on the right-hand side the identity is denoted by 1. A concrete representation for (1) is given by the operators

$$U \mapsto D \equiv \frac{d}{dx}, \quad V \mapsto X$$

acting on a suitable space of functions (where $(X \cdot f)(x) = xf(x)$). In the mathematical literature, the simplification (i.e., normal ordering) of words in $D, X$ can be traced back to at least Scherk [59] (see [3] for a nice discussion of this and several other topics related to normal ordering words in $D, X$) and many similar formulas have appeared, e.g., in connection with operator calculus [16, 56, 57, 58]. Already Scherk derived that the Stirling numbers of second kind $S(n, k)$ (A008277 in [64]) appear in the normal ordering of $(XD)^n$, or, in the variables used here,

$$(VU)^n = \sum_{k=1}^{n} S(n,k) V^k U^k. \tag{2}$$

This relation has been rediscovered countless times. In the physical literature, this connection was rediscovered by Katriel [35] in connection with normal ordering expressions in the boson annihilation operator $a$ and creation operator $a^\dagger$ satisfying the commutation relation $aa^\dagger - a^\dagger a = 1$ of the Weyl algebra. Since the normal ordered form has many desirable properties simplifying many calculations, the normal ordering problem has been discussed in the physical literature extensively; see [3] for a thorough survey of the normal ordering for many functions of $X$ and $D$ with many references to the original literature. Let us point out some classical references [1, 9, 49, 53, 74], some more recent references [2, 5, 26, 31, 40, 62], as well as some references concerned with the $q$-deformed situation [4, 36, 38, 43, 61]. Relation

(2) has been generalized by several authors to the form (here we assume $r \geq s$)

$$(V^r U^s)^n = V^{n(r-s)} \sum_{k=1}^{n} S_{r,s}(n,k) V^k U^k, \tag{3}$$

where the coefficients are, by definition, *generalized Stirling numbers of the second kind*, see, e.g., [3, 6, 10, 11, 37, 46, 47, 48, 51, 59, 60, 67, 68, 69, 70, 71] (e.g., A000369, A035342, A078739, A078740 in [64])). Clearly, one has $S_{1,1}(n,k) = S(n,k)$. Let us briefly mention that also $q$-analogues of these Stirling numbers have been discussed [46, 47, 60, 71].

In another direction, Hsu and Shiue introduced in [34] a three parameter family of generalized Stirling numbers as connection coefficients which unified many of the previous generalizations. This family of generalized Stirling numbers has been treated subsequently in a number of papers, see, e.g., [17, 19, 33]; furthermore, $q$-analogues [18, 20, 65] as well as $p, q$-analogues [7, 55] have been studied.

Two of the present authors considered in [41] variables $U, V$ satisfying the following generalization

$$UV - VU = hV^s \tag{4}$$

of the commutation relation (1), where it was assumed that $h \in \mathbb{C} \setminus \{0\}$ and $s \in \mathbb{R}$. The parameter $h$ should be considered as a free "deformation parameter" (Planck's constant) and we will often consider the special case $h = 1$. Note that in the case $s = 0$ equation (4) reduces to (1) (if $h = 1$). Now, it is very natural to consider in the context of arbitrary $s \in \mathbb{R}$ the expression $(VU)^n$ for variables $U, V$ satisfying (4) and to define the *generalized Stirling numbers* $\mathfrak{S}_{s;h}(n,k)$ by

$$(VU)^n = \sum_{k=1}^{n} \mathfrak{S}_{s;h}(n,k) V^{s(n-k)+k} U^k. \tag{5}$$

The coefficients $\mathfrak{S}_{s;h}(n,k)$ can be interpreted as some kind of generalized Stirling numbers of the second kind. They are very closely related to the generalized Stirling numbers $S_{r,1}(n,k)$ considered by Lang [37] - and already before him by, e.g., Scherk [59], Carlitz [11], Toscano [67, 68, 69] and Comtet [16] - and more recently in [3, 10, 48, 51]. Burde considered in [8] matrices $X, A$ satisfying $XA - AX = X^p$ with $p \in \mathbb{N}$ and discussed the coefficients which appear when $(AX)^n$ is normal ordered. Note that in terms of our variables $U, V$, Burde considered the normal ordered form of $(UV)^n$, which is from our point of view less natural. However, since one can write $(VU)^n = V(UV)^{n-1}U$, these two problems are, of course, intimately related. Diaz and Pariguan [23] described normal ordering in the *meromorphic Weyl algebra*. Recall that for $s = 0$ (and $h = 1$), one has the r epresentation $D, X$ of the variables $U, V$ satisfying the relation $DX - XD = 1$ of the Weyl algebra (1). Considering instead of $X$ the operator $X^{-1}$, one finds the relation $D(X^{-1}) - X^{-1}D = -X^{-2}$ and, thus, a representation of the variables $U, V$ for $s = 2$ and $h = -1$.

The present authors began a serious study of the generalized Stirling numbers $\mathfrak{S}_{s;h}(n,k)$ (and related generalized Bell numbers $\mathfrak{B}_{s;h}(n)$) in [44], where also more references to the literature can be found. In [44], many properties of these numbers were derived, e.g., the recursion relation, an explicit expression and several explicit examples for special choices of the parameters $s, h$. This study was continued in [45], where the particular case $s = 2$

and $h = -1$ corresponding to the meromorphic Weyl algebra was treated briefly as well as related $q$-identities. Furthermore, in [42] it was shown that the exponential generating function of the generalized Bell numbers $\mathfrak{B}_{s;h}(n)$ satisfies an algebraic differential equation as in the conventional case. Thus, although many properties of these generalized Stirling and Bell numbers are known, there still exist a lot of questions. It is the aim of the present paper to fill some of these gaps in our current understanding of these numbers.

The paper is organized as follows. In Section 2, several general results for $\mathfrak{S}_{s;h}(n,k)$ and $\mathfrak{B}_{s;h}(n)$ are established. In particular, it is shown that $\mathfrak{S}_{s;h}(n,k)$ corresponds to a special case of the three parameter family of generalized Stirling numbers introduced by Hsu and Shiue [34], allowing to deduce several properties of $\mathfrak{S}_{s;h}(n,k)$ and $\mathfrak{B}_{s;h}(n)$ quickly. In Section 3, the special case $s = 2$ and $h = -1$ is treated in detail and the connection to Bessel numbers and Bessel polynomials is discussed. Using operator methods, relations between the generalized Stirling numbers are derived in Section 4. Combinatorial proofs of these relations are given in Section 5. In Section 6, a close connection to the Touchard polynomials of higher order introduced recently by Dattoli et al. [21] is discussed, and Touchard polynomials of negative order a re introduced and studied. A $q$-analogue of the numbers $\mathfrak{S}_{s;h}(n,k)$ is introduced in Section 7 and first properties are established, e.g., the recursion relation, a general closed form expression, and explicit formulas in several special cases. Finally, in Section 8, some conclusions are presented.

## 2 Some general results

The generalized Stirling numbers (5) can be defined for $s \in \mathbb{R}$ and $h \in \mathbb{C} \setminus \{0\}$, equivalently, by the recursion relation

$$\mathfrak{S}_{s;h}(n+1,k) = \mathfrak{S}_{s;h}(n,k-1) + h(k+s(n-k))\mathfrak{S}_{s;h}(n,k), \tag{6}$$

if $n \geq 0$ and $k \geq 1$, with $\mathfrak{S}_{s;h}(n,0) = \delta_{n,0}$ and $\mathfrak{S}_{s;h}(0,k) = \delta_{0,k}$ for $n,k \in \mathbb{N}_0$ [44]. The related Bell numbers are defined by $\mathfrak{B}_{s;h}(n) = \sum_{k=0}^{n} \mathfrak{S}_{s;h}(n,k)$. More generally, let us define the generalized Bell polynomials $\mathfrak{B}_{s;h|n}(x)$ by

$$\mathfrak{B}_{s;h|n}(x) := \sum_{k=0}^{n} \mathfrak{S}_{s;h}(n,k)x^k \tag{7}$$

such that $\mathfrak{B}_{s;h|n}(1) = \mathfrak{B}_{s;h}(n)$, the $n$-th generalized Bell number. Choosing $s = 0$ and $h = 1$ yields the Stirling numbers of the second kind $S(n,k)$ (A008277 in [64]), whereas choosing $s = 1$ and $h = -1$ yields the Stirling numbers of the first kind $s(n,k)$ (A008275 in [64]), i.e.,

$$\mathfrak{S}_{0;1}(n,k) = S(n,k), \quad \mathfrak{S}_{1;-1}(n,k) = s(n,k). \tag{8}$$

Hsu and Shiue introduced in the seminal paper [34] a three parameter family of generalized Stirling numbers which unified many of the earlier generalizations of the Stirling numbers. Let us denote the generalized factorial with increment $\alpha$ by

$$(z|\alpha)_n = z(z-\alpha)\cdots(z-n\alpha+\alpha)$$

if $n \geq 1$ with $(z|\alpha)_0 = 1$. In particular, we write $(z|1)_n = (z)_n$. Hsu and Shiue defined a *Stirling-type pair* $\{S^1, S^2\} = \{S^1(n,k), S^2(n,k)\} \equiv \{S(n,k;\alpha,\beta,r), S(n,k;\beta,\alpha,-r)\}$ by the inverse relations

$$(t|\alpha)_n = \sum_{k=0}^{n} S^1(n,k)(t-r|\beta)_k, \tag{9}$$

$$(t|\beta)_n = \sum_{k=0}^{n} S^2(n,k)(t+r|\alpha)_k, \tag{10}$$

where $n \in \mathbb{N}$ and the parameters $\alpha, \beta, r$ are real or complex parameters satisfying $(\alpha, \beta, r) \neq (0,0,0)$. The pair $\{S^1, S^2\}$ is also called an $\langle \alpha, \beta, r\rangle$-pair. The classical Stirling number pair $\{s(n,k), S(n,k)\}$ is the $\langle 1,0,0\rangle$-pair, i.e.,

$$s(n,k) = S(n,k;1,0,0), \qquad S(n,k) = S(n,k;0,1,0). \tag{11}$$

From the definitions, it is clear that one has the orthogonality relations

$$\sum_{k=n}^{m} S^1(m,k)S^2(k,n) = \sum_{k=n}^{m} S^2(m,k)S^1(k,n) = \delta_{m,n}, \tag{12}$$

where $\delta_{m,n}$ is the Kronecker symbol [34]. Furthermore, one can easily derive a recursion relation.

**Theorem 1** ([34], Theorem 1)**.** *The generalized Stirling numbers $S(n,k;\alpha,\beta,r)$ satisfy the recursion relation*

$$S(n+1,k;\alpha,\beta,r) = S(n,k-1;\alpha,\beta,r) + (k\beta - n\alpha + r)S(n,k;\alpha,\beta,r) \tag{13}$$

*with the initial values $S(n,0;\alpha,\beta,r) = (r|\alpha)_n$.*

Comparing the recursion relations (6) and (13), one obtains the following result.

**Theorem 2.** *The generalized Stirling numbers $\mathfrak{S}_{s;h}(n,k)$ correspond to the case $\alpha := -hs, \beta := h(1-s), r := 0$ of the generalized Stirling numbers $S(n,k;\alpha,\beta,r)$ due to Hsu and Shiue, i.e.,*

$$\mathfrak{S}_{s;h}(n,k) = S(n,k;-hs,h(1-s),0). \tag{14}$$

*Conversely, if $r = 0$ and $\alpha \neq \beta$, then the generalized Stirling numbers $S(n,k;\alpha,\beta,0)$ of Hsu and Shiue correspond to the case $s := \frac{\alpha}{\alpha-\beta}$ and $h := \beta - \alpha$ of the generalized Stirling numbers $\mathfrak{S}_{s;h}(n,k)$, i.e.,*

$$S(n,k;\alpha,\beta,0) = \mathfrak{S}_{\frac{\alpha}{\alpha-\beta};\beta-\alpha}(n,k), \quad \alpha \neq \beta. \tag{15}$$

*Proof.* Noting that $h\{k + s(n-k)\} = (hsn + h(1-s)k)$ and comparing (6) and (13), one finds that one has to choose $\alpha = -hs, \beta = h(1-s)$ and $r = 0$. Since $\mathfrak{S}_{s;h}(n,0) = \delta_{n,0}$ and $S(n,0;-hs,h(1-s),0) = (0|-hs)_n = \delta_{n,0}$, the initial values coincide. The other direction is shown in the same fashion by solving $\alpha = -hs$ and $\beta = h(1-s)$ for $s$ and $h$. $\square$

*Remark* 3. Theorem 2 shows that there exists a bijection $\psi$ between the sets of parameters $(s, h)$ and $(\alpha, \beta \neq \alpha, r = 0)$ of the two kinds of generalized Stirling numbers given by $\psi(s, h) = (-hs, h(1-s), 0)$ and $\psi^{-1}(\alpha, \beta, 0) = (\frac{\alpha}{\alpha-\beta}, \beta - \alpha)$ with $\psi^{-1} \circ \psi = \psi \circ \psi^{-1} = \mathrm{Id}$. Note that $\psi(s, h) = (\alpha, \alpha, 0)$ would imply $h = 0$, which is excluded from the beginning. The generalized Stirling numbers $S(n, k; \alpha, \beta, 0)$ have been introduced by Tsylova [70].

Note that due to this identification we can derive some nice consequences for the generalized Stirling numbers $\mathfrak{S}_{s;h}(n, k)$. As a first step, we call a pair

$$\{\mathfrak{S}_{\bar{s};\bar{h}}, \mathfrak{S}_{s;h}\} = \{\mathfrak{S}_{\bar{s};\bar{h}}(n, k), \mathfrak{S}_{s;h}(n, k)\}$$

of (arrays of) generalized Stirling numbers a *dual pair*, if it is a Stirling-type pair when considered as generalized Stirling numbers of Hsu and Shiue.

**Proposition 4.** *The pair $\{\mathfrak{S}_{\bar{s};\bar{h}}, \mathfrak{S}_{s;h}\}$ is a dual pair if and only if $\bar{s} = 1 - s$ and $\bar{h} = -h$, i.e., dual pairs have the form*

$$\{\mathfrak{S}_{1-s;-h}, \mathfrak{S}_{s;h}\}$$

*for $s \in \mathbb{R}$ and $h \in \mathbb{C} \setminus \{0\}$. Furthermore, for a dual pair one has the orthogonality relations*

$$\sum_{k=n}^{m} \mathfrak{S}_{1-s;-h}(m, k)\mathfrak{S}_{s;h}(k, n) = \sum_{k=n}^{m} \mathfrak{S}_{s;h}(m, k)\mathfrak{S}_{1-s;-h}(k, n) = \delta_{m,n}.$$

*Proof.* Let the array $\mathfrak{S}_{s;h} = \{\mathfrak{S}_{s;h}(n, k)\} = \{S(n, k; -hs, h(1-s), 0)\} \equiv \{S^2(n, k)\}$ be given. Its partner $\{S^1(n, k)\}$ in the Stirling-type pair is given by $\{S^1(n, k)\} \equiv \{S(n, k; h(1-s), -hs, 0)\}$. If we want to identify the last array as $\{\mathfrak{S}_{\bar{s};\bar{h}}(n, k)\} = \{S(n, k; -\bar{h}\bar{s}, \bar{h}(1-\bar{s}), 0)\}$, we must have

$$-\bar{h}\bar{s} = h(1-s), \quad \bar{h}(1-\bar{s}) = -hs.$$

From this one finds $\bar{s} = 1 - s$ and $\bar{h} = -h$, as claimed. The orthogonality relations now follow from (12). □

From Theorem 2, it follows that the dual pair given by $\{\mathfrak{S}_{1-s;-h}(n, k), \mathfrak{S}_{s;h}(n, k)\}$ corresponds to the $< h(1-s), -hs, 0 >$-pair $\{S(n, k; h(1-s), -hs, 0), S(n, k; -hs, h(1-s), 0)\}$. Note that there do not exist self-dual arrays $\mathfrak{S}_{s;h}$ in the sense that $\{\mathfrak{S}_{s;h}, \mathfrak{S}_{s;h}\}$ is a dual pair of arrays. If $\mathfrak{S}_{s;h}$ was self-dual, one would have $s = 1 - s$ as well as $h = -h$, implying $(s, h) = (1/2, 0)$. However, $h \neq 0$ is assumed from the beginning since otherwise everything is trivial, see, e.g., the recursion relation (6).

**Example 5.** Let $s = 0$ and $h = 1$; this case corresponds to the Weyl-algebra. Here one has $\mathfrak{S}_{0;1}(n, k) = S(n, k)$. The corresponding dual pair is given by $\{\mathfrak{S}_{1;-1}, \mathfrak{S}_{0;1}\}$. From [44, Equation (14)], one has $\mathfrak{S}_{1;-1}(n, k) = s(n, k)$. Thus, the conventional Stirling pair is reproduced, as was to be expected since the dual pair $\{\mathfrak{S}_{1;-1}, \mathfrak{S}_{0;1}\}$ corresponds to the $< 1, 0, 0 >$-pair, see (11).

