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Sums of Products of s-Fibonacci Polynomial Sequences

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Abstract

We consider s-Fibonacci polynomial sequences $\left(F_0(x), F_s(x), F_{2s}(x), \ldots\right)$, where $s \in \mathbb{N}$ is given. By studying certain z-polynomials involving s-polyfibonomials $\binom{n}{k}_{F_s(x)} = \frac{F_{sn}(x)\cdots F_{s(n-k+1)}(x)}{F_s(x)\cdots F_{ks}(x)}$ and s-Gibonacci polynomial sequences $\left(G_0(x), G_s(x), G_{2s}(x), \ldots\right)$, we generalize some known results (and obtain some new results) concerning sums of products and addition formulas of Fibonacci numbers.

1 Introduction

We use $\mathbb{N}$ for the natural numbers and $\mathbb{N}'$ for $\mathbb{N} \cup \{0\}$. Throughout this article s will denote a natural number.

Recall that Fibonacci polynomials $F_n(x)$ are defined as $F_0(x) = 0$, $F_1(x) = 1$, and $F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$, $n \geq 2$, and extended for negative integers as $F_{-n}(x) = (-1)^{n+1} F_n(x)$. Similarly, Lucas polynomials $L_n(x)$ are defined as $L_0(x) = 2$, $L_1(x) = x$, and $L_n(x) = xL_{n-1}(x) + L_{n-2}(x)$, $n \geq 2$, and extended for negative integers as $L_{-n}(x) = (-1)^n L_n(x)$. It is clear that $F_n(1)$ and $L_n(1)$ correspond to the Fibonacci F_n and Lucas L_n number sequences, respectively (A000045 and A000032 of Sloane's *Encyclopedia*, respectively). A generalized Fibonacci (Gibonacci) polynomial $G_n(x)$ is defined as $G_n(x) = xG_{n-1}(x) + G_{n-2}(x)$, $n \geq 2$, where $G_0(x)$ and $G_1(x)$ are given (arbitrary) initial conditions. We will use also $H_n(x)$ to denote Gibonacci polynomials. It is easy to see that

$$G_n(x) = G_0(x) F_{n-1}(x) + G_1(x) F_n(x),$$

and that we can extend for negative integers as

$$G_{-n}(x) = (-1)^{n+1} G_n(x) + (-1)^n G_0(x) L_n(x).$$

We have Binet's formulas

$$F_n(x) = \frac{1}{\sqrt{x^2+4}}\left(\alpha^n(x) - \beta^n(x)\right) \quad , \quad L_n(x) = \alpha^n(x) + \beta^n(x), \tag{1}$$

where

$$\alpha(x) = \frac{x+\sqrt{x^2+4}}{2} \quad , \quad \beta(x) = \frac{x-\sqrt{x^2+4}}{2}, \tag{2}$$

(the roots of $z^2 - xz - 1 = 0$).

For a Gibonacci polynomial $G_n(x)$ we have

$$G_n(x) = \frac{1}{\sqrt{x^2+4}}\left(c_1(x)\alpha^n(x) - c_2(x)\beta^n(x)\right), \tag{3}$$

where $c_1(x) = G_1(x) - G_0(x)\beta(x)$ and $c_2(x) = G_1(x) - G_0(x)\alpha(x)$.

Some basic relations involving $\alpha(x)$ and $\beta(x)$ (as $\alpha(x)\beta(x) = -1$) will be used throughout the work, as well as some basic Fibonacci polynomial identities (most of times without further comments). About this point we would like to comment the following: some Fibonacci number identities are valid as Fibonacci polynomial identities. For example, the well-known identity

$$F_{n+m} = F_n F_{m+1} + F_{n-1}F_m$$

is just the case $x = 1$ of

$$F_{n+m}(x) = F_n(x) F_{m+1}(x) + F_{n-1}(x) F_m(x). \tag{4}$$

However, it is more natural to accept the existence of Fibonacci number identities that are not valid as identities with Fibonacci polynomials. An example is the Gelin-Cesàro identity

$$F_{n-2}F_{n-1}F_{n+1}F_{n+2} + 1 = F_n^4,$$

which is false for Fibonacci polynomials: for example, for $n = 3$ we have that the left-hand side is the polynomial

$$F_1(x) F_2(x) F_4(x) F_5(x) + 1 = x^8 + 5x^6 + 7x^4 + 2x^2 + 1,$$

while the right-hand side is

$$F_3^4(x) = x^8 + 4x^6 + 6x^4 + 4x^2 + 1.$$

What we can say in general about this example is that

$$F_{n-2}(x) F_{n-1}(x) F_{n+1}(x) F_{n+2}(x) + 1 - F_n^4(x)$$

is a polynomial that has a zero for $x = 1$. For example, for $n = 3$ this polynomial is $x^2(x^2-1)(x^2+2)$.

