Journal of Integer Sequences, Vol. 15 (2012),
Article 12.5.5

Asymptotic Formulae for the n-th Perfect Power

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In memory of my sister Fedra Marina Jakimczuk (1970–2010)

Abstract

Let P_n be the n-th perfect power. In this article we prove asymptotic formulae for P_n. For example, we prove the following formula

$$P_n = n^2 - 2n^{5/3} - 2n^{7/5} + \frac{13}{3}n^{4/3} - 2n^{9/7} + 2n^{6/5} - 2n^{13/11} + o\left(n^{13/11}\right).$$

1 Introduction

A natural number of the form m^n where m is a positive integer and $n \geq 2$ is called a perfect power. The first few terms of the integer sequence of perfect powers are

$$1, 4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, 100, 121, 125, 128 \ldots$$

and they form sequence A001597 in Sloane's *Encyclopedia*.

Let P_n be the n-th perfect power. That is, $P_1 = 1, P_2 = 4, P_3 = 8, P_4 = 9, \ldots$.

In this article we prove asymptotic formulae for P_n. For example,

$$P_n = n^2 - 2n^{5/3} - 2n^{7/5} + \frac{13}{3}n^{4/3} - 2n^{9/7} + 2n^{6/5} - 2n^{13/11} + o\left(n^{13/11}\right).$$

This formula is a corollary of our main theorem (Theorem 6), which can give as many terms in the expansion as desired.

There exist various theorems and conjectures on the sequence P_n. For example, the following theorem:

$$\sum_{n=2}^{\infty}\frac{1}{P_n}=\sum_{k=2}^{\infty}\mu(k)\,(1-\zeta(k))=0,87446\dots$$

where $\mu(k)$ is the Möbius function and $\zeta(k)$ is the Riemann zeta function.

We also have the following theorem called the Goldbach-Euler theorem:

$$\sum_{n=2}^{\infty}\frac{1}{P_n-1}=1.$$

This result was first published by Euler in 1737. Euler attributed the result to a letter (now lost) from Goldbach.

Mihăilescu [4, 5, 6] proved that the only pair of consecutive perfect powers is 8 and 9, thus proving Catalan's conjecture.

The Pillai's conjecture establish the following limit

$$\lim_{n\to\infty}(P_{n+1}-P_n)=\infty.$$

This is an unsolved problem.

There exist algorithms for detecting perfect powers [1, 2].

Let $N(x)$ be the number of perfect powers not exceeding x. M. A. Nyblom [7] proved the following asymptotic formula

$$N(x)\sim\sqrt{x}.$$

M. A. Nyblom [8] also obtained a formula for the exact value of $N(x)$ using the inclusion-exclusion principle.

Let p_h be the h-th prime. Consequently we have,

$$p_1=2,p_2=3,p_3=5,p_4=7,p_5=11,p_6=13,\dots$$

Jakimczuk [3] proved the following theorem where more precise formulae for $N(x)$ are established. This theorem will be used later.

Theorem 1. *Let p_h be the h-th prime with $h\geq 2$, where h is an arbitrary but fixed positive integer. Then*

$$N(x)=\sum_{k=1}^{h-1}(-1)^{k+1}\sum_{\substack{1\leq i_1<\cdots<i_k\leq h-1\\ p_{i_1}\cdots p_{i_k}<p_h}}x^{\frac{1}{p_{i_1}\cdots p_{i_k}}}+(1+o(1))x^{1/p_h},\tag{1}$$

where the inner sum is taken over the k-element subsets $\{i_1,\dots,i_k\}$ of the set $\{1,2,\dots,h-1\}$ such that the inequality $p_{i_1}\cdots p_{i_k}<p_h$ holds.

If $h=5$ then Theorem 1 becomes,

$$N(x)=\sqrt{x}+\sqrt[3]{x}+\sqrt[5]{x}-\sqrt[6]{x}+\sqrt[7]{x}-\sqrt[10]{x}+(1+o(1))\sqrt[11]{x}.\tag{2}$$

Note that equation (2) include the cases $h=2,3,4$. In general, equation (1) for a certain value of $h=k$ include the cases $h=2,3,\dots,k-1$. This fact is a direct consequence of equation (1).