**Example 6.** Let $s = 2$ and $h = -1$; this case corresponds to the meromorphic Weyl-algebra. The corresponding dual pair is given by $\{\mathfrak{S}_{-1;1}, \mathfrak{S}_{2;-1}\}$ and will be considered in more detail in Section 3. It corresponds to the $< 1, 2, 0 >$-pair.

**Example 7.** Let $s = \frac{1}{2}$ and $h = 2$. Here one has $\mathfrak{S}_{\frac{1}{2};2}(n,k) = L(n,k)$, the unsigned Lah numbers [44, Example 3.3] (A008297 in [64]). The corresponding dual pair is given by $\{\mathfrak{S}_{\frac{1}{2};-2}, \mathfrak{S}_{\frac{1}{2};2}\}$. Since $\mathfrak{S}_{\frac{1}{2};-2} = (-1)^{n-k}\mathfrak{S}_{\frac{1}{2};2}(n,k) = (-1)^{n-k}L(n,k)$, one finds that $\{(-1)^{n-k}L(n,k), L(n,k)\}$ constitutes a dual pair and corresponds to the $< 1, -1, 0 >$-pair, as already mentioned in [34].

Before closing this section, let us briefly discuss the convexity of the generalized Bell polynomials and Bell numbers. Defining in close analogy to (7) for the generalized Stirling numbers of Hsu and Shiue

$$S_n(x) \equiv S_n(\alpha, \beta, r; x) := \sum_{k=0}^{n} S(n,k;\alpha,\beta,r)x^k,$$

Corcino and Corcino showed in [19] the following result:

**Theorem 8** ([19], Theorem 2.2)**.** *The sequence $S_n(x)$ with $x > 0$ and $\alpha \le 0$ and $\beta, r \ge 0$ possesses the convexity property, i.e.,*

$$S_{n+1}(x) \le \frac{1}{2}\left(S_n(x) + S_{n+2}(x)\right).$$

Using the bijection from Theorem 2, we can translate this into a convexity result for the polynomials $\mathfrak{B}_{s;h|n}(x)$.

**Theorem 9.** *The sequence of generalized Bell polynomials $\mathfrak{B}_{s;h|n}(x)$ with $x > 0$ and $0 \le s \le 1$ and $h \ge 0$ possesses the convexity property. Consequently, the corresponding sequence of Bell numbers $\mathfrak{B}_{s;h}(n) = \mathfrak{B}_{s;h|n}(1)$ also possesses the convexity property.*

*Proof.* According to Theorem 2, the polynomial $\mathfrak{B}_{s;h|n}(x)$ corresponds to $S_n(-hs, h(1-s), 0; x)$, so we require, by Theorem 8, that $hs \ge 0$ and $h(1-s) \ge 0$. The solutions to these inequalities must satisfy either $h = 0$ or $h > 0$ with $0 \le s \le 1$. $\square$

Some examples of combinations $(s, h)$ satisfying the conditions of the preceding theorem are as follows:

- $(0, 1)$, corresponding to the Stirling numbers of the second kind $S(n,k)$,
- $(1, 1)$, corresponding to the unsigned Stirling numbers of the first kind $c(n,k) \equiv |s(n,k)|$, and
- $(1/2, 2)$, corresponding to the (unsigned) Lah numbers $L(n,k)$.

# 3 The dual pair for $s = 2$ and $h = -1$

It was already mentioned above that the case $s = 2$ and $h = -1$ corresponds to the meromorphic Weyl algebra, see also [45]. Recall that in the conventional Weyl algebra (corresponding to $s = 0$ and $h = 1$), one has the relation $UV - VU = 1$, which is usually represented by the operators $U \mapsto D = d/dx$ and $V \mapsto X$, see the Introduction. In the meromorphic Weyl algebra, one considers [23, 24, 45] instead of the multiplication operator $X$ the multiplication operator $X^{-1}$ and obtains in this fashion a representation of

$$UV - VU = -V^2, \tag{16}$$

i.e., of the case $s = 2$ and $h = -1$. From (5) one obtains in this case, for $n \in \mathbb{N}$,

$$(VU)^n = \sum_{k=1}^{n} \mathfrak{S}_{2;-1}(n,k) V^{2n-k} U^k. \tag{17}$$

It was mentioned briefly in [45] that one has for this case a connection to Bessel polynomials. We now make this connection completely explicit. The generalized Stirling numbers $\mathfrak{S}_{2;1}(n,k) \equiv \mathfrak{S}(n,k)$ were called *meromorphic Stirling numbers* in [45]; note that the case $h = 1$ was considered there, but one has the simple relation $\mathfrak{S}_{2;-1}(n,k) = (-1)^{n-k}\mathfrak{S}_{2;1}(n,k)$ so that we will call $\mathfrak{S}_{2;-1}(n,k)$ also meromorphic Stirling numbers. In Remark 2.4 in [45], it was shown as Equation (2.4) that

$$\mathfrak{S}(n,k) = \frac{(n-1)!}{2^{n-k}(k-1)!}\binom{2n-k-1}{n-1},$$

yielding

$$\mathfrak{S}_{2;-1}(n,k) = (-1)^{n-k}\frac{(n-1)!}{2^{n-k}(k-1)!}\binom{2n-k-1}{n-1}. \tag{18}$$

According to Example 6, one has the dual pair $\{\mathfrak{S}_{-1;1}, \mathfrak{S}_{2;-1}\}$ of arrays. The former array has been considered in [44, Proposition 5.6] and is given by

$$\mathfrak{S}_{-1;1}(n,k) = \frac{(2n-2k)!}{2^{n-k}(n-k)!}\binom{n}{2k-n}, \tag{19}$$

where $\lceil \frac{n}{2} \rceil \leq k \leq n$. Using (14), we see that the dual pair $\{\mathfrak{S}_{-1;1}(n,k), \mathfrak{S}_{2;-1}(n,k)\}$ corresponds to the Stirling-type pair $\{S(n,k;1,2,0), S(n,k;2,1,0)\}$.

In order to draw the explicit connection to Bessel numbers, we recall some of their basic properties which can be found in [76] (see also [14, 32]). The *n-th Bessel polynomial* is defined by

$$y_n(x) := \sum_{k=0}^{n} \frac{(n+k)!}{2^k k!(n-k)!} x^k. \tag{20}$$

The coefficient of $x^{n-k}$ in the $(n-1)$-th Bessel polynomial $y_{n-1}(x)$ is called the *signless Bessel number of the first kind* and is denoted by $a(n,k)$ (A001497 in [64]). The *Bessel number of the first kind* is defined by $b(n,k) = (-1)^{n-k}a(n,k)$ and is given for $1 \leq k \leq n$ by [76, Equation (2)]

$$b(n,k) = (-1)^{n-k}\frac{(2n-k-1)!}{2^{n-k}(k-1)!(n-k)!}. \tag{21}$$

Comparing (18) and (21), we see that the meromorphic Stirling numbers are given by the Bessel numbers of the first kind, i.e.,

$$\mathfrak{S}_{2;-1}(n,k) = b(n,k). \tag{22}$$

Combining this with the identity

$$\mathfrak{S}_{2;-1}(n,k) = (-1)^{n-k}\mathfrak{S}_{2;1}(n,k) = (-1)^{n-k}S(n,k;-2,-1,0),$$

we obtain for the generalized Stirling numbers of Hsu and Shiue that $S(n,k;-2,-1,0) = (-1)^{n-k}b(n,k)$, a connection which was already mentioned by Pitman [54, Equation (18)].

The *Bessel numbers of the second kind* $B(n,k)$ can be defined as the number of partitions of $\{1,2,\ldots,n\}$ into $k$ nonempty blocks of size at most 2, see [76] (A144299 in [64]). Thus, one has for $\lceil \frac{n}{2} \rceil \le k \le n$ the explicit expression [76, Equation (8)]

$$B(n,k) = \frac{n!}{2^{n-k}(2k-n)!(n-k)!}. \tag{23}$$

Comparing (19) and (23), we see that the generalized Stirling numbers $\mathfrak{S}_{-1;1}(n,k)$ are given by the Bessel numbers of the second kind, i.e.,

$$\mathfrak{S}_{-1;1}(n,k) = B(n,k). \tag{24}$$

Since $\{\mathfrak{S}_{-1;1}, \mathfrak{S}_{2;-1}\}$ is a dual pair for which one has orthogonality relations (see Proposition 4), the same is true for the Bessel numbers, i.e., one has

$$\sum_{k=n}^{m} B(m,k)b(k,n) = \sum_{k=n}^{m} b(m,k)B(k,n) = \delta_{m,n}. \tag{25}$$

Of course, these relations are well-known [32] (for example, in [76] they are derived via exponential Riordan arrays and Lagrange inversion). Let us summarize the above observations in the following theorem.

**Theorem 10.** *The dual pair $\{\mathfrak{S}_{-1;1}(n,k), \mathfrak{S}_{2;-1}(n,k)\}$ of arrays corresponding to the meromorphic Stirling numbers $\mathfrak{S}_{2;-1}(n,k)$ is given by the arrays of Bessel numbers of the second and first kind $\{B(n,k), b(n,k)\}$, i.e., $\mathfrak{S}_{-1;1}(n,k) = B(n,k)$ and $\mathfrak{S}_{2;-1}(n,k) = b(n,k)$.*

Now, we discuss the above results in connection with the normal ordering of expressions $(X^{-1}D)^n$. Since $V \mapsto X^{-1}$ and $U \mapsto D$ gives a representation of (16), one obtains from (17) the relation

$$(X^{-1}D)^n = \sum_{k=1}^{n} \mathfrak{S}_{2;-1}(n,k)(X^{-1})^{2n-k}D^k. \tag{26}$$

Hadwiger considered already in 1943 [30] the operator $X^{-1}D$ and derived an expression similar to (26). This equation can also be written as

$$(X^{-1}D)^n = (X^{-1})^{2n} \sum_{k=1}^{n} \mathfrak{S}_{2;-1}(n,k)X^kD^k. \tag{27}$$

Let us change the variable from $x$ to $t = x^{-1}$. It follows that $X^{-1} = T$ as well as $D_x = d/dx = -t^2 d/dt = -T^2 D_t$, thus

$$X^{-1}D_x = -T^3 D_t. \tag{28}$$

Therefore, $(X^{-1}D_x)^n = (-1)^n (T^3 D_t)^n$. Letting $V \equiv T$ and $U \equiv T^2 D_t$ shows that $UV = VU + V^2$, i.e., $T$ and $T^2 D_t$ represent the case $s = 2$ and $h = 1$. Thus,

$$(T^3 D_t)^n = \sum_{k=1}^{n} \mathfrak{S}_{2;1}(n,k) T^{2n-k} (T^2 D_t)^k.$$

It follows that

$$\begin{aligned}
(X^{-1}D_x)^n &= (-1)^n (T^3 D_t)^n \\
&= (-1)^n T^{2n} \sum_{k=1}^{n} \mathfrak{S}_{2;1}(n,k) T^{-k} (T^2 D_t)^k \\
&= (-1)^n (X^{-1})^{2n} \sum_{k=1}^{n} \mathfrak{S}_{2;1}(n,k) X^k (-D_x)^k \\
&= (X^{-1})^{2n} \sum_{k=1}^{n} \mathfrak{S}_{2;1}(n,k) (-1)^{n-k} X^k D_x^k,
\end{aligned}$$

which is equivalent to (27) since $\mathfrak{S}_{2;-1}(n,k) = \mathfrak{S}_{2;1}(n,k)(-1)^{n-k}$. This represents a nice consistency check. Let us consider another example. For this we consider $V \equiv X$ and $U \equiv X^{-1}D$ satisfying $UV = VU + V^{-1}$, i.e., the case $s = -1$ and $h = 1$. Thus,

$$D^n = (X \cdot X^{-1}D)^n = \sum_{k=1}^{n} \mathfrak{S}_{-1;1}(n,k) X^{2k-n} (X^{-1}D)^k. \tag{29}$$

Using (27), this yields

$$\begin{aligned}
D^n &= \sum_{k=1}^{n} \sum_{l=1}^{k} \mathfrak{S}_{-1;1}(n,k) \mathfrak{S}_{2;-1}(k,l) X^{2k-n-2k+l} D^l \\
&= \sum_{l=1}^{n} \left\{ \sum_{k=l}^{n} \mathfrak{S}_{-1;1}(n,k) \mathfrak{S}_{2;-1}(k,l) \right\} X^{l-n} D^l \\
&= \sum_{l=1}^{n} \delta_{n,l} X^{l-n} D^l = D^n,
\end{aligned}$$

where we have used in the third line an orthogonality relation of the dual pair $\{\mathfrak{S}_{-1;1}, \mathfrak{S}_{2;-1}\}$.

Recall that the conventional Stirling numbers of the second kind $S(n,k)$ appear as normal ordering coefficients, i.e., $(XD)^n = \sum_{k=1}^{n} S(n,k) X^k D^k$, and the conventional Stirling numbers of the first kind $s(n,k)$ in the converse expansion, i.e., $X^m D^m = \sum_{k=1}^{m} s(m,k)(XD)^k$. The role of the Stirling numbers is played in the meromorphic case by the Bessel numbers, as the following proposition shows.

**Proposition 11.** *For any $n \in \mathbb{N}$ one has the expansion*

$$(X^{-1}D)^n = (X^{-1})^{2n} \sum_{k=1}^{n} b(n,k) X^k D^k. \tag{30}$$

*Similarly, one has for any $m \in \mathbb{N}$ the expansion*

$$X^m D^m = \sum_{k=1}^{m} B(m,k) X^{2k} (X^{-1}D)^k. \tag{31}$$

*Proof.* The first asserted equation follows from (27) since $\mathfrak{S}_{2;-1}(n,k) = b(n,k)$. The second equation follows similarly from $\mathfrak{S}_{-1;1}(n,k) = B(n,k)$ and (29). $\square$

*Remark* 12. As a consistency check, we insert (31) into (30) and obtain

$$\begin{aligned}
(X^{-1}D)^n &= (X^{-1})^{2n} \sum_{k=1}^{n} b(n,k) \sum_{l=1}^{k} B(k,l) X^{2l} (X^{-1}D)^l \\
&= (X^{-1})^{2n} \sum_{l=1}^{n} \left\{ \sum_{k=l}^{n} b(n,k) B(k,l) \right\} X^{2l} (X^{-1}D)^l \\
&= (X^{-1})^{2n} \sum_{l=1}^{n} \delta_{n,l} X^{2l} (X^{-1}D)^l = (X^{-1}D)^n,
\end{aligned}$$

where we have used in the third line an orthogonality relation of the Bessel numbers.

# 4 Some relations between generalized Stirling numbers

Generalizing (28), we obtain by the change of variables $t = x^{-1}$ for arbitrary $\lambda \in \mathbb{R}$ that

$$X^{\lambda} D_x = -T^{2-\lambda} D_t,$$

generalizing the case $\lambda = -1$ considered above. By taking powers one obtains relations between generalized Stirling numbers corresponding to different sets of parameters. As a first step, note that if $V \equiv X$ and $U \equiv X^{\lambda-1} D_x$, then $UV = VU + V^{\lambda-1}$, i.e., one has a representation of the case $s = \lambda - 1$ and $h = 1$. It follows that

$$(X^{\lambda} D_x)^n = \sum_{k=1}^{n} \mathfrak{S}_{\lambda-1;1}(n,k) X^{(\lambda-1)(n-k)+k} (X^{\lambda-1} D_x)^k.$$

In the same fashion, we obtain

$$(T^{2-\lambda} D_t)^n = \sum_{k=1}^{n} \mathfrak{S}_{1-\lambda;1}(n,k) T^{(1-\lambda)(n-k)+k} (T^{1-\lambda} D_t)^k.$$

Using $T = X^{-1}$ as well as $T^{1-\lambda} D_t = -X^{\lambda+1} D_x$, the right-hand side equals

$$\sum_{k=1}^{n} \mathfrak{S}_{1-\lambda;1}(n,k) X^{(\lambda-1)(n-k)-k} (-X^{\lambda+1} D_x)^k,$$

and it follows that

$$\sum_{k=1}^{n} \mathfrak{S}_{\lambda-1;1}(n,k) X^{k(2-\lambda)} (X^{\lambda-1} D_x)^k = \sum_{k=1}^{n} \mathfrak{S}_{1-\lambda;-1}(n,k) X^{-k\lambda} (X^{\lambda+1} D_x)^k.$$

Now, we wish to express $(X^{\lambda+1} D_x)^k$ in terms of $(X^{\lambda-1} D_x)^m$. For this, we write

$$X^{\lambda+1} D_x = X^2 \cdot X^{\lambda-1} D_x.$$

If we let $V \equiv X^2$ and $U \equiv X^{\lambda-1} D_x$, it follows that

$$UVf(x) = x^{\lambda-1} D_x(x^2 f(x)) = 2x^{\lambda} f(x) + x^{\lambda+1} D_x f(x) = 2V^{\lambda/2} f(x) + VUf(x),$$

i.e., one has a representation of the case $s = \lambda/2$ and $h = 2$. This implies

$$(X^{\lambda+1} D_x)^k = (X^2 \cdot X^{\lambda-1} D_x)^k = \sum_{m=1}^{k} \mathfrak{S}_{\frac{\lambda}{2};2}(k,m) X^{\lambda(k-m)+2m} (X^{\lambda-1} D_x)^m,$$

showing

$$\begin{aligned}
&\sum_{k=1}^{n} \mathfrak{S}_{\lambda-1;1}(n,k) X^{k(2-\lambda)} (X^{\lambda-1} D_x)^k \\
&\qquad = \sum_{k=1}^{n} \sum_{m=1}^{k} \mathfrak{S}_{1-\lambda;-1}(n,k) \mathfrak{S}_{\frac{\lambda}{2};2}(k,m) X^{(2-\lambda)m} (X^{\lambda-1} D_x)^m \\
&\qquad = \sum_{m=1}^{n} \left\{ \sum_{k=m}^{n} \mathfrak{S}_{1-\lambda;-1}(n,k) \mathfrak{S}_{\frac{\lambda}{2};2}(k,m) \right\} X^{(2-\lambda)m} (X^{\lambda-1} D_x)^m.
\end{aligned}$$

Comparing coefficients, one obtains the following identity:

$$\mathfrak{S}_{\lambda-1;1}(n,k) = \sum_{l=k}^{n} \mathfrak{S}_{1-\lambda;-1}(n,l) \mathfrak{S}_{\frac{\lambda}{2};2}(l,k). \tag{32}$$

Denoting $s = 1 - \lambda$, this shows the following proposition.