For a given Gibonacci polynomial sequence $G_n(x)$, the corresponding s-Gibonacci polynomial sequence $G_{sn}(x)$ is given by $G_{sn}(x) = (G_0(x), G_s(x), G_{2s}(x), \ldots)$. We will be dealing with *s-Gibonacci polynomial factorials*, denoted as $(G_n(x)!)_s$, where $n \in \mathbb{N}'$, and defined as $(G_0(x)!)_s = 1$ and $(G_n(x)!)_s = G_s(x)\, G_{2s}(x) \cdots G_{ns}(x)$ for $n \in \mathbb{N}$. Also we will be working with s-Gibonomials

$$\binom{n}{k}_{G_s} = \frac{(G_n!)_s}{(G_k!)_s\,(G_{n-k}!)_s},$$

(see [7]), where the s-Gibonacci sequences are replaced by s-Gibonacci polynomial sequences $G_{sn}(x)$. We will refer to these objects as *s-polygibonomials*, and we will use the natural notation $\binom{n}{k}_{G_s(x)}$ (with $n, k \in \mathbb{N}'$) for them. That is, we have that

$$\binom{n}{k}_{G_s(x)} = \frac{(G_n(x)!)_s}{(G_k(x)!)_s\,(G_{n-k}(x)!)_s},$$

for $0 \le k \le n$, and $\binom{n}{k}_{G_s(x)} = 0$ otherwise. In other words, if $0 \le k \le n$ we have

$$\binom{n}{k}_{G_s(x)} = \frac{G_{sn}(x)\, G_{s(n-1)}(x) \cdots G_{s(n-k+1)}(x)}{G_s(x)\, G_{2s}(x) \cdots G_{ks}(x)}. \tag{5}$$

Plainly it is valid the symmetry property $\binom{n}{k}_{G_s(x)} = \binom{n}{n-k}_{G_s(x)}$, and then we have also

$$\sum_{k=0}^{2n-1} (-1)^k \binom{2n-1}{k}_{G_s(x)} = 0. \tag{6}$$

We call the attention to the fact that in the case $G_n(x) = F_n(x)$, the s-polyfibonomials $\binom{n}{k}_{F_s(x)}$ *are indeed polynomials* (despite the polynomial quotients in the definition). To see this we use the same argument that shows that s-Fibonomials $\binom{n}{k}_{F_s}$ are integers (see [1]): from (4) we have that

$$F_{s(n-k)+1}(x)\, F_{sk}(x) + F_{sk-1}(x)\, F_{s(n-k)}(x) = F_{sn}(x),$$

and then we can write

$$\binom{n}{k}_{F_s(x)} = F_{s(n-k)+1}(x) \binom{n-1}{k-1}_{F_s(x)} + F_{sk-1}(x) \binom{n-1}{k}_{F_s(x)}.$$

Thus, with a simple induction argument we obtain the desired conclusion. In fact, it is easy to see that the degree of $\binom{n}{k}_{F_s(x)}$ is $sk(n-k)$. (Of course, the number sequences $\binom{n}{k}_{F_s(1)}$ correspond to s-Fibonomial sequences (see [7]).) However, s-polygibonomials are in general rational functions. For example, the 2-polylucanomial $\binom{4}{2}_{L_2(x)}$ is

$$\begin{aligned}\binom{4}{2}_{L_2(x)} &= \frac{L_8(x)\, L_6(x)}{L_2(x)\, L_4(x)} \\ &= \frac{(x^4+4x^2+1)(x^8+8x^6+20x^4+16x^2+2)}{x^4+4x^2+2}.\end{aligned}$$

Two examples of s-polyfibonomials $\binom{n}{k}_{F_s(x)}$, as triangular arrays, with n for lines and $k = 0, 1, \ldots, n$ for columns, are the following:

For $s = 1$ we have

$$\begin{array}{c}
1 \\
1 \qquad 1 \\
1 \qquad x \qquad 1 \\
1 \qquad x^2+1 \qquad x^2+1 \qquad 1 \\
1 \qquad x^3+2x \qquad x^4+3x^2+2 \qquad x^3+2x \qquad 1 \\
1 \qquad x^4+3x^2+1 \qquad x^6+5x^4+7x^2+2 \qquad x^6+5x^4+7x^2+2 \qquad x^4+3x^2+1 \qquad 1 \\
\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots
\end{array}$$

For $s = 2$ we have

$$\begin{array}{c}
1 \\
1 \qquad 1 \\
1 \qquad x^2+2 \qquad 1 \\
1 \qquad x^4+4x^2+3 \qquad x^4+4x^2+3 \qquad 1 \\
1 \qquad x^6+6x^4+10x^2+4 \qquad x^8+8x^6+21x^4+20x^2+6 \qquad x^6+6x^4+10x^2+4 \qquad 1 \\
1 \qquad x^8+8x^6+21x^4+20x^2+5 \qquad x^{12}+12x^{10}+55x^8+120x^6+127x^4+60x^2+10 \qquad x^{12}+12x^{10}+55x^8+120x^6+127x^4+60x^2+10 \qquad x^8+8x^6+21x^4+20x^2+5 \qquad 1 \\
\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots
\end{array}$$

(The case $x = 1$ of the previous arrays corresponds to Fibonomials and 2-Fibonomials, A010048 and A034801 of Sloane's *Encyclopedia*, respectively.)

In this work we are concerned with sums of products of s-Fibonacci polynomial sequences. The problem of finding closed formulas for these sums in the case $s = x = 1$ has a quite long story. Lots of results for different particular cases are now available. Some examples are the works of Melham [4, 5]; Prodinger [8]; Seibert & Trojovský [9]; and Wituła and Słota [11], among many others.

The motivation for this work came from identity (5) in [10] (and all the nice and important formulas that can be derived from it). A version of this identity can be stated as follows: for $m \in \mathbb{N}$ and $a_1, \ldots, a_m \in \mathbb{Z}$ given, one has that

$$\sum_{r=0}^{m} (-1)^{\frac{r(r+1)}{2}+r} \binom{m}{r}_G H_{n+m-r} \prod_{i=1}^{m} G_{a_i+m-r} = G_m! H_{n+a_1+\cdots+a_m+\frac{m(m+1)}{2}}, \tag{7}$$

where H_n and G_n are two given Gibonacci sequences, with $G_0 = 0$. What we present in this work are closed formulas for sums that resembles the left-hand side of (7), where the involved Gibonomial $\binom{m}{r}_G$ is replaced by the s-polyfibonomial $\binom{m}{r}_{F_s(x)}$, and the remaining Gibonacci sequences H_n and G_n are replaced for the corresponding s-Gibonacci polynomial sequences $H_{sn}(x)$ and $G_{sn}(x)$. (Thus, identity (7) becomes the case $x = s = 1$ of some of our results.) While the proof of (7) presented in [10] is an induction argument, in this work we proceed explicitly to study certain polynomials, their factorizations and the evaluations of them in certain powers of $\alpha(x)$ and $\beta(x)$, that eventually will lead us to the required proofs.

The main results presented in this work are the following polynomial identities: for given $m \in \mathbb{N}$ and $k, l, p_1, \ldots, p_{2m+1} \in \mathbb{Z}$, we have