2 Some Lemmas

The following lemma is an immediate consequence of the binomial theorem.

Lemma 2. *We have*

$$(1+x)^\alpha = 1 + (\alpha + o(1))x \qquad (x \to 0),$$
$$(1+x)^\alpha = 1 + \alpha x + O(x^2) \qquad (x \to 0),$$
$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + O(x^3) \qquad (x \to 0).$$

Lemma 3. *Let P_n be the n-th perfect power. We have*

$$P_n \sim n^2.$$

Proof. Equation (2) gives $N(x) \sim \sqrt{x}$. Consequently $N(P_n) = n \sim \sqrt{P_n}$. Therefore $P_n \sim n^2$. □

Lemma 4. *Let p_h be the h-th prime. If $h \geq 3$ then we have*

$$\frac{2}{p_{h-1}} - \frac{1}{3} < \frac{2}{p_h}.$$

Proof. We have

$$\frac{2}{p_{h-1}} - \frac{1}{3} < \frac{2}{p_h} \Leftrightarrow \frac{2}{p_{h-1}} - \frac{2}{p_h} < \frac{1}{3} \Leftrightarrow \frac{1}{p_{h-1}} - \frac{1}{p_h} < \frac{1}{6}.$$

Clearly, the last inequality is true if $h \geq 5$ since $p_{h-1} \geq 7$.

On the other hand, we have

$$\frac{1}{p_2} - \frac{1}{p_3} = \frac{1}{3} - \frac{1}{5} = \frac{2}{15} < \frac{1}{6}$$
$$\frac{1}{p_3} - \frac{1}{p_4} = \frac{1}{5} - \frac{1}{7} = \frac{2}{35} < \frac{1}{6}$$

□

3 The Fundamental Lemma

The following lemma is a characterization of asymptotic formulae for P_n. The lemma prove the existence of asymptotic formulae for P_n.

Lemma 5. *Let p_h $(h \geq 3)$ be the h-th prime. We have*

$$P_n = n^2 - 2n^{5/3} + \sum_{i=1}^{m} d_i n^{g_i} + (-2 + o(1))n^{1+\frac{2}{p_h}}, \tag{3}$$

where $2 > 5/3 > g_1 > \cdots > g_m > 1 + \frac{2}{p_h}$, the d_i are rational coefficients and in equation (3) appear the terms $-2n^{1+\frac{2}{p_i}}$ $(i = 2, \ldots, h-1)$. Besides the rational exponents $5/3$ and g_i $(i = 1, \ldots, m)$ are of the form $\frac{b_i}{c_i}$ where b_i and c_i are relatively prime and the c_i are squarefree integers with prime divisors bounded by p_{h-1}.

Proof. We shall use mathematical induction. First, we shall prove that the lemma is true for $h = 3$.

If $h = 2$ then Theorem 1 becomes (see (2))

$$N(x) = \sqrt{x} + (1 + o(1))\sqrt[3]{x}.$$

Substituting $x = P_n$ into this equation and using Lemma 3 we obtain

$$N(P_n) = n = \sqrt{P_n} + (1 + o(1))\sqrt[3]{P_n} = \sqrt{P_n} + (1 + o(1))n^{2/3}.$$

That is

$$\sqrt{P_n} = n + (-1 + o(1))n^{2/3}.$$

Therefore

$$\begin{aligned} P_n &= \left(n + (-1 + o(1))n^{2/3}\right)^2 \\ &= n^2 + 2(-1 + o(1))n^{5/3} + (-1 + o(1))^2 n^{4/3} \\ &= n^2 + (-2 + o(1))n^{5/3}. \end{aligned} \tag{4}$$