**Proposition 13.** *For arbitrary $s \in \mathbb{R}$, one has the following identity between generalized Stirling numbers:*

$$\mathfrak{S}_{-s;1}(n,k) = \frac{(-1)^n}{2^k} \sum_{l=k}^{n} (-2)^l \mathfrak{S}_{s;1}(n,l) \mathfrak{S}_{\frac{1-s}{2};1}(l,k). \tag{33}$$

**Corollary 14.** *For $s = 1$ one obtains from Proposition 13 a relation between Bessel numbers of the second kind and Stirling numbers of the first and second kind (see [76, Equation (19)]):*

$$B(n,k) = \sum_{l=k}^{n} 2^{l-k} s(n,l) S(l,k).$$

*Proof.* This follows from (33) since $\mathfrak{S}_{-1;1}(n,k) = B(n,k)$ as well as $\mathfrak{S}_{1;1}(n,l) = (-1)^{n-l}s(n,l)$ and $\mathfrak{S}_{0;1}(l,k) = S(l,k)$. $\square$

In another direction, it is possible to consider $(X^\lambda D)^n$ in several different ways. One is as $(X \cdot X^{\lambda-1}D)^n$ as above, leading to $\mathfrak{S}_{\lambda-1;1}(n,k)$. Another way is to consider $V \equiv X^\lambda$ and $U \equiv D$.

**Lemma 15.** *The operators $X^\lambda$ and $D$ define for any $\lambda \in \mathbb{R}$ via $V \mapsto X^\lambda$ and $U \mapsto D$ a representation of variables $U, V$ satisfying $UV = VU + \lambda V^{\frac{\lambda-1}{\lambda}}$, i.e., of the case $s = \frac{\lambda-1}{\lambda}$ and $h = \lambda$.*

*Proof.* Since $D(x^\lambda f(x)) = \lambda x^{\lambda-1} f(x) + x^\lambda Df(x)$, one has

$$\left\{D \circ X^\lambda - X^\lambda \circ D\right\} f(x) = \lambda X^{\lambda-1} f(x) = \lambda (X^\lambda)^{\frac{\lambda-1}{\lambda}} f(x),$$

showing the assertion. $\square$

More generally, splitting the exponent one can write $(X^\lambda D)^n$ also as $(X^\nu \cdot X^{\lambda-\nu}D)^n$ with $\nu \neq 0$. In this fashion, one calculates $x^{\lambda-\nu}D(x^\nu f(x)) = \nu x^{\lambda-1} f(x) + x^\lambda Df(x)$ so that $V \mapsto X^\nu$ and $U \mapsto X^{\lambda-\nu}D$ yields a representation of

$$UV - VU = \nu V^{\frac{\lambda-1}{\nu}},$$

i.e., of the case $s = \frac{\lambda-1}{\nu}$ and $h = \nu$. Here the numbers $\mathfrak{S}_{\frac{\lambda-1}{\nu};\nu}(n,k)$ will be involved. Splitting a given $\lambda$ in two different ways, one obtains from $(X^\nu \cdot X^{\lambda-\nu}D)^n = (X^\lambda D)^n = (X^\kappa \cdot X^{\lambda-\kappa}D)^n$ the relation

$$\sum_{k=1}^{n} \mathfrak{S}_{\frac{\lambda-1}{\nu};\nu}(n,k) X^{(\lambda-1)(n-k)+k\nu}(X^{\lambda-\nu}D)^k = \sum_{k=1}^{n} \mathfrak{S}_{\frac{\lambda-1}{\kappa};\kappa}(n,k) X^{(\lambda-1)(n-k)+k\kappa}(X^{\lambda-\kappa}D)^k. \quad (34)$$

Let us write $\kappa = \nu - \sigma$ with $\sigma > 0$, so that $(X^{\lambda-\kappa}D)^k = (X^\sigma \cdot X^{\lambda-\nu}D)^k$. Noting that

$$(X^\sigma \cdot X^{\lambda-\nu}D)^k = \sum_{l=1}^{k} \mathfrak{S}_{\frac{\sigma+\lambda-\nu-1}{\sigma};\sigma}(k,l) X^{(\sigma+\lambda-\nu-1)(k-l)+l\sigma}(X^{\lambda-\nu}D)^l,$$

the right-hand side of (34) becomes

$$\sum_{k=1}^{n}\sum_{l=1}^{k} \mathfrak{S}_{\frac{\lambda-1}{\nu-\sigma};\nu-\sigma}(n,k)\mathfrak{S}_{\frac{\sigma+\lambda-\nu-1}{\sigma};\sigma}(k,l) X^{(\lambda-1)(n-k)+k(\nu-\sigma)} X^{(\sigma+\lambda-\nu-1)(k-l)+l\sigma}(X^{\lambda-\nu}D)^l,$$

or

$$\sum_{l=1}^{n} \left\{ \sum_{k=l}^{n} \mathfrak{S}_{\frac{\lambda-1}{\nu-\sigma};\nu-\sigma}(n,k)\mathfrak{S}_{\frac{\sigma+\lambda-\nu-1}{\sigma};\sigma}(k,l) \right\} X^{(\lambda-1)(n-l)+l\nu}(X^{\lambda-\nu}D)^l.$$

Comparing this with the left-hand side of (34), one has shown the following generalization of (33):

$$\mathfrak{S}_{\frac{\lambda-1}{\nu};\nu}(n,k) = \sum_{l=k}^{n} \mathfrak{S}_{\frac{\lambda-1}{\nu-\sigma};\nu-\sigma}(n,l)\mathfrak{S}_{\frac{\sigma+\lambda-\nu-1}{\sigma};\sigma}(l,k). \quad (35)$$

For example, fixing $\nu = 1$ and considering the dependence on $\sigma$, one finds

$$\mathfrak{S}_{\lambda-1;1}(n,k) = \sum_{l=k}^{n} \mathfrak{S}_{\frac{\lambda-1}{1-\sigma};1-\sigma}(n,l)\mathfrak{S}_{\frac{\sigma+\lambda-2}{\sigma};\sigma}(l,k).$$

Furthermore, considering $\sigma = 2$, this gives

$$\mathfrak{S}_{\lambda-1;1}(n,k) = \sum_{l=k}^{n} \mathfrak{S}_{1-\lambda;-1}(n,l)\mathfrak{S}_{\frac{\lambda}{2};2}(l,k),$$

i.e., (32) from above. Switching to $s = \lambda - 1$, one obtains from (35) the following proposition.

**Proposition 16.** *Let $s \in \mathbb{R}, \nu \neq 0$, and $\sigma > 0$. Then one has the following identity between generalized Stirling numbers:*

$$\mathfrak{S}_{\frac{s}{\nu};\nu}(n,k) = \sum_{l=k}^{n} \mathfrak{S}_{\frac{s}{\nu-\sigma};\nu-\sigma}(n,l)\mathfrak{S}_{\frac{s+\sigma-\nu}{\sigma};\sigma}(l,k).$$

**Corollary 17.** *Choosing $s = 1$, $\nu = 2$, and $\sigma = 1$ in the identity of Proposition 16 gives the following relation between (unsigned) Lah numbers and Stirling numbers of the first and second kind (see [73]):*

$$L(n,k) = \sum_{l=k}^{n} c(n,l)S(l,k),$$

*where $c(n,l) = (-1)^{n-l}s(n,l)$ denotes the unsigned Stirling number of the first kind.*

# 5 Combinatorial proofs

In this section, we provide combinatorial proofs of Propositions 4, 16, and 13. It will be more convenient to let $a = hs$ and $b = h(1-s)$ and consider $G_{a;b}(n,k)$ given by the equivalent recurrence

$$G_{a;b}(n,k) = G_{a;b}(n-1,k-1) + [a(n-1)+bk]G_{a;b}(n-1,k), \qquad n,k \geq 1, \qquad (36)$$

with $G_{a;b}(n,0) = \delta_{n,0}$ and $G_{a;b}(0,k) = \delta_{0,k}$ for all $n,k \geq 0$. Note that $G_{a;b}(n,k) = S(n,k;-a,b,0)$ from above.

When $a = b = 1$, we see from (36) that the $G_{a;b}(n,k)$ reduce to the (unsigned) Lah numbers $L(n,k)$. It is well known that $L(n,k) = |\mathcal{L}(n,k)|$, where $\mathcal{L}(n,k)$ denotes the set of all distributions of $n$ labeled balls, denoted $1,2,\ldots,n$, among $k$ unlabeled, *contents-ordered* boxes such that no box is left empty, which are often termed *Lah distributions*. See [63] and [72]. For example, if $n = 3$ and $k = 2$, then $L(3,2) = 6$, the distributions being $\{1,2\},\{3\}$; $\{2,1\},\{3\}$; $\{1,3\},\{2\}$; $\{3,1\},\{2\}$; $\{2,3\},\{1\}$; and $\{3,2\},\{1\}$. Let $L(n) = \sum_{k=0}^{n} L(n,k)$ and $\mathcal{L}(n) = \cup_{k=0}^{n}\mathcal{L}(n,k)$. Then $L(n) = |\mathcal{L}(n)|$, the cardinality of the set of all distributions of $n$ labeled balls in unlabeled, contents-ordered boxes. See, e.g., [52], where the $L(n)$ are described as counting *sets of lists* having size $n$.

We now recall a combinatorial interpretation for $G_{a;b}(n,k)$ given in [44] which we will make use of. We first will need the following definition, where $[n] = \{1,2,\ldots,n\}$ if $n$ is a positive integer, with $[0] = \varnothing$.

**Definition 18.** If $\lambda \in \mathcal{L}(n)$ and $i \in [n]$, then we say that $i$ is a record low of $\lambda$ if there are no elements $j < i$ to the left of $i$ within its block in $\lambda$.

For example, if $n = 10$ and $\lambda = \{6, 5, 8, 1, 9\}, \{4, 10, 2, 3\}, \{7\} \in \mathcal{L}(10)$, then the elements 6, 5, and 1 are record lows in the first block, 4 and 2 are record lows in the second, and 7 is a record low in the third block for a total of six record lows altogether. Note that the smallest element within a block as well as the left-most one are always record lows.

**Definition 19.** Given $\lambda \in \mathcal{L}(n)$, let $rec^*(\lambda)$ denote the total number of record lows of $\lambda$ not counting those corresponding to the smallest member of some block. Let $nrec(\lambda)$ denote the number of elements of $[n]$ which are not record lows of $\lambda$.

For example, if $\lambda$ is as above, then $rec^*(\lambda) = 3$ (corresponding to 6, 5, and 4) and $nrec(\lambda) = 4$ (corresponding 8, 9, 10, and 3). Given $\lambda \in \mathcal{L}(n)$ and indeterminates $a$ and $b$, define the $(a, b)$-weight of $\lambda$, which we'll denote $w_{a;b}(\lambda)$, by

$$w_{a;b}(\lambda) = a^{nrec(\lambda)} b^{rec^*(\lambda)}.$$

For each $n$ and $k$, we have that $G_{a;b}(n, k)$ is the joint distribution polynomial for the $nrec$ and $rec^*$ statistics on $\mathcal{L}(n, k)$.

**Theorem 20.** *[44, Theorem 5.1] If $n, k \geq 0$, then*

$$G_{a;b}(n, k) = \sum_{\lambda \in \mathcal{L}(n,k)} w_{a;b}(\lambda). \tag{37}$$

Using this interpretation, we now provide bijective proofs of Propositions 4, 16, and 13.

**Combinatorial proof of Proposition 4.**

We prove, equivalently,

$$\sum_{k=m}^{n} G_{a;b}(n, k) G_{-b;-a}(k, m) = \sum_{k=m}^{n} G_{-b;-a}(n, k) G_{a;b}(k, m) = \delta_{m,n}, \tag{38}$$

where $m$ and $n$ are given integers with $0 \leq m \leq n$ and $G_{a;b}(n, k)$ is defined by (36). We treat only the first equality, the proof of the second being similar. To do so, we consider a collection of Lah distributions whose elements themselves are Lah distributions. More precisely, given $m \leq k \leq n$, let $\mathcal{A}_k$ denote the set of ordered pairs $\rho = (\alpha, \beta)$, where $\alpha \in \mathcal{L}(n, k)$ and $\beta$ is a Lah distribution having $m$ blocks whose elements are the blocks of $\alpha$. Here, we order the blocks of $\alpha$ according to the size of smallest elements when ordering them within $\beta$. Define the weight of $\rho$, which we will denote $v(\rho)$, by

$$v(\rho) = (-1)^{k-m} w_{a;b}(\alpha) w_{b;a}(\beta).$$

For example, if $n = 20$, $m = 5$, and $k = 10$, and $\rho = (\alpha, \beta)$, where

$$\begin{aligned} \alpha = &\{1\},\ \{8, 15, 19, 2, 17\},\ \{3, 5\},\ \{4, 18\},\ \{6\},\ \{7, 20\},\ \{13, 9\},\ \{16, 10\},\ \{11\}, \\ &\{14, 12\} \end{aligned}$$

and

$$\beta = \{\{1\}\},\ \{\{4,18\},\{8,15,19,2,17\},\{7,20\}\},\ \{\{6\},\{14,12\},\{3,5\}\},\ \{\{13,9\}\},\\ \{\{11\},\{16,10\}\},$$

then we have

$$v(\rho) = (-1)^{10-5}(a^6b^4)(b^2a^3) = -a^9b^6.$$

Let $\mathcal{A} = \cup_{k=m}^{n}\mathcal{A}_k$. By (37), the left-hand side of (38) gives the total weight of all the members of $\mathcal{A}$, each summand giving the total weight of $\mathcal{A}_k$, upon noting $nrec(\beta)+rec^*(\beta) = k-m$ for all $\beta \in \mathcal{L}(k,m)$. To complete the proof, we will define an involution of $\mathcal{A}$ which pairs each member of $\mathcal{A}$ with another of opposite weight when $m<n$. (Note that if $m=n$, then the identity is trivial, both sides reducing to one and $\mathcal{A}$ containing only a single member, namely, $\{\{1\}\},\{\{2\}\},\ldots,\{\{n\}\}$.)

Let $\rho = (\alpha,\beta) \in \mathcal{A}$. Let us assume further, for convenience, that the blocks of $\beta$ are arranged from left to right in increasing order according to the size of the smallest element of $[n]$ lying within. Let $C$ denote the left-most block of $\beta$ containing at least two elements of $[n]$ altogether. Note that the blocks of $\alpha$ within $C$ may come in any order and suppose $C$ contains $r$ elements of $[n]$ altogether, which we'll denote by $c_1 < c_2 < \cdots < c_r$.

We now define an involution of $\mathcal{A}$ in two steps. Given $\rho = (\alpha,\beta)$, let $i_o$ denote the *largest* index $i \in [r]-\{1\}$, if it exists, such that one of the following conditions holds:

(I) the element $c_i$ is the first element of a block of $\alpha$ within $C$ containing at least two elements of $[n]$ and is not the smallest element of that block;

(II) the element $c_i$ is the first and smallest element of a block of $\alpha$ within $C$ which comes directly to the right of a block whose first element is smaller than $c_i$.