$$\begin{aligned}
&\sum_{r=0}^{2m+1} (-1)^{\frac{(sr+2(s+1))(r+1)}{2}} \binom{2m+1}{r}_{F_s(x)} H_{s(2m-r+k)+l}(x) \prod_{i=1}^{2m+1} G_{s(n+r+p_i)}(x) \qquad (8)\\
&= (-1)^{s(m+k)+l+1} \left(x^2+4\right)^{-\frac{1}{2}} \left(F_{2m+1}(x)!\right)_s \\
&\quad \times \begin{pmatrix} \left(H_1(x) - H_0(x)\alpha(x)\right)\left(G_1(x) - G_0(x)\beta(x)\right)^{2m+1} \\ \times \alpha^{s((2m+1)(n+m)+p_1+\cdots+p_{2m+1}-k+1)-l}(x) \\ -\left(H_1(x) - H_0(x)\beta(x)\right)\left(G_1(x) - G_0(x)\alpha(x)\right)^{2m+1} \\ \times \beta^{s((2m+1)(n+m)+p_1+\cdots+p_{2m+1}-k+1)-l}(x) \end{pmatrix},
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{r=0}^{2m} (-1)^{\frac{(sr+2(s+1))(r+1)}{2}} \binom{2m}{r}_{F_s(x)} H_{s(2m+r+k)+l}(x) \prod_{i=1}^{2m} G_{s(n+r+p_i)}(x) \qquad (9)\\
&= (-1)^{s(m+1)+1} \left(x^2+4\right)^{-\frac{1}{2}} \left(F_{2m}(x)!\right)_s \\
&\quad \times \begin{pmatrix} \left(H_1(x) - H_0(x)\beta(x)\right)\left(G_1(x) - G_0(x)\beta(x)\right)^{2m} \\ \times \alpha^{s(2m(m+n)+p_1+\cdots+p_{2m}+3m+k)+l}(x) \\ -\left(H_1(x) - H_0(x)\alpha(x)\right)\left(G_1(x) - G_0(x)\alpha(x)\right)^{2m} \\ \times \beta^{s(2m(m+n)+p_1+\cdots+p_{2m}+3m+k)+l}(x) \end{pmatrix}.
\end{aligned}$$

In section 3 we present the proofs of (8) and (9). What we do to prove these formulas is studying the factorization of certain polynomials involving s-polyfibonomials and s-Gibonacci polynomial sequences, and the evaluation of these polynomials in certain powers of $\alpha(x)$ and $\beta(x)$ as well. This is done in section 2. In section 4 we give some examples of the results proved in section 3. Finally, in the appendix we give the proofs of some identities used in section 2.

2 Preliminary results

We begin this section by showing the factorization of a z-polynomial with s-polyfibonomials as coefficients.

Proposition 1. *For $t,m\in\mathbb{N}'$ we have that*

$$\begin{aligned}
&(-1)^{s(m+1)+1}\sum_{r=0}^{2m}(-1)^{\frac{(sr+2(s+1))(r+1)}{2}+sr(t+1)}\binom{2m}{r}_{F_s(x)}z^r \qquad (10)\\
=&\prod_{p=1}^{m}\left(z^2+(-1)^{st+1}L_{s(4p-2m-1)}(x)\,z+(-1)^s\right).
\end{aligned}$$

Proof. We proceed by induction on m. The case $m=0$ (and $m=1$) can be verified easily. If we suppose the result is true for a given $m\in\mathbb{N}$, then we consider the right-hand side of (10) with $m+1$ replacing m and write

$$\begin{aligned}
&\prod_{p=1}^{m+1}\left(z^2+(-1)^{st+1}L_{s(4p-2m-3)}(x)\,z+(-1)^s\right)\\
=&\left(z^2+(-1)^{st+1}L_{s(2m+1)}(x)\,z+(-1)^s\right)\prod_{p=1}^{m}\left(z^2+(-1)^{st+1}L_{s(4p-2m-3)}(x)\,z+(-1)^s\right).
\end{aligned}$$

Observe that

$$\begin{aligned}
&\prod_{p=1}^{m}\left(z^2+(-1)^{st+1}L_{s(4p-2m-3)}(x)\,z+(-1)^s\right)\\
=&\prod_{p=1}^{m}\left(z^2+(-1)^{st+1}L_{-s(4p-2m-1)}(x)\,z+(-1)^s\right)\\
=&\prod_{p=1}^{m}\left(z^2+(-1)^{s(t+1)+1}L_{s(4p-2m-1)}(x)\,z+(-1)^s\right).
\end{aligned}$$