If $h = 3$ then Theorem 1 becomes (see (2))

$$N(x) = \sqrt{x} + \sqrt[3]{x} + (1 + o(1))\sqrt[5]{x}.$$

Substituting $x = P_n$ into this equation and using equation (4), Lemma 3 and Lemma 2 we obtain

$$\begin{aligned} N(P_n) &= n = P_n^{1/2} + P_n^{1/3} + (1 + o(1))n^{2/5} = P_n^{1/2} + \left(n^2 + (-2 + o(1))n^{5/3}\right)^{1/3} \\ &+ (1 + o(1))n^{2/5} = P_n^{1/2} + n^{2/3}\left(1 + (-2 + o(1))n^{-1/3}\right)^{1/3} + (1 + o(1))n^{2/5} \\ &= P_n^{1/2} + n^{2/3}\left(1 + ((-2/3) + o(1))n^{-1/3}\right) \\ &+ (1 + o(1))n^{2/5} = P_n^{1/2} + n^{2/3} + (1 + o(1))n^{2/5}. \end{aligned}$$

That is

$$P_n^{1/2} = n - n^{2/3} + (-1 + o(1))n^{2/5}.$$

Therefore

$$P_n = \left(n - n^{2/3} + (-1 + o(1))n^{2/5}\right)^2 = n^2 - 2n^{5/3} + (-2 + o(1))n^{7/5}.$$

That is

$$P_n = n^2 - 2n^{5/3} + (-2 + o(1))n^{7/5}. \tag{5}$$

Equation (5) is Lemma 5 for $h = 3$. Consequently the lemma is true for $h = 3$.

Suppose that the lemma is true for $h - 1 \geq 3$. We shall prove that the lemma is also true for $h \geq 4$.

We have (see (1))

$$
\begin{aligned}
N(x) &= \sum_{k=1}^{h-1}(-1)^{k+1}\sum_{\substack{1\le i_1<\cdots<i_k\le h-1\\ p_{i_1}\cdots p_{i_k}<p_h}} x^{\frac{1}{p_{i_1}\cdots p_{i_k}}} + (1+o(1))x^{1/p_h} = x^{1/2}\\
&+ \sum_{i=2}^{h-1} x^{1/p_i} + \sum_{k=2}^{h-1}(-1)^{k+1}\sum_{\substack{1\le i_1<\cdots<i_k\le h-1\\ p_{i_1}\cdots p_{i_k}<p_h}} x^{\frac{1}{p_{i_1}\cdots p_{i_k}}}\\
&+ (1+o(1))x^{1/p_h} \qquad (h\ge 4).
\end{aligned} \tag{6}
$$

Substituting $x = P_n$ into (6) and using Lemma 3 we obtain

$$
\begin{aligned}
n &= P_n^{1/2} + \sum_{i=2}^{h-1} P_n^{1/p_i} + \sum_{k=2}^{h-1}(-1)^{k+1}\sum_{\substack{1\le i_1<\cdots<i_k\le h-1\\ p_{i_1}\cdots p_{i_k}<p_h}} P_n^{\frac{1}{p_{i_1}\cdots p_{i_k}}}\\
&+ (1+o(1))n^{2/p_h} \qquad (h\ge 4).
\end{aligned} \tag{7}
$$

By inductive hypothesis we have

$$
P_n = n^2 - 2n^{5/3} + \sum_{i=1}^{s} a_i n^{r_i} + (-2+o(1))n^{1+\frac{2}{p_{h-1}}} \qquad (h\ge 4), \tag{8}
$$

where $2 > 5/3 > r_1 > \cdots > r_s > 1 + \frac{2}{p_{h-1}}$, the a_i are rational coefficients and in equation (8) appear the terms $-2n^{1+\frac{2}{p_i}}$ $(i = 2, \ldots, h-2)$. Besides the rational exponents $5/3$ and r_i $(i = 1, \ldots, s)$ are of the form $\frac{l_i}{f_i}$ where l_i and f_i are relatively prime and the f_i are squarefree integers with prime divisors bounded by p_{h-2}.