If condition (I) holds, and the block containing $c_{i_o}$ is of the form $\{c_{i_o}, x_1, x_2, \ldots, d, y_1, y_2, \ldots\}$, where $d$ is the second left-to-right minima from the left, then replace the single block with two blocks $\{d, y_1, y_2, \ldots\}$, $\{c_{i_o}, x_1, x_2, \ldots\}$. Conversely, if (II) holds, merge the block containing $c_{i_o}$ with the one directly before it by writing its elements prior to the elements of its predecessor. Let $\rho' = (\alpha',\beta')$ denote the resulting member of $\mathcal{A}$ obtained by changing the block $C$ in either of the two ways described. Note that $\alpha'$ has one more or one fewer blocks than $\alpha$ and $\beta'$ has one more or one fewer (block) elements than $\beta$. Observe further that changing $C$ as described above when the first condition holds takes away a factor of $b$ since $c_{i_o}$ is no longer counted in $rec^*(\alpha)$ (as it is now a block minimum), but introduces a factor of $-b$ since the new block $\{c_{i_o}, x_1, x_2, \ldots\}$ is a non-record low and is thus counted in $nrec(\beta')$, whence $\rho$ and $\rho'$ have opposite weight.

For example, if $\rho = (\alpha,\beta)$ is as given above, then $C$ is the second block of $\beta$, with $r=9$, $i_o = 4$ (condition (I) holding), and $c_{i_o} = 8$. We then have $\rho' = (\alpha',\beta')$, where

$$\alpha' = \{1\},\ \{2,17\},\ \{3,5\},\ \{4,18\},\ \{6\},\ \{7,20\},\ \{8,15,19\},\ \{13,9\},\ \{16,10\},\ \{11\},\\ \{14,12\}$$

and

$$\beta' = \{\{1\}\}, \ \{\{4,18\},\{2,17\},\{8,15,19\},\{7,20\}\}, \ \{\{6\},\{14,12\},\{3,5\}\},\{\{13,9\}\},$$
$$\{\{11\},\{16,10\}\},$$

whence

$$v(\rho') = (-1)^{11-5}(a^6b^3)(b^3a^3) = a^9b^6 = -v(\rho).$$

The mapping $\rho \mapsto \rho'$ is seen to be an involution of $\mathcal{A}$ which is not defined in the case when the block $C$ is either of the following forms:

$$(i)\ C = \{E_1, E_2, \ldots, E_t, \{c_2\gamma_2\}, \{c_1\gamma_1\}\},$$
$$(ii)\ C = \{E_1, E_2, \ldots, E_t, \{c_1\gamma_1 c_2\gamma_2\}\},$$

where $\gamma_1$ and $\gamma_2$ are possibly empty sequences and the $E_i$ are contents-ordered blocks which occur in decreasing order according to the size of the first element and in which the first element is also the smallest one within each block. However, exchanging (i) for (ii), and vice-versa, defines an involution in this case that reverses the weight, which completes the proof of (38). □

**Combinatorial proof of Proposition 16.**

Equivalently, we prove the identity

$$G_{a;b}(n,k) = \sum_{\ell=k}^{n} G_{a;t}(n,\ell)G_{-t;b}(\ell,k), \tag{39}$$

where $n$ and $k$ are given integers with $0 \leq k \leq n$. If $k \leq \ell \leq n$, then let $\mathcal{A}_\ell$ consist of the ordered pairs $\rho = (\alpha, \beta)$ as described in the preceding proof. Define the weight $u(\rho)$ by

$$u(\rho) = w_{a;t}(\alpha)w_{-t;b}(\beta).$$

Let $\mathcal{A} = \cup_{\ell=k}^{n}\mathcal{A}_\ell$. By (37), the right-hand side of (39) gives the total $u$-weight of all the members of $\mathcal{A}$.

We now define an involution on $\mathcal{A}$ as follows. Let $\rho = (\alpha, \beta) \in \mathcal{A}$, where we assume that the blocks of $\alpha$ are ordered according to the size of smallest elements and that the blocks of $\beta$ are arranged from left to right in increasing order according to the size of the smallest element of $[n]$ contained within them. Let $D$ denote a block of $\beta$ and suppose $D$ contains $r$ members of $[n]$ altogether, which we denote $c_1 < c_2 < \cdots < c_r$. Assume that an index $i$ exists such that $c_i \in D$ satisfies either condition (I) or (II) in the proof of Proposition 4 above, letting $i_o$ denote the largest such $i$. Assume further that $D$ is the left-most block of $\beta$ for which $i_o$ exists.

Now apply the first involution of the preceding proof using the block $D$. This pairs each member of $\mathcal{A} - \mathcal{A}^*$ with another of opposite weight, where $\mathcal{A}^*$ consists of those members $\rho = (\alpha, \beta)$ of $\mathcal{A}$ in which the blocks of $\alpha$ contained within any block of $\beta$ occur from left to right in decreasing order according to the size of the first element, with the first element also the smallest within each of these blocks. To complete the proof, we define a weight-preserving bijection between $\mathcal{A}^*$ and $\mathcal{L}(n,k)$. To do so, given $\rho = (\alpha, \beta) \in \mathcal{A}^*$, simply erase parentheses enclosing the blocks of $\alpha$ lying within each block of $\beta$ and concatenate words. To reverse this, within each block of $\lambda \in \mathcal{L}(n,k)$, place a divider just before each left-to-right minimum. Note that each left-to-right minimum in $\lambda$ (except for those corresponding to block minima ) contributes a factor of $b$ towards the weight of $\lambda$, just as each block of $\alpha$, excepting the smallest, lying within a block of $\beta$ contributes a factor of $b$ towards the weight $u(\rho)$ since these blocks occur in decreasing order. □

A similar argument can be applied to Proposition 13.

**Combinatorial proof of Proposition 13.**

We prove, equivalently,

$$G_{a;b}(n,k) = \frac{(-1)^n}{2^k} \sum_{\ell=k}^{n} (-2)^\ell G_{-a;t}(n,\ell) G_{\frac{t}{2};\frac{b}{2}}(\ell,k), \tag{40}$$

where $G_{a;b}(n,k)$ satisfies (36) and $a=-s$, $b=1+s$, and $t=1-s$ for some parameter $s$. Let $\mathcal{A}_\ell$ and $\mathcal{A}$ be as in the proof of Proposition 16 above. If $\rho = (\alpha,\beta) \in \mathcal{A}_\ell$, then define the weight $r(\rho)$ by

$$r(\rho) = (-1)^{n-\ell} 2^{\ell-k} w_{-a;t}(\alpha) w_{\frac{t}{2};\frac{b}{2}}(\beta).$$

By (37), the right-hand side of (40) gives the total weight with respect to $r$ of all the members of $\mathcal{A}$. But since $nrec(\alpha) + rec^*(\alpha) = n - \ell$ and $nrec(\beta) + rec^*(\beta) = \ell - k$ for all $\alpha \in \mathcal{L}(n,\ell)$ and $\beta \in \mathcal{L}(\ell,k)$, we may rewrite the $r$-weight more simply as

$$r(\rho) = w_{a;-t}(\alpha) w_{t;b}(\beta).$$

Identity (40) now follows from the proof of Proposition 16. □

# 6 Touchard polynomials of arbitrary integer order and generalized Bell polynomials

Following [21], we define the *Touchard polynomials* (also called *exponential polynomials*) for $n \in \mathbb{N}$ by

$$T_n(x) := e^{-x}(XD)^n e^x. \tag{41}$$

Note that one can obtain from this definition of the Touchard polynomials and the fact that $(XD)^n = \sum_{k=0}^{n} S(n,k) X^k D^k$ the relation

$$T_n(x) = \sum_{k=0}^{n} S(n,k) x^k = B_n(x), \tag{42}$$

where the second equality corresponds to the definition of the conventional Bell polynomials.

In [21], *Touchard polynomials of higher order* are considered. They are defined for $m \in \mathbb{N}$ (and $n \in \mathbb{N}$) by

$$T_n^{(m)}(x) := e^{-x}(X^m D)^n e^x \tag{43}$$

and reduce for $m = 1$ to the case above. Many of their properties are discussed in [21]. In particular, noting that the normal ordering of $(X^m D)^n$ leads to the generalized Stirling numbers $S_{m,1}(n,k)$, one has a close connection between the Touchard polynomials of order $m$ and the Stirling numbers $S_{m,1}(n,k)$. On the other hand, from Lemma 15 we know that $V \mapsto X^m$ and $U \mapsto D$ defines a representation for $UV = VU + mV^{\frac{m-1}{m}}$, i.e., of the case $s = \frac{m-1}{m}$ and $h = m$. Thus,

$$(X^m D)^n = \sum_{k=0}^{n} \mathfrak{S}_{\frac{m-1}{m};m}(n,k) X^{(m-1)(n-k)+mk} D^k = X^{(m-1)n} \sum_{k=0}^{n} \mathfrak{S}_{\frac{m-1}{m};m}(n,k) X^k D^k.$$

Inserting this into (43) shows the first part of the following theorem.

**Theorem 21.** *Let $m \in \mathbb{N}$. The Touchard polynomials of order $m$ are given as*

$$T_n^{(m)}(x) = x^{(m-1)n} \sum_{k=0}^{n} \mathfrak{S}_{\frac{m-1}{m};m}(n,k) x^k. \tag{44}$$

*The polynomial $T_n^{(m)}$ has degree $mn$ in $x$ and can be written in terms of generalized Bell polynomials as*

$$T_n^{(m)}(x) = x^{(m-1)n} \mathfrak{B}_{\frac{m-1}{m};m|n}(x). \tag{45}$$

*In particular, $T_n^{(m)}(1) = \mathfrak{B}_{\frac{m-1}{m};m}(n)$.*

*Proof.* The first equation (44) was already derived above. Inserting the definition of the generalized Bell polynomials shows the second equation. The last assertion follows from $\mathfrak{B}_{\frac{m-1}{m};m|n}(1) = \mathfrak{B}_{\frac{m-1}{m};m}(n)$. □

**Example 22.** Let us consider $m = 1$. In that case we obtain $T_n^{(1)}(1) = \mathfrak{B}_{0;1}(n) \equiv B(n)$, the conventional Bell numbers (A000110 in [64]), as was to be expected. For $m = 2$ we obtain $T_n^{(2)}(1) = \mathfrak{B}_{\frac{1}{2};2}(n)$, where $\mathfrak{B}_{\frac{1}{2};2}(n) = \sum_{k=0}^{n} L(n,k)$ since $\mathfrak{S}_{\frac{1}{2};2}(n,k) = L(n,k)$, where we have denoted the (unsigned) Lah number by $L(n,k)$. For $m = 3$ one obtains $T_n^{(3)}(1) = \mathfrak{B}_{\frac{2}{3};3}(n)$.

**Corollary 23** ([21])**.** *The Touchard polynomials of order $m \geq 2$ have the explicit expression*

$$T_n^{(m)}(x) = [(m-1)x^{(m-1)}]^n \sum_{k=0}^{n} \sum_{j=0}^{k} \frac{(-1)^{k-j}}{k!} \binom{k}{j} \frac{\Gamma(n + \frac{j}{m-1})}{\Gamma(\frac{j}{m-1})} x^k. \tag{46}$$

*Proof.* In [44, Theorem 3.9], the following expression was derived for the generalized Stirling numbers (where $s \neq 0$):

$$\mathfrak{S}_{s;h}(n,k) = \frac{h^{n-k} s^n n!}{(1-s)^k k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \binom{n + \frac{j}{s} - j - 1}{n}. \tag{47}$$

Inserting $s = \frac{m-1}{m}$ and $h = m$, this equals

$$\mathfrak{S}_{\frac{m-1}{m};m}(n,k) = \frac{(m-1)^n n!}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \binom{n + \frac{j}{m-1} - 1}{n}.$$

Using

$$\binom{n+\frac{j}{m-1}-1}{n} = \frac{\Gamma(n+\frac{j}{m-1})}{\Gamma(\frac{j}{m-1})n!},$$

inserting this into (44), and simplifying the resulting expression shows the claimed equation. □

The explicit form of the Touchard polynomials of order $m \geq 2$ was also derived in [21], see Equation (40) and the subsequent discussion. It was mentioned in [21, Equation (37)] that the higher order Touchard polynomials satisfy the recursion relation

$$(x^m + x^m D)T_n^{(m)}(x) = T_{n+1}^{(m)}(x). \tag{48}$$

Let us consider Touchard polynomials for a fixed order $m \in \mathbb{N}$ with $m \geq 2$. Recalling Theorem 9, we observe that the sequence of generalized Bell polynomials $\mathfrak{B}_{\frac{m-1}{m};m|n}(x)$ satisfies for $x > 0$ the convexity property. From the obvious inequality

$$(x^{m-1}-1)^2 \geq 0,$$

we obtain $2x^{m-1} \leq 1 + x^{2(m-1)}$ and, after multiplication by $x^{(m-1)n}$ (here $x \geq 0$ is used), the inequality

$$2x^{(m-1)(n+1)} \leq x^{(m-1)n} + x^{(m-1)(n+2)},$$

which shows that the sequence $x^{(m-1)n}$ also possesses the convexity property for $x > 0$.

**Corollary 24.** *For a fixed order $m \geq 2$, the sequence $(T_n^{(m)}(x))_{n\in\mathbb{N}}$ of Touchard polynomials of order $m$ can be written as a product of two sequences which both possess for $x > 0$ the convexity property.*

Recall that a sequence $(a_n)_{n\in\mathbb{N}}$ is called *log concave* if and only if $a_n^2 \leq a_{n-1}a_{n+1}$ for all $n \geq 2$. Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be given sequences. We will say that the two sequences are *2-log concave* if and only if

$$\frac{a_{n-1}}{a_n} \leq \frac{b_n}{b_{n+1}} \quad \text{and} \quad \frac{b_{n-1}}{b_n} \leq \frac{a_n}{a_{n+1}} \tag{49}$$

for all $n \geq 2$.

**Proposition 25.** *Let two convex sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be given and assume that they are in addition 2-log concave. Then the product sequence $(c_n)_{n\in\mathbb{N}}$ with $c_n = a_n b_n$ is also convex.*

*Proof.* Since $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are both convex, one has $2a_n \leq a_{n-1} + a_{n+1}$ as well as $2b_n \leq b_{n-1} + b_{n+1}$, implying

$$4c_n \leq c_{n-1} + c_{n+1} + a_{n-1}b_{n+1} + a_{n+1}b_{n-1}.$$

Thus,

$$2c_n \leq c_{n-1} + c_{n+1} + (a_{n-1}b_{n+1} + a_{n+1}b_{n-1} - 2a_nb_n),$$

and $c_n$ is convex if $a_{n-1}b_{n+1} + a_{n+1}b_{n-1} - 2a_nb_n \leq 0$. Since the two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are 2-log concave, they satisfy $a_{n-1}b_{n+1} \leq a_nb_n$ as well as $b_{n-1}a_{n+1} \leq a_nb_n$, showing the desired inequality. □

In Corollary 24, it was shown that the sequences of Touchard polynomials of higher order can be written as products of two convex sequences of polynomials. In view of Proposition 25, one could hope that the sequences $(x^{(m-1)n})_{n\in\mathbb{N}}$ and $(\mathfrak{B}_{\frac{m-1}{m};m|n}(x))_{n\in\mathbb{N}}$ are 2-log concave, implying that the sequence $(T_n^{(m)}(x))_{n\in\mathbb{N}}$ is convex (for $x>0$). Unfortunately, this is not true. The two inequalities (49) give for $a_n = x^{(m-1)n}$ and $b_n = \mathfrak{B}_{\frac{m-1}{m};m|n}(x)$

$$\mathfrak{B}_{\frac{m-1}{m};m|n+1}(x) \le x^{m-1}\mathfrak{B}_{\frac{m-1}{m};m|n}(x) \quad \text{and} \quad x^{m-1}\mathfrak{B}_{\frac{m-1}{m};m|n-1}(x) \le \mathfrak{B}_{\frac{m-1}{m};m|n}(x).$$

Together they imply

$$\mathfrak{B}_{\frac{m-1}{m};m|n+1}(x) = x^{m-1}\mathfrak{B}_{\frac{m-1}{m};m|n}(x),$$

which is not true.

It is now very tempting to introduce Touchard polynomials of negative order by using (43) for $-m$ with $m \in \mathbb{N}$.

**Definition 26.** Let $m \in \mathbb{N}$. The *Touchard polynomials of negative order $-m$* are defined for all $n \in \mathbb{N}$ by

$$T_n^{(-m)}(x) := e^{-x}(X^{-m}D)^n e^x. \tag{50}$$

*Remark* 27. One can also define the Touchard polynomials of order zero, but due to $T_n^{(0)}(x) = e^{-x}D^n e^x = 1$ for all $n \in \mathbb{N}$, no interesting polynomials result.

From the definition above, it is easy to see that the analogue of (48) holds true, i.e.,

$$(x^{-m} + x^{-m}D)T_n^{(-m)}(x) = T_{n+1}^{(-m)}(x). \tag{51}$$

In complete analogy to the case of positive order, one has the following theorem.