Equation (8) gives

$$
P_n = n^2\left(1 - 2n^{-1/3} + \sum_{i=1}^{s} a_i n^{-2+r_i} + (-2+o(1))n^{-1+\frac{2}{p_{h-1}}}\right), \tag{9}
$$

where

$$
-2n^{-1/3} + \sum_{i=1}^{s} a_i n^{-2+r_i} + (-2+o(1))n^{-1+\frac{2}{p_{h-1}}} \sim -2n^{-1/3}.
$$

Consequently

$$
-2n^{-1/3} + \sum_{i=1}^{s} a_i n^{-2+r_i} + (-2+o(1))n^{-1+\frac{2}{p_{h-1}}} = O\left(n^{-1/3}\right) = o(1). \tag{10}
$$

Let $t \geq 3$ be a positive integer. Equations (9), (10) and Lemma 2 give

$$\begin{aligned} P_n^{1/t} &= n^{2/t}\left(1-2n^{-1/3}+\sum_{i=1}^{s} a_i n^{-2+r_i}+(-2+o(1))n^{-1+\frac{2}{p_{h-1}}}\right)^{1/t} \\ &= n^{2/t}\left(1+\frac{1}{t}\left(-2n^{-1/3}+\sum_{i=1}^{s} a_i n^{-2+r_i}+(-2+o(1))n^{-1+\frac{2}{p_{h-1}}}\right)\right) \\ &+ O\left(n^{-2/3}\right)\Big) = n^{2/t}-\frac{2}{t}n^{-\frac{1}{3}+\frac{2}{t}}+\sum_{i=1}^{s}\frac{1}{t}a_i n^{-2+r_i+\frac{2}{t}} \\ &+ \frac{1}{t}(-2+o(1))n^{-1+\frac{2}{p_{h-1}}+\frac{2}{t}}+O\left(n^{-\frac{2}{3}+\frac{2}{t}}\right). \end{aligned} \tag{11}$$

Note that if $t \geq 3$ then (see Lemma 4)

$$\frac{1}{t}(-2+o(1))n^{-1+\frac{2}{p_{h-1}}+\frac{2}{t}} = o\left(n^{2/p_h}\right), \tag{12}$$

and if $t \geq 3$ then

$$O\left(n^{-\frac{2}{3}+\frac{2}{t}}\right) = o\left(n^{2/p_h}\right). \tag{13}$$

Consequently (11) becomes (see (12) and (13))

$$P_n^{1/t} = n^{2/t}-\frac{2}{t}n^{-\frac{1}{3}+\frac{2}{t}}+\sum_{i=1}^{s}\frac{1}{t}a_i n^{-2+r_i+\frac{2}{t}}+o\left(n^{2/p_h}\right). \tag{14}$$

Note that (see (14)) if $t \geq 3$ the exponent $\frac{2}{t} < 1$ and consequently also $-\frac{1}{3}+\frac{2}{t} < 1$ and $-2+r_i+\frac{2}{t} < 1$ since $-2+r_i < 0$ (see (8))

Substituting (14) into (7) we find that

$$\begin{aligned} n &= P_n^{1/2}+\sum_{j=2}^{h-1}\left(n^{2/p_j}-\frac{2}{p_j}n^{-\frac{1}{3}+\frac{2}{p_j}}+\sum_{i=1}^{s}\frac{1}{p_j}a_i n^{-2+r_i+\frac{2}{p_j}}\right)+\sum_{k=2}^{h-1}(-1)^{k+1} \\ & \sum_{\substack{1\leq i_1<\cdots<i_k\leq h-1 \\ p_{i_1}\cdots p_{i_k}<p_h}}\left(n^{\frac{2}{p_{i_1}\cdots p_{i_k}}}-\frac{2}{p_{i_1}\cdots p_{i_k}}n^{-\frac{1}{3}+\frac{2}{p_{i_1}\cdots p_{i_k}}}\right. \\ &+ \left.\sum_{i=1}^{s}\frac{1}{p_{i_1}\cdots p_{i_k}}a_i n^{-2+r_i+\frac{2}{p_{i_1}\cdots p_{i_k}}}\right)+(1+o(1))n^{2/p_h} \\ &= P_n^{1/2}+\sum_{i=1}^{l} b_i n^{s_i}+(1+o(1))n^{2/p_h} \qquad (h \geq 4), \end{aligned} \tag{15}$$

where $1 > s_1 > \cdots > s_l > \frac{2}{p_h}$. That is

$$P_n^{1/2} = n-\sum_{i=1}^{l} b_i n^{s_i}+(-1+o(1))n^{2/p_h}. \tag{16}$$

Note that all positive exponents in equation (15), that is, the positive exponents of the form

$$\frac{2}{p_j}, \quad -\frac{1}{3}+\frac{2}{p_j}, \quad -2+r_i+\frac{2}{p_j}$$

$$\frac{2}{p_{i_1}\cdots p_{i_k}}, \quad -\frac{1}{3}+\frac{2}{p_{i_1}\cdots p_{i_k}}, \quad -2+r_i+\frac{2}{p_{i_1}\cdots p_{i_k}}$$

are (see (8)) of the form $\frac{m_i}{n_i}$ where m_i and n_i are relatively prime and the n_i are squarefree integers with prime divisors bounded by p_{h-1}. Therefore these exponents are different from $\frac{2}{p_h}$ and consequently the exponents s_i $(i=1,\ldots,l)$ in (16) are of this same form.

Note that $1+\frac{2}{p_h} > 2\,\frac{2}{p_h}$, since $\frac{2}{p_h}<1$. Consequently equation (16) gives

$$\begin{aligned} P_n &= \left(n-\sum_{i=1}^{l} b_i n^{s_i} + (-1+o(1))n^{2/p_h}\right)^2 = \left(n-\sum_{i=1}^{l} b_i n^{s_i}\right)^2 \\ &+ \ (-2+o(1))n^{1+\frac{2}{p_h}} = n^2-2n^{5/3}+\sum_{i=1}^{s} a_i n^{r_i} - 2n^{1+\frac{2}{p_{h-1}}} + \sum_{i=1}^{q} c_i n^{k_i} \\ &+ \ (-2+o(1))n^{1+\frac{2}{p_h}} \quad (h\geq 4), \end{aligned} \tag{17}$$

where $2>5/3>r_1>\cdots>r_s>1+\frac{2}{p_{h-1}}>k_1>\cdots>k_q>1+\frac{2}{p_h}$.

Note also that the first terms in equation (17) are the terms of equation (8). On the other hand in equation (17) appear the term $-2n^{1+\frac{2}{p_{h-1}}}$ (see equation (8)). We now prove these facts.

Equation (17) can be written in the form

$$P_n = Q(n)+\sum_{i=1}^{q} c_i n^{k_i} + (-2+o(1))n^{1+\frac{2}{p_h}} = Q(n)+o\left(n^{1+\frac{2}{p_{h-1}}}\right), \tag{18}$$

where $Q(n)$ is a sum of terms of the form $e_i n^{q_i}$ $\left(q_i \geq 1+\frac{2}{p_{h-1}}\right)$.

On the other hand, equation (8) can be written in the form

$$P_n = n^2-2n^{5/3}+\sum_{i=1}^{s} a_i n^{r_i} - 2n^{1+\frac{2}{p_{h-1}}} + o\left(n^{1+\frac{2}{p_{h-1}}}\right). \tag{19}$$

Equations (18) and (19) give

$$0 = P_n - P_n = \left(Q(n) - \left(n^2-2n^{5/3}+\sum_{i=1}^{s} a_i n^{r_i} - 2n^{1+\frac{2}{p_{h-1}}}\right)\right) + o\left(n^{1+\frac{2}{p_{h-1}}}\right).$$

If

$$Q(n) \neq n^2-2n^{5/3}+\sum_{i=1}^{s} a_i n^{r_i} - 2n^{1+\frac{2}{p_{h-1}}}$$

then we obtain

$$0 = (P_n - P_n) \sim a n^q \qquad (a \neq 0) \qquad \left(q \geq 1 + \frac{2}{p_{h-1}}\right).$$

That is, an evident contradiction. Consequently

$$Q(n) = n^2 - 2n^{5/3} + \sum_{i=1}^{s} a_i n^{r_i} - 2n^{1+\frac{2}{p_{h-1}}}. \tag{20}$$

Finally, equations (18) and (20) give (17). $\square$

Lemma 5 is constructive, we can build the next formula using the former formula. Next, we build the formula that correspond to $h = 4$. We shall need this formula.