**Theorem 28.** *The Touchard polynomials of negative order $-m$ are given as*

$$T_n^{(-m)}(x) = x^{-(m+1)n}\sum_{k=0}^{n}\mathfrak{S}_{\frac{m+1}{m};-m}(n,k)x^k. \tag{52}$$

*Furthermore, $T_n^{(-m)}$ is a polynomial in $\frac{1}{x}$ of degree $(m+1)n-1$ and can be written in terms of the generalized Bell polynomials as*

$$T_n^{(-m)}(x) = x^{-(m+1)n}\mathfrak{B}_{\frac{m+1}{m};-m|n}(x). \tag{53}$$

*In particular, $T_n^{(-m)}(1) = \mathfrak{B}_{\frac{m+1}{m};-m}(n)$.*

*Proof.* Using Lemma 15 for $\lambda = -m$, we find that

$$(X^{-m}D)^n = X^{-(m+1)n}\sum_{k=0}^{n}\mathfrak{S}_{\frac{m+1}{m};-m}(n,k)X^kD^k.$$

Inserting this into the definition of $T_n^{(-m)}(x)$, one has shown that

$$T_n^{(-m)}(x) = e^{-x}x^{-(m+1)n}\sum_{k=0}^{n}\mathfrak{S}_{\frac{m+1}{m};-m}(n,k)x^ke^x,$$

which is equivalent to the first assertion. The other assertions are shown in analogy to Theorem 21. $\square$

As in the case of positive order, one can obtain an explicit expression for the Touchard polynomials of negative order.

**Corollary 29.** *The Touchard polynomials of negative order $-m$ have the explicit expression*

$$T_n^{(-m)}(x) = \left(-\frac{(m+1)}{x^{(m+1)}}\right)^n \sum_{k=0}^{n}\sum_{j=0}^{k} \frac{(-1)^{k-j}}{k!}\binom{k}{j}\frac{\Gamma(n-\frac{j}{m+1})}{\Gamma(1-\frac{j}{m+1})}x^k. \tag{54}$$

It seems to be nontrivial to find a general expression for $T_n^{(-m)}(x)$ that is more explicit than this. When $m=2$, for example, we get

$$T_n^{(-2)}(x) = x^{-3n}\sum_{k=1}^{n} \mathfrak{S}_{\frac{3}{2};-2}(n,k)x^k = x^{-3n}\mathfrak{B}_{\frac{3}{2};-2|n}(x).$$

In the case $m=1$, however, one obtains a more pleasing result.

**Theorem 30.** *The Touchard polynomials of order $-1$ may be expressed by Bessel polynomials, i.e.,*

$$T_n^{(-1)}(x) = x^{-n}y_{n-1}(-\frac{1}{x}), \tag{55}$$

*where $y_{n-1}$ is the $(n-1)$-th Bessel polynomial defined in (20). In particular, $T_n^{(-1)}(x)$ is a polynomial of degree $2n-1$ in $\frac{1}{x}$.*

*Proof.* From (52) we obtain for $m=1$ the expression

$$T_n^{(-1)}(x) = x^{-2n}\sum_{k=1}^{n}\mathfrak{S}_{2;-1}(n,k)x^k = x^{-2n}\sum_{k=1}^{n} b(n,k)x^k, \tag{56}$$

where we have used (22) in the second equation. Using the definition of the Bessel numbers $b(n,k)$ and the explicit form of the Bessel polynomials given in (20), we obtain

$$y_{n-1}(z) = \sum_{k=1}^{n} b(n,k)(-z)^{n-k},$$

hence

$$y_{n-1}(-\frac{1}{x}) = x^{-n}\sum_{k=1}^{n} b(n,k)x^k.$$

Inserting this into the above equation for $T_n^{(-1)}(x)$ yields the assertion. ◻

**Example 31.** The first few Touchard polynomials of order $-1$ are given explicitly as

$$\begin{aligned}
T_1^{(-1)}(x) &= x^{-2}(x),\\
T_2^{(-1)}(x) &= x^{-4}(x^2-x),\\
T_3^{(-1)}(x) &= x^{-6}(x^3-3x^2+3x),\\
T_4^{(-1)}(x) &= x^{-8}(x^4-6x^3+15x^2-15x).
\end{aligned}$$

**Proposition 32.** *Let $s = 2$ and $h = -1$. The $n$-th meromorphic Bell polynomial $\mathfrak{B}_{2;-1|n}(x)$ can be expressed by Bessel polynomials, i.e.,*

$$\mathfrak{B}_{2;-1|n}(x) = x^n y_{n-1}(-\frac{1}{x}) = x^{2n} T_n^{(-1)}(x). \tag{57}$$

*In particular, the $n$-th meromorphic Bell number $\mathfrak{B}_{2;-1}(n)$ is given by $\mathfrak{B}_{2;-1}(n) = y_{n-1}(-1) = T_n^{(-1)}(1)$.*

*Proof.* From the definition and (56), one has

$$\mathfrak{B}_{2;-1|n}(x) = \sum_{k=1}^{n} \mathfrak{S}_{2;-1}(n,k) x^k = x^{2n} T_n^{(-1)}(x) = x^n y_{n-1}(-\frac{1}{x}),$$

showing the first assertion. Recalling $\mathfrak{B}_{s;h}(n) = \mathfrak{B}_{s;h|n}(1)$, the remaining assertions follow. □

In the case $s = 2$ and $h = -1$, one can also write $\mathfrak{B}_{2;-1|n}(x) = \sum_{k=1}^{n} b(n,k) x^k$. For the dual parameters $s = -1$ and $h = 1$, one has in a similar fashion

$$\mathfrak{B}_{-1;1|n}(x) = \sum_{k=1}^{n} \mathfrak{S}_{-1;1}(n,k) x^k = \sum_{k=1}^{n} B(n,k) x^k.$$

Defining the *Hermite polynomials* $H_n$ by their exponential generating function [16, Page 50],

$$e^{2tz-t^2} = \sum_{n\geq 0} H_n(z) \frac{t^n}{n!},$$

it was shown in [44, Example 4.2] that

$$\mathfrak{B}_{-1;1}(n) = \left(\frac{i}{\sqrt{2}}\right)^n H_n\left(\frac{1}{i\sqrt{2}}\right).$$

This is equivalent to

$$\sum_{k=1}^{n} B(n,k) = \left(\frac{i}{\sqrt{2}}\right)^n H_n\left(\frac{1}{i\sqrt{2}}\right),$$

which is also mentioned in [76, Page 631]. For Hermite polynomials one has the classical Rodriguez formula [58, Page 45], yielding the following property analogous to (41):

$$H_n(x) = (-1)^n e^{x^2} D^n e^{-x^2}. \tag{58}$$

Note that considering successive derivatives of $e^{-x^2}$ involves the combinatorics of the product rule. On the other hand, considering $(X^{-1}D)$ instead of $D$ shows that $(X^{-1}D)e^{-x^2} = (-2)e^{-x^2}$, hence

$$(X^{-1}D)^k e^{-x^2} = (-2)^k e^{-x^2}. \tag{59}$$

This shows that - although considering $D^n e^{-x^2}$ is difficult - $(X^{-1}D)^n e^{-x^2}$ is simple! Recall that (29) allows us to transform the derivative $D^n$ into a sum of "derivatives" $(X^{-1}D)^k$. It follows that

$$\begin{aligned} H_n(x) &= (-1)^n e^{x^2} D^n e^{-x^2} \\ &= (-1)^n e^{x^2} \sum_{k=1}^{n} \mathfrak{S}_{-1;1}(n,k) X^{2k-n} (X^{-1}D)^k e^{-x^2} \\ &= (-x)^{-n} \sum_{k=1}^{n} \mathfrak{S}_{-1;1}(n,k)(-2x^2)^k, \end{aligned}$$

where we have used (59) in the last line. Letting $z = -2x^2$, hence $x = -i\sqrt{z/2}$ (we choose the negative root), we may write

$$\mathfrak{B}_{-1;1|n}(z) = \sum_{k=1}^{n} \mathfrak{S}_{-1;1}(n,k) z^k = \left(\frac{i\sqrt{z}}{\sqrt{2}}\right)^n H_n\left(\frac{\sqrt{z}}{i\sqrt{2}}\right).$$

Thus, we have shown the following analogue to Proposition 32.

**Proposition 33.** *Let $s = -1$ and $h = 1$. The $n$-th generalized Bell polynomial $\mathfrak{B}_{-1;1|n}(x)$ can be expressed by Hermite polynomials, i.e.,*

$$\mathfrak{B}_{-1;1|n}(x) = \left(\frac{i\sqrt{x}}{\sqrt{2}}\right)^n H_n\left(\frac{\sqrt{x}}{i\sqrt{2}}\right).$$

*In particular, the $n$-th generalized Bell number $\mathfrak{B}_{-1;1}(n)$ is given by*

$$\mathfrak{B}_{-1;1}(n) = \left(\frac{i}{\sqrt{2}}\right)^n H_n\left(\frac{1}{i\sqrt{2}}\right).$$

As the last point of this section, we wish to determine the exponential generating function for the Touchard polynomials of negative order. To do so, we first recall some results concerning the higher order case discussed in [21] as well as some related operational formulas. Note that the case $m = 1$ corresponds to the conventional Bell numbers, yielding the well known result [57, Page 64]

$$\sum_{n\geq 0} \frac{\lambda^n}{n!} T_n^{(1)}(x) = \sum_{n\geq 0} \frac{\lambda^n}{n!} B_n(x) = e^{x(e^{\lambda}-1)}.$$

Now, let us consider the case $m \geq 2$. Directly from the definition of $T_n^{(m)}(x)$ in (43), we obtain

$$\sum_{n\geq 0} \frac{\lambda^n}{n!} T_n^{(m)}(x) = e^{-x} e^{\lambda x^m D} e^x, \tag{60}$$

where we have denoted $D = \frac{d}{dx}$ as above. For such generalized exponential operators, the action of $e^{\lambda q(x) D}$ on a given function $f$ is given by [22, Equation (4)]

$$e^{\lambda q(x) D} f(x) = f[F_q^{-1}(\lambda + F_q(x))], \tag{61}$$

where

$$F_q(x) = \int^x \frac{d\zeta}{q(\zeta)} \tag{62}$$

and $F_q^{-1}$ is its inverse. Note that for $q(x) = 1$ one obtains $F_q(x) = x$ and, consequently, $e^{\lambda D} f(x) = f(x+\lambda)$. As another example, consider $q(x) = x$. It follows that $F_q(x) = \ln(x)$ as well as $F_q^{-1}(x) = e^x$, yielding $e^{\lambda x D} f(x) = f(e^{\lambda} x)$. Now, we turn to $q(x) = x^m$ with $m \geq 2$ an integer. It follows that $F_q(x) = -\frac{x^{-(m-1)}}{m-1}$ and $F_q^{-1}(x) = \sqrt[m-1]{-\frac{1}{(m-1)x}}$. Inserting this into (61) yields, after some rearranging,

$$e^{\lambda x^m D} f(x) = f\left(\frac{x}{\sqrt[m-1]{1-(m-1)\lambda x^{m-1}}}\right), \tag{63}$$

which is mentioned in [21, Equation (35)] and was already known to the Reverend Charles Graves in the early 1850s [29]! Inserting this into (60) gives the final result

$$\sum_{n \geq 0} \frac{\lambda^n}{n!} T_n^{(m)}(x) = e^{x\left[\left\{1-(m-1)\lambda x^{m-1}\right\}^{-\frac{1}{m-1}} - 1\right]}, \tag{64}$$

which can also be found in [21, Equation (38)] (with $\ell = 0$). To derive this result we could have used, alternatively, the connection (45) to generalized Bell polynomials. It was shown in [44, Corollary 4.1] that the exponential generating function of the generalized Bell polynomials is given for $s \in \mathbb{R} \setminus \{0, 1\}$ by

$$\sum_{n \geq 0} \frac{\mu^n}{n!} \mathfrak{B}_{s;h|n}(x) = e^{\left\{1-(1-hs\mu)^{\frac{s-1}{s}}\right\}\frac{x}{h(s-1)}}. \tag{65}$$

Using (45), one finds

$$\sum_{n \geq 0} \frac{\lambda^n}{n!} T_n^{(m)}(x) = \sum_{n \geq 0} \frac{(\lambda x^{m-1})^n}{n!} \mathfrak{B}_{\frac{m-1}{m};m|n}(x).$$

On the right-hand side of the last equation, one can use (65) since $\frac{m-1}{m} \neq 0, 1$ for $m \geq 2$ and obtain exactly (64).

Let us now turn to Touchard polynomials of order $-m$ with $m \in \mathbb{N}$. Here we have the following result.

**Theorem 34.** *The exponential generating function of the Touchard polynomials of negative order $-m$ is given by*

$$\sum_{n \geq 0} \frac{\lambda^n}{n!} T_n^{(-m)}(x) = e^{x\left[\left\{1+(m+1)\lambda x^{-(m+1)}\right\}^{\frac{1}{m+1}} - 1\right]} = e^{\sqrt[m+1]{x^{m+1}+(m+1)\lambda} - x}. \tag{66}$$

*Proof.* In this case, one has the connection (53) to generalized Bell polynomials, implying

$$\sum_{n\geq 0} \frac{\lambda^n}{n!} T_n^{(-m)}(x) = \sum_{n\geq 0} \frac{1}{n!} \left(\frac{\lambda}{x^{m+1}}\right)^n \mathfrak{B}_{\frac{m+1}{m};-m|n}(x).$$

Applying (65) to the right-hand side (with $\mu = \frac{\lambda}{x^{m+1}}$, $s = \frac{m+1}{m}$ and $h = -m$) yields the first asserted equation. The second equation follows easily. $\square$

Let us discuss briefly the preceding result from the operational point of view. Here we have to consider $e^{\lambda x^{-m} D} f(x)$ so that $q(x) = x^{-m}$. It follows that $F_q(x) = \frac{x^{m+1}}{m+1}$ and $F_q^{-1}(x) = \sqrt[m+1]{(m+1)x}$. Inserting this into (61) yields, after some rearranging,

$$e^{\lambda x^{-m} D} f(x) = f(\sqrt[m+1]{x^{m+1} + (m+1)\lambda}). \tag{67}$$

Using this, it follows in analogy to the case $m \geq 2$ from the definition of $T_n^{(-m)}(x)$ in (50) that

$$\sum_{n\geq 0} \frac{\lambda^n}{n!} T_n^{(-m)}(x) = e^{-x} e^{\lambda x^{-m} D} e^x = e^{\sqrt[m+1]{x^{m+1}+(m+1)\lambda} - x},$$

and the expression on the right-hand side equals the one given in (66).

**Corollary 35.** *One has the operational rule*

$$e^{\lambda x^{-1} D} f(x) = f(\sqrt{x^2 + 2\lambda}).$$

*The exponential generating function of the Touchard polynomials of order $-1$ is given by*

$$\sum_{n\geq 0} \frac{\lambda^n}{n!} T_n^{(-1)}(x) = e^{\sqrt{x^2+2\lambda} - x}. \tag{68}$$

*Proof.* The first assertion follows from (67), while the second follows from (66). $\square$

*Remark* 36. For the Touchard polynomials of order $-1$, we can use another relation to obtain (68). For this, we recall the connection to Bessel polynomials established in Theorem 30, i.e., $T_n^{(-1)}(x) = x^{-n} y_{n-1}(-\frac{1}{x})$. Let us define the related polynomials $f_n(x) := x^n y_{n-1}(\frac{1}{x})$. Carlitz has shown [13, Equation (2.5)] that one has for the $f_n(x)$ the exponential generating function

$$\sum_{n\geq 0} \frac{\lambda^n}{n!} f_n(x) = e^{x\left\{1-\sqrt{1-2\lambda}\right\}}.$$

Using the relation $T_n^{(-1)}(x) = \left(-\frac{1}{x^2}\right)^n f_n(-x)$, one therefore finds

$$\sum_{n\geq 0} \frac{\lambda^n}{n!} T_n^{(-1)}(x) = \sum_{n\geq 0} \frac{1}{n!} \left(-\frac{\lambda}{x^2}\right)^n f_n(-x) = e^{-x\left\{1-\sqrt{1+2\frac{\lambda}{x^2}}\right\}},$$

showing (68).

# 7 The $q$-analogue of the generalized Stirling numbers

The main motivation for the definition of the generalized Stirling numbers came from the combinatorial aspects of two variables $U, V$ satisfying the commutation relation

$$UV = VU + hV^s.$$

Clearly, the case $s = 0$ corresponds to the Weyl algebra $UV = VU + h$, see (1). The commutation relation of the Weyl algebra has been $q$-deformed by postulating the commutation relation

$$UV = qVU + h, \tag{69}$$

i.e., the commutator $[U, V] \equiv UV - VU = h$ has been replaced by the $q$-commutator $[U, V]_q \equiv UV - qVU = h$. In the following, we assume again $h \in \mathbb{C} \setminus \{0\}$. Let $s \in \mathbb{R}$. We consider variables $U, V$ satisfying the commutation relation

$$UV = qVU + hV^s \tag{70}$$

(we assume a generic $q \in \mathbb{C}$, but for the calculations it is only important that $q$ is central in the algebra generated by $U, V$). This commutation relation is the $q$-analogue of (4).