If $h = 4$ equation (6) is (see (2))

$$N(x) = x^{1/2} + x^{1/3} + x^{1/5} - x^{1/6} + (1 + o(1))x^{1/7}. \tag{21}$$

On the other hand (Lemma 5) equation (8) is (see (5))

$$P_n = n^2 - 2n^{5/3} + (-2 + o(1))n^{7/5}.$$

Consequently equation (15) is

$$\begin{aligned} n &= P_n^{1/2} + \left(n^{2/3} - \frac{2}{3}n^{-\frac{1}{3}+\frac{2}{3}}\right) + \left(n^{2/5} - \frac{2}{5}n^{-\frac{1}{3}+\frac{2}{5}}\right) - \left(n^{2/6} - \frac{2}{6}n^{-\frac{1}{3}+\frac{2}{6}}\right) \\ &+ (1 + o(1))n^{2/7} = P_n^{1/2} + n^{2/3} - \frac{2}{3}n^{1/3} + n^{2/5} - \frac{2}{5}n^{1/15} - n^{1/3} + \frac{1}{3} + (1 + o(1))n^{2/7} \\ &= P_n^{1/2} + n^{2/3} + n^{2/5} - \frac{5}{3}n^{1/3} + (1 + o(1))n^{2/7}. \end{aligned}$$

Therefore

$$P_n^{1/2} = n - n^{2/3} - n^{2/5} + \frac{5}{3}n^{1/3} + (-1 + o(1))n^{2/7}.$$

Consequently (see (17))

$$\begin{aligned} P_n &= \left(n - n^{2/3} - n^{2/5} + \frac{5}{3}n^{1/3}\right)^2 + (-2 + o(1))n^{9/7} \\ &= n^2 - 2n^{5/3} - 2n^{7/5} + \frac{13}{3}n^{8/6} + (-2 + o(1))n^{9/7}. \end{aligned}$$

That is

$$P_n = n^2 - 2n^{5/3} - 2n^{7/5} + \frac{13}{3}n^{8/6} + (-2 + o(1))n^{9/7}. \tag{22}$$

4 Main Result

The following theorem is the main result of this article. In this theorem we obtain explicit formulae for P_n.

Theorem 6. *Let p_h be the h-th prime with $h \geq 3$, where h is an arbitrary but fixed positive integer.*

Let us consider the formula (see (1))

$$N(x) = \sum_{k=1}^{h-1} (-1)^{k+1} \sum_{\substack{1 \leq i_1 < \cdots < i_k \leq h-1 \\ p_{i_1} \cdots p_{i_k} < p_h}} x^{\frac{1}{p_{i_1} \cdots p_{i_k}}} + (1 + o(1)) x^{1/p_h}. \tag{23}$$

We have

$$\begin{aligned} P_n &= n^2 + \frac{13}{3} n^{8/6} + \frac{32}{15} n^{32/30} + \sum_{k=1}^{h-1} (-1)^k \sum_{\substack{1 \leq i_1 < \cdots < i_k \leq h-1 \\ p_{i_1} \cdots p_{i_k} < p_h, \ p_{i_1} \cdots p_{i_k} \neq 2, \ 6, \ 30}} 2n^{1 + \frac{2}{p_{i_1} \cdots p_{i_k}}} \\ &+ (-2 + o(1)) n^{1 + \frac{2}{p_h}}. \end{aligned} \tag{24}$$

Proof. We shall see that everything relies on Theorem 1. The theorem is true for $h = 3$ (see Lemma 5) and for $h = 4$ (see (21) and (22)). Suppose $h \geq 5$, that is $p_h \geq 11$. Equation (23) can be written in the form (see (21))

$$N(x) = x^{1/2} + x^{1/3} + x^{1/5} - x^{1/6} + \sum_{i=1}^{s} (-1)^{1 + a_i} x^{1/n_i} + (1 + o(1)) x^{1/p_h}, \tag{25}$$

where a_i is the number of different prime factors in n_i and the exponents are in decreasing order,

$$\frac{1}{2} > \frac{1}{3} > \frac{1}{5} > \frac{1}{6} > \frac{1}{n_1} > \cdots > \frac{1}{n_s} > \frac{1}{p_h}. \tag{26}$$

For example, if $h = 5$ then equation (25) becomes equation (2).