If $q$ is generic and $x$ is a complex number or an indeterminate, then define the *$q$-basic number* $[x]_q$ by

$$[x]_q := \frac{1 - q^x}{1 - q}.$$

When $x = n$ is a non-negative integer, then $[n]_q$ is also given by $1 + q + \cdots + q^{n-1}$ if $n \geq 1$, with $[0]_q := 0$. The associated factorial is given by $[n]_q! := [n]_q[n-1]_q \cdots [2]_q[1]_q$ if $n \geq 1$, with $[0]_q! := 1$. Furthermore, the $q$-binomial coefficient (or *Gaussian binomial coefficient*) is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q![k]_q!}$$

if $0 \leq k \leq n$, and is zero if $k < 0$ or if $k > n \geq 0$. In the ring of polynomials in the $q$-variable $x$, one can introduce the *Jackson-derivative* $\mathcal{D}_q$ by

$$\mathcal{D}_q w(x) := \frac{w(qx) - w(x)}{(q-1)x}, \tag{71}$$

and it is a simple consequence that

$$\mathcal{D}_q x^n = [n]_q x^{n-1}. \tag{72}$$

## 7.1 Definition of the $q$-analogue of $\mathfrak{S}_{s;h}(n, k)$ and first results

The following lemma was already derived in [41] and can be shown by a simple induction.

**Lemma 37.** *[41, Lemma 4.2] Let $U$ and $V$ be variables satisfying (70). Then the following identity holds true for all $k \in \mathbb{N}$:*

$$UV^k = q^k V^k U + h[k]_q V^{s+k-1}. \tag{73}$$

In close analogy to the case $q=1$ considered in (5), we define the corresponding *$q$-deformed generalized Stirling numbers* $\mathfrak{S}_{s;h}(n,k|q)$ as normal ordering coefficients of $(VU)^n$, i.e.,

$$(VU)^n = \sum_{k=0}^{n} \mathfrak{S}_{s;h}(n,k|q) V^{s(n-k)+k} U^k, \tag{74}$$

where $U$ and $V$ satisfy (70). The corresponding *$q$-deformed generalized Bell numbers* are defined in analogy to the conventional case by

$$\mathfrak{B}_{s;h}(n|q) := \sum_{k=0}^{n} \mathfrak{S}_{s;h}(n,k|q). \tag{75}$$

**Example 38.** The case $s=0$ and $h=1$ corresponds to the $q$-deformed Weyl algebra (69) and one has a representation by operators $V \mapsto X$ and $U \mapsto \mathcal{D}_q$, where $\mathcal{D}_q$ is the Jackson-derivative defined in (71). It is well-known (see, e.g., [15, Equation (42)]) that

$$(X\mathcal{D}_q)^n = \sum_{k=0}^{n} S_q(n,k) X^k \mathcal{D}_q^k, \tag{76}$$

where $S_q(n,k)$ are the $q$-deformed Stirling numbers of the second kind satisfying [50, Equation (1.14)]

$$[x]_q^n = \sum_{k=0}^{n} S_q(n,k)[x]_q^{\underline{k}},$$

with $[x]_q^{\underline{k}} = [x]_q[x-1]_q \cdots [x-k+1]_q$, as well as the recursion relation [50, Equation (1.15)]

$$S_q(n+1,k) = q^{k-1} S_q(n,k-1) + [k]_q S_q(n,k).$$

Sometimes also the numbers $\tilde{S}_q(n,k) = q^{-\binom{k}{2}} S_q(n,k)$ are considered which arose in the work of Carlitz [12] and Gould [28]. Comparing (74) and (76), we obtain

$$\mathfrak{S}_{0;1}(n,k|q) = S_q(n,k), \tag{77}$$

in analogy to the case $q=1$ mentioned in (8).

**Example 39.** The case $s=2$ and $h=-1$ corresponds to the $q$-meromorphic Weyl algebra introduced by Diaz and Pariguan [24], where one has the commutation relation

$$UV = qVU - V^2.$$

Clearly, for $q=1$, the above commutation relation reduces to the one of the meromorphic Weyl algebra given in (16). According to [24, Theorem 5], one has a representation by operators[1] $V \mapsto X^{-1}$ and $U \mapsto \mathfrak{D}_q := q^{-1}\mathcal{D}_{q^{-1}}$. Let us check this. Using the $q$-Leibniz rule $\mathcal{D}_q(fg) = \mathcal{D}_q(f)g + I_q(f)\mathcal{D}_q(g)$, where $I_q(f)(x) = f(qx)$, we find

$$\begin{aligned} \mathfrak{D}_q(x^{-1}f(x)) &= q^{-1}\mathcal{D}_{q^{-1}}(x^{-1}f(x)) \\ &= q^{-1}\left(\mathcal{D}_{q^{-1}}(x^{-1})f(x) + (q^{-1}x)^{-1}\mathcal{D}_{q^{-1}}(f(x))\right) \\ &= -x^{-2}f(x) + qx^{-1}\mathfrak{D}_q(f(x)), \end{aligned}$$

[1] Note that Diaz and Pariguan [24] consider the commutation relation $UV = qVU + V^2$ which implies an additional sign in the representation.

where we have used in the last line $\mathcal{D}_{q^{-1}}(x^{-1}) = -qx^{-2}$. Thus,

$$\mathfrak{D}_q X^{-1} = qX^{-1}\mathfrak{D}_q - (X^{-1})^2,$$

as was to be verified. According to (74), the corresponding $q$-meromorphic Stirling numbers $\mathfrak{S}_{2;-1}(n,k|q)$ are defined by

$$(X^{-1}\mathfrak{D}_q)^n = \sum_{k=0}^{n} \mathfrak{S}_{2;-1}(n,k|q)(X^{-1})^{2n-k}\mathfrak{D}_q^k.$$

This relation reduces for $q = 1$ to (27).

The first few instances of the $q$-deformed generalized Stirling numbers can be determined directly from their definition. Clearly, $(VU)^1 = VU$, so $\mathfrak{S}_{s;h}(1,1|q) = 1$ (and, consequently, $\mathfrak{B}_{s;h}(1|q) = 1$). The first interesting case is $n = 2$. Directly from the commutation relation and using (70), one finds

$$(VU)^2 = VUVU = V\{qVU + hV^s\}U = qV^2U^2 + hV^{s+1}U,$$

implying $\mathfrak{S}_{s;h}(2,1|q) = h, \mathfrak{S}_{s;h}(2,2|q) = q$ (and, consequently, $\mathfrak{B}_{s;h}(2|q) = q + h$). The next instance $n = 3$ is slightly more tedious, but completely analogous,

$$\begin{aligned}
(VU)^3 &= VU\{qV^2U^2 + hV^{s+1}U\} \\
&= qV\{UV^2\}U^2 + hV\{UV^{s+1}\}U \\
&= qV\{q^2V^2U + h[2]_qV^{s+1}\}U^2 + hV\{q^{s+1}V^{s+1}U + h[s+1]_qV^{2s}\}U \\
&= q^3V^3U^3 + hq\{[2]_q + q^s\}V^{s+2}U^2 + h^2[s+1]_qV^{2s+1}U,
\end{aligned}$$

where we have used in the third line (73). This implies

$$\mathfrak{S}_{s;h}(3,1|q) = h^2[s+1]_q, \quad \mathfrak{S}_{s;h}(3,2|q) = hq\{[2]_q + q^s\}, \quad \mathfrak{S}_{s;h}(3,3|q) = q^3,$$

and, consequently, $\mathfrak{B}_{s;h}(3|q) = h^2[s+1]_q + hq\{[2]_q + q^s\} + q^3$.

As in the case $q = 1$, one can determine the recursion relation of the $q$-deformed generalized Stirling numbers.

**Theorem 40.** *The numbers $\mathfrak{S}_{s;h}(n,k|q)$ defined by (74) satisfy the recursion relation*

$$\mathfrak{S}_{s;h}(n+1,k|q) = q^{s(n+1-k)+k-1}\mathfrak{S}_{s;h}(n,k-1|q) + h[s(n-k)+k]_q\mathfrak{S}_{s;h}(n,k|q) \tag{78}$$

*for all $n \geq 0$ and $k \geq 1$, with $\mathfrak{S}_{s;h}(n,0|q) = \delta_{n,0}$ and $\mathfrak{S}_{s;h}(0,k|q) = \delta_{0,k}$ for all $n,k \in \mathbb{N}_0$.*

*Proof.* Starting from (74), one has $(VU)^{n+1} = \sum_{k=1}^{n+1} \mathfrak{S}_{s;h}(n+1,k|q)V^{s(n+1-k)+k}U^k$. On the other hand, one has

$$\begin{aligned}
&(VU)^{n+1} \\
&= \sum_{k=1}^{n} \mathfrak{S}_{s;h}(n,k|q)VUV^{s(n-k)+k}U^k \\
&= \sum_{k=1}^{n} \mathfrak{S}_{s;h}(n,k|q)V\{q^{s(n-k)+k}V^{s(n-k)+k}U + h\left[s(n-k)+k\right]_q V^{s(n-k)+k-1+s}\}U^k \\
&= \sum_{k=1}^{n+1} \mathfrak{S}_{s;h}(n,k|q)\{q^{s(n-k)+k}V^{s(n-k)+k+1}U^{k+1} + h\left[s(n-k)+k\right]_q V^{s(n+1-k)+k}U^k\},
\end{aligned}$$

where we have used (73) in the second line. Comparing coefficients yields the assertion. □

Recursion (78) for $\mathfrak{S}_{s;h}(n,k|q)$ is the $q$-analogue of the recursion (6) for $\mathfrak{S}_{s;h}(n,k)$. The following corollary can be derived directly from (78).

**Corollary 41.** *The $q$-deformed generalized Stirling numbers satisfy the relation*

$$\mathfrak{S}_{s;h}(n,k|q) = h^{n-k}\mathfrak{S}_{s;1}(n,k|q). \tag{79}$$

Let us consider some special choices of the parameters $s$ and $h$.

**Example 42.** Let $s=0$ and $h=1$. It follows from (78) that

$$\mathfrak{S}_{0;1}(n+1,k|q) = q^{k-1}\mathfrak{S}_{0;1}(n,k-1|q) + [k]_q\mathfrak{S}_{0;1}(n,k|q),$$

which is the recursion relation of the conventional $q$-Stirling numbers of the second kind, see Example 38. Thus, $\mathfrak{S}_{0;1}(n,k|q) \equiv S_q(n,k)$.

**Example 43.** Let $s=1$ and $h=-1$. It follows from (78) that

$$\mathfrak{S}_{1;-1}(n+1,k|q) = q^{n}\mathfrak{S}_{1;-1}(n,k-1|q) - [n]_q\mathfrak{S}_{1;-1}(n,k|q).$$

Gould defined in [28] the $q$-Stirling numbers of the first kind $\tilde{s}_q(n,k)$ (building on the treatment of Carlitz [12]). In our context, the slight variant $s_q(n,k) = q^{-\binom{n}{2}}\tilde{s}_q(n,k)$ is more convenient since then one has in analogy to the undeformed case the relation

$$[x]_{\underline{q}}^{\underline{n}} = \sum_{k=0}^{n} s_q(n,k)[x]_q^k.$$

The recursion relation for these $q$-Stirling numbers of the first kind $s_q(n,k)$ is given by

$$s_q(n+1,k) = q^{-n}s_q(n,k-1) - q^{-n}[n]_q s_q(n,k),$$

from which one can show, using the recursions, that $s_q(n,k) = q^{k-n}\mathfrak{S}_{1;-1}(n,k|1/q)$. Thus,

$$\mathfrak{S}_{1;-1}(n,k|q) = \left(\frac{1}{q}\right)^{n-k} s_{\frac{1}{q}}(n,k). \tag{80}$$

Using (79), one obtains for arbitrary $h \neq 0$ the relation

$$\mathfrak{S}_{1;h}(n,k|q) = \left(-\frac{h}{q}\right)^{n-k} s_{\frac{1}{q}}(n,k).$$

Using (78), it is possible to determine an explicit formula for $\mathfrak{S}_{s;h}(n,k|q)$ for arbitrary $s \neq 1$ and $h \neq 0$. To do so, we will need the following result.

**Theorem 44.** *[39, Theorem 1.1] Suppose $(a_i)_{i\geq 0}$ and $(b_i)_{i\geq 0}$ are sequences of complex numbers where the $b_i$ are distinct. Let $\{u(n,k)\}_{n,k\geq 0}$ be the array defined by the recurrence*

$$u(n+1,k) = u(n,k-1) + (a_n + b_k)u(n,k) \tag{81}$$

*for all $n \geq 0$ and $k \geq 1$, subject to the boundary conditions $u(n,0) = \prod_{i=0}^{n-1}(a_i + b_0)$ and $u(0,k) = \delta_{0,k}$ for all $n,k \geq 0$. Then one has*

$$u(n,k) = \sum_{j=0}^{k} \left( \frac{\prod_{i=0}^{n-1}(b_j + a_i)}{\prod_{\substack{i=0\\ i\neq j}}^{k}(b_j - b_i)} \right), \qquad \forall\ n,k \geq 0. \tag{82}$$

We now derive an explicit formula for $\mathfrak{S}_{s;h}(n,k|q)$ in the case $s \neq 1$. An explicit expression for $\mathfrak{S}_{s;h}(n,k|q)$ when $s = 1$ can be found in Example 43.

**Theorem 45.** *If $s \neq 1$ and $h \neq 0$ are arbitrary, then*

$$\mathfrak{S}_{s;h}(n,k|q) = h^{n-k} q^{s\binom{n}{2} - (s-1)\binom{k}{2} - (n-k)} \sum_{j=0}^{k} \left( \frac{\prod_{i=0}^{n-1}([si]_{1/q} - [(s-1)j]_{1/q})}{\prod_{\substack{i=0\\ i\neq j}}^{k}([(s-1)i]_{1/q} - [(s-1)j]_{1/q})} \right), \tag{83}$$

*for all $n \geq k \geq 1$.*

*Proof.* Let $a_{n,k} = \mathfrak{S}_{s;h}(n,k|q)$. Multiplying (78) by $q^{(s-1)\binom{k}{2} - s\binom{n+1}{2}}$, and letting

$$b_{n,k} = q^{(s-1)\binom{k}{2} - s\binom{n}{2}} a_{n,k},$$

gives the recurrence

$$b_{n+1,k} = b_{n,k-1} + hq^{-sn}[sn - (s-1)k]_q b_{n,k},$$

which may be rewritten as

$$b_{n+1,k} = b_{n,k-1} + \frac{h}{q}([sn]_{1/q} - [(s-1)k]_{1/q}) b_{n,k}. \tag{84}$$

Applying Theorem 44 with

$$a_i = \frac{h[si]_{1/q}}{q} \qquad \text{and} \qquad b_i = -\frac{h[(s-1)i]_{1/q}}{q},$$

and observing that $s \neq 1, h \neq 0$ implies that the $b_i$ are all distinct, gives

$$b_{n,k} = \left(\frac{h}{q}\right)^{n-k} \sum_{j=0}^{k} \left( \frac{\prod_{i=0}^{n-1}([si]_{1/q} - [(s-1)j]_{1/q})}{\prod_{\substack{i=0\\ i\neq j}}^{k}([(s-1)i]_{1/q} - [(s-1)j]_{1/q})} \right). \tag{85}$$

Noting $a_{n,k} = q^{s\binom{n}{2} - (s-1)\binom{k}{2}} b_{n,k}$ gives the requested formula (83). $\square$

*Remark* 46. If $s \neq 0, 1$ and $h \neq 0$, then (83) reduces for $q = 1$ to (47):

$$\mathfrak{S}_{s;h}(n,k|q)\ |_{q=1} = h^{n-k} \sum_{j=0}^{k} \left( \frac{\prod_{i=0}^{n-1}[s(i-j)+j]}{\prod_{\substack{i=0\\ i\neq j}}^{k}[(s-1)(i-j)]} \right) = \frac{h^{n-k}s^n}{(1-s)^k} \sum_{j=0}^{k} \left( \frac{\prod_{i=0}^{n-1}[i-j+\frac{j}{s}]}{(-1)^{k-j}j!(k-j)!} \right)$$

$$= \frac{h^{n-k}s^n n!}{(1-s)^k k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \binom{n+\frac{j}{s}-j-1}{n} = \mathfrak{S}_{s;h}(n,k).$$

Recall from Theorem 10 that in the case $q = 1$ one has the dual pair

$$\{\mathfrak{S}_{-1;1}(n,k), \mathfrak{S}_{2;-1}(n,k)\}$$

corresponding to Bessel numbers (of second and first kinds, respectively). Thus, the corresponding $q$-analogues might be interesting objects. One obtains from (78) the recursion relations

$$\mathfrak{S}_{-1;1}(n+1,k|q) = q^{2k-n-2}\mathfrak{S}_{-1;1}(n,k-1|q) + [2k-n]_q \mathfrak{S}_{-1;1}(n,k|q) \tag{86}$$

and

$$\mathfrak{S}_{2;-1}(n+1,k|q) = q^{2n-k+1}\mathfrak{S}_{2;-1}(n,k-1|q) - [2n-k]_q \mathfrak{S}_{2;-1}(n,k|q). \tag{87}$$

We may give the following explicit formulas. If $n$ is a positive integer, then let $[2n]_q!! = [2n]_q[2n-2]_q \cdots [2]_q$ and $[2n-1]_q!! = [2n-1]_q[2n-3]_q \cdots [1]_q$.