On the other hand, we have (Lemma 5 and equation (22))

$$P_n = n^2 - 2n^{5/3} - 2n^{7/5} + \frac{13}{3} n^{8/6} + \sum_{i=1}^{t} d_i n^{r_i} + (-2 + o(1)) n^{1 + \frac{2}{p_h}}, \tag{27}$$

where the exponents are in decreasing order,

$$2 > \frac{5}{3} > \frac{7}{5} > \frac{8}{6} > r_1 > \cdots > r_t > 1 + \frac{2}{p_h}. \tag{28}$$

Equation (27) gives

$$P_n = n^2 \left(1 - 2n^{-1/3} - 2n^{-3/5} + \frac{13}{3} n^{-4/6} + \sum_{i=1}^{t} d_i n^{r_i - 2} + (-2 + o(1)) n^{-1 + \frac{2}{p_h}} \right) \tag{29}$$

where

$$-2n^{-1/3}-2n^{-3/5}+\frac{13}{3}n^{-4/6}+\sum_{i=1}^{t}d_in^{r_i-2}+(-2+o(1))n^{-1+\frac{2}{p_h}}\sim -2n^{-1/3},$$

since (see (28))

$$-\frac{1}{3}>-\frac{3}{5}>-\frac{4}{6}>r_1-2>\cdots>r_t-2>-1+\frac{2}{p_h}. \tag{30}$$

Consequently

$$\begin{aligned} A_n &= -2n^{-1/3}-2n^{-3/5}+\frac{13}{3}n^{-4/6}+\sum_{i=1}^{t}d_in^{r_i-2}+(-2+o(1))n^{-1+\frac{2}{p_h}} \\ &= O\left(n^{-1/3}\right)=o(1). \end{aligned} \tag{31}$$

Besides

$$\begin{aligned} B_n &= \left(-2n^{-1/3}-2n^{-3/5}+\frac{13}{3}n^{-4/6}+\sum_{i=1}^{t}d_in^{r_i-2}+(-2+o(1))n^{-1+\frac{2}{p_h}}\right)^2 \\ &= \left(-2n^{-1/3}-2n^{-3/5}+\left(\frac{13}{3}+o(1)\right)n^{-4/6}\right)^2 \\ &= 4n^{-2/3}+8n^{-14/15}+O\left(n^{-1}\right). \end{aligned} \tag{32}$$

Substituting $x=P_n$ into equation (25) and using Lemma 3 we obtain

$$n=P_n^{1/2}+P_n^{1/3}+P_n^{1/5}-P_n^{1/6}+\sum_{i=1}^{s}(-1)^{1+a_i}P_n^{1/n_i}+n^{2/p_h}+o\left(n^{2/p_h}\right). \tag{33}$$

Equations (29), (31), (32) and Lemma 2 give

$$\begin{aligned} P_n^{1/2} &= n\left(1+\frac{1}{2}A_n+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2}B_n+O\left(n^{-1}\right)\right) \\ &= n-n^{2/3}-n^{2/5}+\frac{13}{6}n^{2/6}+\sum_{i=1}^{t}\frac{d_i}{2}n^{r_i-1} \\ &- n^{2/p_h}+o\left(n^{2/p_h}\right)-\frac{1}{2}n^{2/6}-n^{2/30}+O(1). \end{aligned} \tag{34}$$

Equations (29), (31), (30) and Lemma 2 give

$$P_n^{1/3}=n^{2/3}\left(1+\frac{1}{3}A_n+O\left(n^{-2/3}\right)\right)=n^{2/3}-\frac{2}{3}n^{2/6}-\frac{2}{3}n^{2/30}+O(1). \tag{35}$$

$$P_n^{1/5}=n^{2/5}\left(1+\frac{1}{5}A_n+O\left(n^{-2/3}\right)\right)=n^{2/5}-\frac{2}{5}n^{2/30}+o(1). \tag{36}$$

$$P_n^{1/6} = n^{2/6}\left(1+\frac{1}{6}A_n + O\left(n^{-2/3}\right)\right) = n^{2/6} + O(1). \tag{37}$$

$$P_n^{1/n_i} = n^{2/n_i}\left(1+\frac{1}{n_i}A_n + O\left(n^{-2/3}\right)\right) = n^{2/n_i} + o(1) \qquad (i=1,\dots,s). \tag{38}$$

Substituting equations (34), (35), (36), (37) and (38) into equation (33) we find that

$$0 = \sum_{i=1}^{t}\frac{d_i}{2}n^{r_i-1} + \sum_{i=1}^{s}(-1)^{1+a_i}n^{2/n_i} - \frac{31}{15}n^{2/30} + o\left(n^{2/p_h}\right). \tag{39}$$

Note that (see (28) and (26)) $r_i - 1 > \frac{2}{p_h}$ and $\frac{2}{n_i} > \frac{2}{p_h}$.

If $p_h \leq 29$ then $-\frac{31}{15}n^{2/30} = o\left(n^{2/p_h}\right)$. Consequently we have

$$\sum_{i=1}^{t}\frac{d_i}{2}n^{r_i-1} = \sum_{i=1}^{s}(-1)^{a_i}n^{2/n_i},$$

where $t=s$, $d_i = (-1)^{a_i}2$ $(i=1,\dots,s)$ and $r_i = 1+\frac{2}{n_i}$ $(i=1,\dots,s)$. Since in contrary case we have $0 \sim an^b$ where $a \neq 0$ and $b > \frac{2}{p_h}$, an evident contradiction. Substituting these values into (27) we obtain (24) (see (25)).

If $p_h \geq 31$ then $\frac{2}{30} > \frac{2}{p_h}$ and there exists k such that $n_k = 30 = 2.3.5$ (see (23)). Consequently we have

$$\sum_{i=1}^{t}\frac{d_i}{2}n^{r_i-1} = \sum_{i=1}^{s}(-1)^{a_i}n^{2/n_i} + \frac{31}{15}n^{2/30},$$

where $t=s$, $d_i = (-1)^{a_i}2$ $(i \neq k)$, $d_k = 2(-1+\frac{31}{15}) = \frac{32}{15}$ and $r_i = 1+\frac{2}{n_i}$ $(i=1,\dots,s)$. Since in contrary case we have $0 \sim an^b$ where $a \neq 0$ and $b > \frac{2}{p_h}$, an evident contradiction. Substituting these values into (27) we obtain (24) (see (25)). $\square$

Example 7. If $h=5$ equation (23) is (see (2))

$$N(x) = x^{1/2} + x^{1/3} + x^{1/5} - x^{1/6} + x^{1/7} - x^{1/10} + (1+o(1))x^{1/11}.$$

Consequently Theorem 6 gives

$$P_n = n^2 - 2n^{5/3} - 2n^{7/5} + \frac{13}{3}n^{4/3} - 2n^{9/7} + 2n^{6/5} + (-2+o(1))n^{13/11}$$

5 Acknowledgements

The author would like to thank the anonymous referee for his/her valuable comments and suggestions for improving the original version of this article. The author is also very grateful to Universidad Nacional de Luján.

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2000 *Mathematics Subject Classification*: Primary 11A99; Secondary 11B99.
Keywords: n-th perfect power, asymptotic formula.

(Concerned with sequence A001597.)

Received October 1 2011; revised versions received January 14 2012; March 20 2012; May 29 2012. Published in *Journal of Integer Sequences*, May 29 2012.

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