**Proposition 47.** *If $n \geq k \geq 1$, then*

$$\mathfrak{S}_{-1;1}(n,k|q) = q^{k(k-1)}[n]_q! \sum_{j=\lfloor \frac{n+1}{2} \rfloor}^{k} (-1)^{k-j} \frac{q^{j(j+1-2k)}}{[2j]_q!![2k-2j]_q!!} \begin{bmatrix} 2j \\ n \end{bmatrix}_q \tag{88}$$

*and*

$$\mathfrak{S}_{2;-1}(n,k|q) = \frac{(-1)^{n-k}}{[k]_q!} \sum_{\substack{j=1\\ j\ odd}}^{k} (-1)^{\frac{j-1}{2}} q^{\frac{j^2-1}{4}+k-j} [j]_q!![2n-2-j]_q!! \begin{bmatrix} k \\ j \end{bmatrix}_q. \tag{89}$$

*Proof.* Let $c_{n,k} = \mathfrak{S}_{-1;1}(n,k|q)$. Applying (83) when $s = -1$ and $h = 1$, and observing the fact $[-i]_{1/q} = -q[i]_q$, gives

$$c_{n,k} = q^{k(k-1)-\binom{n}{2}} \sum_{j=0}^{k} \left( \frac{\prod_{i=0}^{n-1}([2j]_q - [i]_q)}{\prod_{\substack{i=0\\ i\neq j}}^{k}([2j]_q - [2i]_q)} \right).$$

Noting $[r]_q - [s]_q = q^s[r-s]_q$, and rearranging factors, we obtain the requested formula for $c_{n,k}$.

Let $d_{n,k} = \mathfrak{S}_{2;-1}(n,k|q)$. Applying (83) in the case when $s = 2$ and $h = -1$, gives

$$d_{n,k} = q^{n(n-1)-\binom{k}{2}-(n-k)} \sum_{j=0}^{k} \left( \frac{\prod_{i=0}^{n-1}([j]_{1/q} - [2i]_{1/q})}{\prod_{\substack{i=0\\ i\neq j}}^{k}([j]_{1/q} - [i]_{1/q})} \right),$$

which may be rewritten as

$$d_{n,k} = \frac{(-1)^{n-k} q^{n(n-1)-\binom{k}{2}}}{[k]_{1/q}!} \sum_{\substack{j=1 \\ j \text{ odd}}}^{k} (-1)^{\frac{j-1}{2}} \left(\frac{1}{q}\right)^{\frac{j^2-1}{4}+(j+1)(n-k)} [j]_{1/q}!![2n-2-j]_{1/q}!! \begin{bmatrix} k \\ j \end{bmatrix}_{1/q}. \tag{90}$$

Using the facts

$$[j]_{1/q}!! = \left(\frac{1}{q}\right)^{\frac{j^2-1}{4}} [j]_q!!$$

when $j$ is odd and

$$\begin{bmatrix} k \\ j \end{bmatrix}_{1/q} = \frac{1}{q^{j(k-j)}} \begin{bmatrix} k \\ j \end{bmatrix}_q,$$

and simplifying, yields the requested formula for $d_{n,k}$. $\square$

Dulucq introduced in [25] $q$-analogues of the Bessel polynomials $y_n(x;q)$ in a combinatorial fashion and he gave the following expression generalizing (20).

**Theorem 48.** *[25, Theorem 2.3] The $q$-analogue of the Bessel polynomial is given by*

$$y_n(x;q) = \sum_{k=0}^{n} \begin{bmatrix} n+k \\ n-k \end{bmatrix}_q [2k-1]_q!! q^{\binom{n-k}{2}} x^k. \tag{91}$$

*Furthermore, one has the recursion relation [25, Theorem 2.2]*

$$y_{n+1}(x;q) = [2n+1]_q x y_n(x;q) + q^{2n-1} y_{n-1}(x;q), \tag{92}$$

*with the initial values $y_0(x;q) = 1$ and $y_1(x;q) = 1 + x$.*

Recall from Section 3 that the Bessel numbers of the first kind are defined by $b(n,k) = (-1)^{n-k} a(n,k)$, where $a(n,k)$ is the coefficient of $x^{n-k}$ in $y_{n-1}(x)$. Thus, we define in a similar fashion the *$q$-deformed Bessel number of the first kind* $b(n,k|q)$ to be $(-1)^{n-k}$ times the coefficient of $x^{n-k}$ in $y_{n-1}(x;q)$. Thus,

$$b(n,k|q) = (-1)^{n-k} q^{\binom{k-1}{2}} \begin{bmatrix} 2n-k-1 \\ k-1 \end{bmatrix}_q [2(n-k)-1]_q!!. \tag{93}$$

Using

$$[2(n-k)-1]_q!! = \frac{[2(n-k)-1]_q!}{[2(n-k)-2]_q!!},$$

this can be written, equivalently, as

$$b(n,k|q) = (-1)^{n-k} q^{\binom{k-1}{2}} \frac{[2n-k-1]_q!}{[k-1]_q![2n-2k]_q!!}. \tag{94}$$

It is the $q$-analogue to (21).

**Proposition 49.** *The q-deformed Bessel numbers of the first kind satisfy the recursion relation*

$$b(n+1,k|q) = q^{k-2}b(n,k-1|q) - q^{k-1}[2n-k]_q b(n,k|q). \tag{95}$$

*Proof.* From (94), we obtain

$$\begin{aligned} b(n+1,k|q) &= (-1)^{n+1-k} q^{\binom{k-1}{2}} \frac{[2n-k+1]_q!}{[k-1]_q![2n-2k+2]_q!!} \\ &= -(-1)^{n-k} q^{\binom{k-1}{2}} \frac{[2n-k-1]_q![2n-k]_q[2n-k+1]_q}{[k-1]_q![2n-2k]_q!![2n-2k+2]_q}. \end{aligned}$$

Using $[2n-k+1]_q = [k-1]_q + q^{k-1}[2n-2k+2]_q$, this gives

$$\begin{aligned} b(n+1,k|q) &= -(-1)^{n-k} q^{\binom{k-1}{2}} \frac{[2n-k-1]_q![2n-k]_q}{[k-1]_q![2n-2k]_q!!} \left( \frac{[k-1]_q}{[2n-2k+2]_q} + q^{k-1} \right) \\ &= (-1)^{n-k+1} q^{\binom{k-2}{2}+k-2} \frac{[2n-k]_q!}{[k-2]_q![2n-2k+2]_q!!} - q^{k-1}[2n-k]_q b(n,k|q) \\ &= q^{k-2}b(n,k-1|q) - q^{k-1}[2n-k]_q b(n,k|q), \end{aligned}$$

as was to be shown. □

Note that the recursion relation of the $q$-Bessel numbers of the first kind $b(n,k|q)$ looks similar to the one for $\mathfrak{S}_{2;-1}(n,k|q)$ given in (87), with only the powers of $q$ differing, and we seek a direct connection between these two numbers. Solving recurrence (95) in another way, and equating the expression that results with the one in (94), yields the following $q$-identity which seems to be new in the $q=1$ case as well.

**Corollary 50.** *If $n \geq k \geq 1$, then*

$$\frac{[k]_q[2n-k-1]_q!}{[2n-2k]_q!!} = \sum_{\substack{j=1 \\ j \text{ odd}}}^{k} (-1)^{\frac{j-1}{2}} q^{\frac{j^2-1}{4}+(j-1)(n-k)} [j]_q!![2n-2-j]_q!! \begin{bmatrix} k \\ j \end{bmatrix}_q. \tag{96}$$

*Proof.* Let $b_{n,k} = b(n,k|q)$. Dividing both sides of (95) by $q^{\binom{k-1}{2}}$, and letting $c_{n,k} = q^{-\binom{k-1}{2}} b_{n,k}$, gives the recurrence

$$c_{n+1,k} = c_{n,k-1} - \frac{1}{q}([2n]_q - [k]_q)c_{n,k}.$$

Applying Theorem 44 with

$$a_i = -\frac{[2i]_q}{q} \qquad \text{and} \qquad b_i = \frac{[i]_q}{q}$$

gives

$$c_{n,k} = q^{k-n} \sum_{j=0}^{k} \left( \frac{\prod_{i=0}^{n-1}([j]_q - [2i]_q)}{\prod_{\substack{i=0 \\ i \neq j}}^{k}([j]_q - [i]_q)} \right),$$

which may be rewritten as

$$c_{n,k} = \frac{(-1)^{n-k}}{[k]_q!} \sum_{\substack{j=1\\ j\ odd}}^{k} (-1)^{\frac{j-1}{2}} q^{\frac{j^2-1}{4}+(j-1)(n-k)} [j]_q!![2n-2-j]_q!! \begin{bmatrix} k \\ j \end{bmatrix}_q .$$

Noting $b_{n,k} = q^{\binom{k-1}{2}} c_{n,k}$, equating the resulting expression for $b_{n,k}$ with the one given in (94), and rearranging factors gives (96). $\square$

We can now state the relation between $\mathfrak{S}_{2;-1}(n,k|q)$ and the $q$-deformed Bessel numbers $b(n,k|q)$. It is the $q$-analogue of (22).

**Proposition 51.** *The $q$-meromorphic Stirling numbers can be expressed by $(1/q)$-deformed Bessel numbers of the first kind, i.e.,*

$$\mathfrak{S}_{2;-1}(n,k|q) = q^{(n-1)^2} (q^{-1})^{n-k} b(n,k|q^{-1}). \tag{97}$$

*Proof.* It remains to collect several results already shown above. Recall from the proof of Proposition 47 that $\mathfrak{S}_{2;-1}(n,k|q)$ is given by (90). Thus, denoting $\tilde{q} = 1/q$, we have

$$\mathfrak{S}_{2;-1}(n,k|q) = \tilde{q}^{\binom{k}{2}-n(n-1)} \frac{(-1)^{n-k}}{[k]_{\tilde{q}}!} \sum_{\substack{j=1\\ j\ odd}}^{k} (-1)^{\frac{j-1}{2}} \tilde{q}^{\frac{j^2-1}{4}+(j+1)(n-k)} [j]_{\tilde{q}}!![2n-2-j]_{\tilde{q}}!! \begin{bmatrix} k \\ j \end{bmatrix}_{\tilde{q}} .$$

The sum on the right-hand side equals - apart from a factor $\tilde{q}^{2(n-k)}$ - precisely the sum on the right-hand side of identity (96), where $q$ is replaced by $\tilde{q}$, so that we obtain

$$\mathfrak{S}_{2;-1}(n,k|q) = \tilde{q}^{\binom{k}{2}-n(n-1)} \frac{(-1)^{n-k}}{[k]_{\tilde{q}}!} \tilde{q}^{2(n-k)} \frac{[k]_{\tilde{q}}[2n-k-1]_{\tilde{q}}!}{[2n-2k]_{\tilde{q}}!!}.$$

This can be written as

$$\mathfrak{S}_{2;-1}(n,k|q) = \tilde{q}^{\binom{k}{2}-n(n-1)-\binom{k-1}{2}} \tilde{q}^{2(n-k)} \left( (-1)^{n-k} \tilde{q}^{\binom{k-1}{2}} \frac{[2n-k-1]_{\tilde{q}}!}{[k-1]_{\tilde{q}}![2n-2k]_{\tilde{q}}!!} \right).$$

By (94), the term enclosed by parentheses equals $b(n,k|\tilde{q})$. Simplifying the exponent of $\tilde{q}$ and recalling $\tilde{q} = 1/q$ yields the assertion. $\square$

Let us consider for $n \in \mathbb{N}$ the associated $q$-meromorphic Bell polynomial

$$\mathfrak{B}_{2;-1|n}(x;q) := \sum_{k=1}^{n} \mathfrak{S}_{2;-1}(n,k|q) x^k.$$

Using Proposition 51, this can be written as

$$\mathfrak{B}_{2;-1|n}(x;q) = q^{(n-1)^2-n} \sum_{k=1}^{n} b(n,k|q^{-1})(qx)^k. \tag{98}$$

Exactly as in the case $q=1$ (see the proof of Theorem 30), we have for the $q$-analogue of the Bessel polynomials the relation

$$y_{n-1}(-\frac{1}{x};q) = x^{-n}\sum_{k=1}^{n} b(n,k|q)x^k.$$

Considering $1/q$ instead of $q$, and then $qx$ instead of $x$, yields

$$(qx)^n y_{n-1}(-\frac{1}{qx};\frac{1}{q}) = \sum_{k=1}^{n} b(n,k|q^{-1})(qx)^k. \tag{99}$$

Is is now straightforward to show the following $q$-analogue of Proposition 32.

**Proposition 52.** *The $n$-th $q$-meromorphic Bell polynomial $\mathfrak{B}_{2;-1|n}(x;q)$ can be expressed by $(1/q)$-deformed Bessel polynomials, i.e.,*

$$\mathfrak{B}_{2;-1|n}(x;q) = q^{(n-1)^2} x^n y_{n-1}(-\frac{1}{qx};\frac{1}{q}). \tag{100}$$

*In particular, the corresponding $q$-meromorphic Bell numbers are given by*

$$\mathfrak{B}_{2;-1}(n|q) = q^{(n-1)^2} y_{n-1}(-\frac{1}{q};\frac{1}{q}).$$

*Proof.* Inserting (99) into (98) yields

$$\mathfrak{B}_{2;-1|n}(x;q) = q^{(n-1)^2-n}(qx)^n y_{n-1}(-\frac{1}{qx};\frac{1}{q}),$$

showing the first asserted equation. The second equation follows from

$$\mathfrak{B}_{2;-1}(n|q) = \mathfrak{B}_{2;-1|n}(1;q).$$

□

Recall that the $q$-deformed Lah numbers $L_q(n,k)$, see [27], are given by

$$L_q(n,k) = q^{k(k-1)}\frac{[n]_q!}{[k]_q!}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$$

and satisfy the recursion relation

$$L_q(n+1,k) = q^{n+k-1}L_q(n,k-1) + [n+k]_q L_q(n,k). \tag{101}$$

From (78), one obtains for the choices $s=1/2$ and $h=2$ the recursion relation

$$\mathfrak{S}_{\frac{1}{2};2}(n+1,k|q) = q^{\frac{n+k-1}{2}}\mathfrak{S}_{\frac{1}{2};2}(n,k-1|q) + 2\left[\frac{n+k}{2}\right]_q \mathfrak{S}_{\frac{1}{2};2}(n,k|q), \tag{102}$$

which is very similar to (101). In fact, using

$$\left[\frac{x}{2}\right]_q = \frac{[x]_{\sqrt{q}}}{[2]_{\sqrt{q}}},$$

we can introduce $\tilde{q} = \sqrt{q}$ and get

$$\mathfrak{S}_{\frac{1}{2};2}(n+1,k|q) = \tilde{q}^{n+k-1}\mathfrak{S}_{\frac{1}{2};2}(n,k-1|q) + \frac{2}{[2]_{\tilde{q}}}\,[n+k]_{\tilde{q}}\,\mathfrak{S}_{\frac{1}{2};2}(n,k|q).$$

Thus, we can easily show the following.

**Proposition 53.** *The $q$-deformed generalized Stirling numbers $\mathfrak{S}_{\frac{1}{2};2}(n,k|q)$ can be expressed by $\sqrt{q}$-deformed Lah numbers, i.e.,*

$$\mathfrak{S}_{\frac{1}{2};2}(n,k|q) = \left(\frac{2}{1+\sqrt{q}}\right)^{n-k} L_{\sqrt{q}}(n,k). \tag{103}$$

*Proof.* Let us introduce $T(n,k) := (\frac{2}{1+\sqrt{q}})^{n-k}L_{\sqrt{q}}(n,k)$. It follows that

$$\begin{aligned}
&T(n+1,k)\\
&= \left(\frac{2}{1+\sqrt{q}}\right)^{n+1-k} L_{\sqrt{q}}(n+1,k)\\
&= \left(\frac{2}{1+\sqrt{q}}\right)^{n-(k-1)} \sqrt{q}^{n+k-1}L_{\sqrt{q}}(n,k-1) + \left(\frac{2}{1+\sqrt{q}}\right)^{n-k+1} [n+k]_{\sqrt{q}}L_{\sqrt{q}}(n,k)\\
&= \sqrt{q}^{n+k-1}T(n,k-1) + \left(\frac{2}{1+\sqrt{q}}\right)[n+k]_{\sqrt{q}}T(n,k),
\end{aligned}$$

for all $n \geq 0$ and $k \geq 1$. Thus, $T(n,k)$ satisfies the same recursion relation as $\mathfrak{S}_{\frac{1}{2};2}(n,k|q)$. Since $T(n,0) = \delta_{n,0}$ and $T(0,k) = \delta_{0,k}$ for all $n,k \in \mathbb{N}_0$, the initial values also coincide, completing the proof. ☐

Before closing this section, let us collect the results for the cases we have considered explicitly in the following table.

| $(s,h)$ | $\mathfrak{S}_{s;h}(n,k\|q)$ | Comment |
|---|---|---|
| $(0,1)$ | $S_q(n,k)$ | $q$-deformed Stirling numbers of the second kind (77) |
| $(1,-1)$ | $(1/q)^{n-k}s_{1/q}(n,k)$ | $(1/q)$-deformed Stirling numbers of the first kind (80) |
| $(2,-1)$ | $q^{(n-1)^2}(1/q)^{n-k}b(n,k\|q^{-1})$ | $(1/q)$-deformed Bessel numbers of the first kind (97) |
| $(\frac{1}{2},2)$ | $(2/(1+\sqrt{q}))^{n-k}L_{\sqrt{q}}(n,k)$ | $\sqrt{q}$-deformed Lah numbers (103) |

## 7.2 A comparison with the literature

The Stirling numbers introduced by Hsu and Shiue in [34] have been generalized in different directions. Let us first mention the $q$-analogue introduced by Corcino, Hsu and Tan [20] and considered further in [18]. Introducing an *exponential factorial of $t$ with base $a$* by

$$[t|a]_n := \prod_{j=0}^{n-1}(t - a^j),$$

with $[t|a]_0 = 1$ and $[t|a]_1 = t - 1$, a *pair of exponential-type Stirling numbers*

$$\{S^1[n,k], S^2[n,k]\} \equiv \{S^1[n,k;a,b,c], S^2[n,k;b,a,-c]\}$$

was introduced in [20] by the inverse relations

$$[t|a]_n = \sum_{k=0}^{n} S^1[n,k][t-c|b]_k$$

and

$$[t|b]_n = \sum_{k=0}^{n} S^2[n,k][t+c|a]_k.$$

To obtain a kind of $q$-analogue, one may set $a = q^\alpha, b = q^\beta$ and $c = q^\gamma - 1$. Then the resulting $q$-Stirling numbers $\sigma^1[n,k]$ may be introduced via

$$\sigma^1[n,k] \equiv \sigma^1[n,k;\alpha,\beta,\gamma]_q := S^1[n,k;q^\alpha,q^\beta,q^\gamma - 1](q-1)^{n-k}, \tag{104}$$

with $\sigma^1[0,0] = 1$, where the case $\alpha = 0$ or $\beta = 0$ is treated as the limit $\alpha \to 0$ or $\beta \to 0$ whenever the limit exists; the case $\sigma^2[n,k]$ is similar. It has been shown, see [20, Proposition 6], that

$$\lim_{q\to 1} \sigma^1[n,k;\alpha,\beta,\gamma]_q = S(n,k;\alpha,\beta,\gamma).$$

Thus, the numbers $\sigma^1[n,k;\alpha,\beta,\gamma]_q$ are indeed a $q$-generalization of the generalized Stirling numbers $S(n,k;\alpha,\beta,\gamma)$ due to Hsu and Shiue. They satisfy the recursion relation [20, Proposition 8]

$$\sigma^1[n+1,k;\alpha,\beta,\gamma]_q = \sigma^1[n,k-1;\alpha,\beta,\gamma]_q + ([k\beta]_q - [n\alpha]_q - [\gamma]_q)\sigma^1[n,k;\alpha,\beta,\gamma]_q. \tag{105}$$

Comparing this relation with the one for the $q$-deformed generalized Stirling numbers

$$\mathfrak{S}_{s;h}(n,k|q)$$

shows the following proposition.

**Proposition 54.** *The $q$-deformed generalized Stirling numbers $\mathfrak{S}_{s;h}(n,k|q)$ do not coincide with the $q$-generalized Stirling numbers $\sigma^1[n,k;\alpha,\beta,\gamma]_q$ of Corcino, Hsu and Tan.*

The generalized Stirling numbers $S(n,k;\alpha,\beta,r)$ were generalized in a slightly different fashion by Remmel and Wachs [55], see also [7]. Remmel and Wachs presented two natural ways to give $p,q$-analogues of these generalized Stirling numbers. Before presenting their results, let us recall that the $p,q$-analogue of a real number $x$ is given by

$$[x]_{p,q} = \frac{p^x - q^x}{p - q},$$

and the factorials $[n]_{p,q}!$ and binomial coefficients are defined in the natural fashion. Note that choosing $p = 1$ gives the $q$-numbers considered above,

$$[x]_{1,q} = \frac{1-q^x}{1-q} = [x]_q.$$

Now, the first type of generalization Remmel and Wachs introduced consists in replacing $(t-r)$ by $([t]_{p,q} - [r]_{p,q})$ in (9) and (10). For example, they defined $S^{1,p,q}_{n,k}(\alpha,\beta,r)$ in [55, Equation (19)] by

$$\begin{aligned}&([t]_{p,q} - [r]_{p,q})([t]_{p,q} - [r+\alpha]_{p,q}) \cdots ([t]_{p,q} - [r+(n-1)\alpha]_{p,q})\\ &\qquad = \sum_{k=0}^{n} S^{1,p,q}_{n,k}(\alpha,\beta,r)([t]_{p,q})([t]_{p,q} - [\beta]_{p,q}) \cdots ([t]_{p,q} - [(k-1)\beta]_{p,q}),\end{aligned}$$

and, similarly, for $S^{2,p,q}_{n,k}(\alpha,\beta,r)$. Furthermore, they showed the recursion relation [55, Equation (22)]

$$S^{1,p,q}_{n+1,k}(\alpha,\beta,r) = S^{1,p,q}_{n,k-1}(\alpha,\beta,r) + ([k\beta]_{p,q} - [n\alpha - r]_{p,q}) S^{1,p,q}_{n,k}(\alpha,\beta,r), \tag{106}$$

with $S^{1,p,q}_{0,0}(\alpha,\beta,r) = 1$ and $S^{1,p,q}_{n,k}(\alpha,\beta,r) = 0$ if $k < 0$ or $k > n$.

**Proposition 55.** *The $q$-generalized Stirling numbers $\sigma^1[n,k;\alpha,\beta,\gamma]_q$ of Corcino, Hsu and Tan coincide for $\gamma = 0$ with the type-I $p,q$-generalized Stirling numbers $S^{1,p,q}_{n,k}(\alpha,\beta,r)$ for $r = 0$ and $p = 1$, i.e.,*

$$\sigma^1[n,k;\alpha,\beta,0]_q = S^{1,1,q}_{n,k}(\alpha,\beta,0).$$

*Proof.* Comparing the respective recursion relations (105) (for $\gamma = 0$) and (106) (for $r = 0$ and $p = 1$), we see that they are equal. ∎

From our point of view, more interesting is the type-II $p,q$-analogue $\tilde{S}^{1,p,q}_{n,k}(\alpha,\beta,r)$, which Remmel and Wachs introduced by replacing $(t-r)$ by $[t-r]_{p,q}$ in (9) and (10). More precisely (see the discussion in [55, Page 6]), they defined $\tilde{S}^{1,p,q}_{n,k}(\alpha,\beta,r)$ in [55, Equation (41)] as the solution of the recursion relation

$$\tilde{S}^{1,p,q}_{n+1,k}(\alpha,\beta,r) = q^{(k-1)\beta - n\alpha - r}\tilde{S}^{1,p,q}_{n,k-1}(\alpha,\beta,r) + p^{t-k\beta}[k\beta - n\alpha - r]_{p,q}\tilde{S}^{1,p,q}_{n,k}(\alpha,\beta,r), \tag{107}$$

with the initial conditions $\tilde{S}^{1,p,q}_{0,0}(\alpha,\beta,r) = 1$ and $\tilde{S}^{1,p,q}_{n,k}(\alpha,\beta,r) = 0$ if $k < 0$ or $k > n$, and showed that the numbers so defined satisfy [55, Equation (38)]

$$\begin{aligned}&[t-r]_{p,q}[t-r-\alpha]_{p,q} \cdots [t-r-(n-1)\alpha]_{p,q}\\ &\qquad = \sum_{k=0}^{n} \tilde{S}^{1,p,q}_{n,k}(\alpha,\beta,r)[t]_{p,q}[t-\beta]_{p,q} \cdots [t-(k-1)\beta]_{p,q}.\end{aligned} \tag{108}$$

A similar result holds for $\tilde{S}^{2,p,q}_{n,k}(\alpha,\beta,r)$. As Remmel and Wachs noted, the variable $t$ is an extra parameter and one should write $\tilde{S}^{1,p,q}_{n,k}(\alpha,\beta,r,t)$ instead of $\tilde{S}^{1,p,q}_{n,k}(\alpha,\beta,r)$ to specify the dependence on $t$. However, since we are interested in the case $p = 1$, the parameter $t$ will play no role for us.

Now, we can show the $q$-analogue of Theorem 2.

**Theorem 56.** *The q-deformed generalized Stirling numbers $\mathfrak{S}_{s;h}(n,k|\tilde{q})$ are given by the type-II $p,q$-analogue $\tilde{S}^{1,p,q}_{n,k}(\alpha,\beta,r)$ of Remmel and Wachs for $p=1$ and $q=\tilde{q}^{\frac{1}{h}}$, where the parameters are chosen as $\alpha=-hs, \beta=h(1-s)$ and $r=0$, i.e.,*

$$\mathfrak{S}_{s;h}(n,k|q) = \left(\frac{h}{[h]_{q^{\frac{1}{h}}}}\right)^{n-k} \tilde{S}^{1,1,q^{\frac{1}{h}}}_{n,k}(-hs,h(1-s),0). \tag{109}$$

*In particular, for $h=1$ one has the identification*

$$\mathfrak{S}_{s;1}(n,k|q) = \tilde{S}^{1,1,q}_{n,k}(-s,1-s,0). \tag{110}$$

*Proof.* Considering (107) for $p=1$ with the choice of parameters $\alpha=-hs, \beta=h(1-s)$ and $r=0$ yields the recursion relation

$$\begin{aligned}\tilde{S}^{1,1,q}_{n+1,k}(-hs,h(1-s),0) &= q^{h[s(n-k+1)+k-1]}\tilde{S}^{1,1,q}_{n,k-1}(-hs,h(1-s),0)\\ &\quad +[h\{s(n-k)+k\}]_q\tilde{S}^{1,1,q}_{n,k}(-hs,h(1-s),0),\end{aligned}$$

which equals for $h=1$ the one for $\mathfrak{S}_{s;1}(n,k|q)$ given in (78), showing (110). Now, let us assume $h\neq 1$. Recalling that we can write for any $x$,

$$[hx]_q = [h]_q[x]_{q^h},$$

the above recursion relation can be written with $\tilde{q}:=q^h$ and $T(n,k):=\tilde{S}^{1,1,q}_{n,k}(-hs,h(1-s),0)$ as

$$T(n+1,k) = \tilde{q}^{s(n-k+1)+k-1}T(n,k-1)+[h]_q[s(n-k)+k]_{\tilde{q}}T(n,k).$$

To get rid of the factor $[h]_q$, we define

$$U(n,k) := \left(\frac{h}{[h]_q}\right)^{n-k} T(n,k),$$

and obtain for $U(n,k)$ the recursion relation

$$U(n+1,k) = \tilde{q}^{s(n-k+1)+k-1}U(n,k-1)+h[s(n-k)+k]_{\tilde{q}}U(n,k).$$

Comparing this with (78), we see that $U(n,k)$ satisfies the same recursion relation as $\mathfrak{S}_{s;h}(n,k|\tilde{q})$ and, therefore, they are equal, upon comparing initial values. Thus, we have

$$\mathfrak{S}_{s;h}(n,k|\tilde{q}) = U(n,k) = \left(\frac{h}{[h]_q}\right)^{n-k} T(n,k) = \left(\frac{h}{[h]_q}\right)^{n-k} \tilde{S}^{1,1,q}_{n,k}(-hs,h(1-s),0).$$

Considering $\mathfrak{S}_{s;h}(n,k|q)$ instead of $\mathfrak{S}_{s;h}(n,k|\tilde{q})$ yields the assertion. □

**Example 57.** Let us consider $s=1/2$ and $h=2$. The identification (109) reduces in this case to

$$\mathfrak{S}_{\frac{1}{2};2}(n,k|q) = \left(\frac{2}{[2]_{\sqrt{q}}}\right)^{n-k} \tilde{S}^{1,1,\sqrt{q}}_{n,k}(-1,1,0).$$

Using $[2]_{\sqrt{q}} = 1+\sqrt{q}$ as well as Proposition 53, we find that

$$\tilde{S}^{1,1,\sqrt{q}}_{n,k}(-1,1,0) = L_{\sqrt{q}}(n,k),$$

which is the $q$-analogue of the relation $S(n,k;-1,1,0)=L(n,k)$ discussed in Example 7.

**Corollary 58.** *The $q$-deformed generalized Stirling numbers $\mathfrak{S}_{s;1}(n,k|q)$ can be written as connection coefficients*

$$[t]_q[t+s]_q\cdots[t+(n-1)s]_q = \sum_{k=0}^{n}\mathfrak{S}_{s;1}(n,k|q)[t]_q[t+(s-1)]_q\cdots[t+(k-1)(s-1)]_q. \quad (111)$$

*Proof.* Using the identification of $\mathfrak{S}_{s;1}(n,k|q)$ with $\tilde{S}^{1,1,q}_{n,k}(-s,1-s,0)$ according to (110), the asserted equation is just equation (108) for $\tilde{S}^{1,1,q}_{n,k}(-s,1-s,0)$. $\square$

Introducing the notation

$$[t]_q^{(r,s)} := [t]_q[t+s]_q\cdots[t+(r-1)s]_q,$$

we can write (111) briefly in the form

$$[t]_q^{(n,s)} = \sum_{k=0}^{n}\mathfrak{S}_{s;1}(n,k|q)[t]_q^{(k,s-1)},$$

which is the $q$-analogue of the corresponding identity for the case $q=1$, see [44, Theorem 5.8].

# 8 Conclusion

In the present paper, we considered further the generalized Stirling numbers $\mathfrak{S}_{s;h}(n,k)$ as well as the corresponding Bell numbers $\mathfrak{B}_{s;h}(n)$ introduced and discussed by the present authors in [41, 42, 44, 45]. As a first point, it was shown that $\mathfrak{S}_{s;h}(n,k)$ corresponds to the particular case $S(n,k;-hs,h(1-s),0)$ of the three-parameter family of generalized Stirling numbers introduced earlier by Hsu and Shiue [34]. From this, we obtained immediately that the arrays of numbers $\mathfrak{S}_{s;h}(n,k)$ and $\mathfrak{S}_{1-s;-h}(n,k)$ form a dual pair of inverse arrays, thereby giving orthogonality relations. Furthermore, using a recent result of Corcino and Corcino [19], this also shows that the family of generalized Bell polynomials $\mathfrak{B}_{s;h|n}(x)$ is convex for $h\geq 0$ and $0\leq s\leq 1$ (with $x>0$). The special case $s=0$ and $h=1$ corresponds to the conventional Weyl algebra and is well-known. Th e case $s=2$ and $h=-1$ corresponds to the meromorphic Weyl algebra introduced by Diaz and Pariguan [23] and was already briefly considered in [45]. In the present paper, this case was treated explicitly and the connection to Bessel numbers and Bessel polynomials was established. For the dual case $s=-1$ and $h=1$, a similar connection to Hermite polynomials was shown. The generalized Stirling and Bell numbers were also shown to be very closely connected to the Touchard polynomials of higher order introduced recently by Dattoli et al. [21]. This allowed us to derive some properties of the Touchard polynomials of higher order rather quickly and led to the introduction of Touchard polynomials of negative order, for which similar properties were shown. In particular, the Touchard polynomials of order $-1$ can be expressed through Bessel polynomials. As a final aspect, certain natural $q$-analogues $\mathfrak{S}_{s;h}(n,k|q)$ of the generalized Stirling numbe rs were introduced by modifying the underlying commutation relation and first properties were shown, e.g., the recursion relation. It was also shown that these numbers are not given by the $q$-analogue of the generalized Stirling numbers due to Hsu and Shiue introduced by Corcino, Hsu and Tan [20], but rather as a special case of the type-II $p, q$-analogue of the generalized Stirling numbers introduced by Remmel and Wachs [55]. Several special choices for the parameters $s$ and $h$ were considered explicitly, in particular, the case $s = 2$ and $h = -1$ corresponding to the $q$-meromorphic Weyl algebra introduced by Diaz and Pariguan [24].

Let us point out a few other aspects which warrant further study. Clearly, there are many properties of the generalized Stirling and Bell numbers to be unearthed (e.g., unimodality, congruence properties). Also, it would be nice to find new connections to well-known combinatorial numbers and polynomials. As a next point, we would like to mention the Touchard polynomials of higher or negative order for which most of their properties are not yet known. Finally, the $q$-analogue $\mathfrak{S}_{s;h}(n, k|q)$ (as well as the associated Bell numbers) should be considered in more depth. For example, it would be nice to exhibit more special cases explicitly, to determine the parameters of dual pairs (thereby also obtaining orthogonality relations), and to find $q$-analogues of other identities satisfied by $\mathfrak{S}_{s;h}(n, k)$.

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