Then, by using the induction hypothesis we have

$$
\begin{aligned}
&\prod_{p=1}^{m+1}\left(z^2+(-1)^{st+1}L_{s(4p-2m-3)}(x)z+(-1)^s\right)\\
=\;&\left(z^2+(-1)^{st+1}L_{s(2m+1)}(x)z+(-1)^s\right)\prod_{p=1}^{m}\left(z^2+(-1)^{st+1}L_{s(4p-2m-3)}(x)z+(-1)^s\right)\\
=\;&(-1)^{s(m+1)+1}\left(z^2+(-1)^{st+1}L_{s(2m+1)}(x)z+(-1)^s\right)\\
&\times\sum_{r=0}^{2m}(-1)^{\frac{(sr+2(s+1))(r+1)}{2}+srt}\binom{2m}{r}_{F_s(x)}z^r\\
=\;&(-1)^{s(m+1)+1}\left(\begin{array}{c}\sum_{r=2}^{2m+2}(-1)^{\frac{(s(r-2)+2(s+1))(r-1)}{2}+s(r-2)t}\binom{2m}{r-2}_{F_s(x)}\\
+\sum_{r=1}^{2m+1}(-1)^{\frac{(s(r-1)+2(s+1))r}{2}+s(r-1)t+st+1}\binom{2m}{r-1}_{F_s(x)}L_{s(2m+1)}(x)\\
+\sum_{r=0}^{2m}(-1)^{\frac{(sr+2(s+1))(r+1)}{2}+srt+s}\binom{2m}{r}_{F_s(x)}\end{array}\right)z^r\\
=\;&(-1)^{s(m+1)+1}\sum_{r=0}^{2m+2}(-1)^{\frac{(sr+2(s+1))(r+1)}{2}+sr(t+1)}\binom{2m+2}{r}_{F_s(x)}z^r\frac{1}{F_{s(2m+2)}(x)F_{s(2m+1)}(x)}\\
&\times\left(\begin{array}{c}(-1)^{s(r+1)}F_{sr}(x)F_{s(r-1)}(x)+(-1)^sF_{sr}(x)F_{s(2m+2-r)}(x)L_{s(2m+1)}(x)\\
+(-1)^{s(r+1)}F_{s(2m+2-r)}(x)F_{s(2m+1-r)}(x)\end{array}\right)\\
=\;&(-1)^{s(m+2)+1}\sum_{r=0}^{2m+2}(-1)^{\frac{(sr+2(s+1))(r+1)}{2}+sr(t+1)}\binom{2m+2}{r}_{F_s(x)}z^r,
\end{aligned}
$$

as wanted. In the last step we used that

$$
\begin{aligned}
&(-1)^{s(r+1)}F_{sr}(x)F_{s(r-1)}(x)+(-1)^sF_{sr}(x)F_{s(2m+2-r)}(x)L_{s(2m+1)}(x)\\
&+(-1)^{s(r+1)}F_{s(2m+2-r)}(x)F_{s(2m+1-r)}(x)\\
=\;&(-1)^sF_{s(2m+2)}(x)F_{s(2m+1)}(x),
\end{aligned}
\tag{11}
$$

(see appendix for the proof). □

The following identities (factorizations of signed sums of s-polyfibonomials) can be obtained as corollaries of (10)

$$\sum_{r=0}^{2m}(-1)^{\frac{r(r-1+2s+2t+2rs)}{2}}\binom{2m}{r}_{F_{2s-1}(x)}=(-1)^{mt}\prod_{p=1}^{m}L_{(2s-1)(4p-2m-1)}(x). \tag{12}$$

$$\sum_{r=0}^{2m}(-1)^r\binom{2m}{r}_{F_{4s}(x)}=\left(-x^2-4\right)^m\prod_{p=1}^{m}F^2_{2s(4p-2m-1)}(x). \tag{13}$$

$$\sum_{r=0}^{2m}\binom{2m}{r}_{F_{4s}(x)}=\prod_{p=1}^{m}L^2_{2s(4p-2m-1)}(x). \tag{14}$$

We continue, in propositions (2), (3) and (4), with other algebraic properties of certain z-polynomials, that will be used in the remaining results of this section (propositions (5) to (8)).

Proposition 2. *For $t, m \in \mathbb{N}'$ and $k, l \in \mathbb{Z}$, we have that*

$$\begin{aligned}
& \sum_{r=0}^{2m+1} (-1)^{\frac{(sr+2(s+1))(r+1)}{2}+sr(t+1)} \binom{2m+1}{r}_{F_s(x)} H_{s(2m-r+k)+l}(x)\, z^r \\
= \; & (-1)^{sm+1} \left((-1)^{st+1} H_{s(k-1)+l}(x)\, z + (-1)^s H_{s(2m+k)+l}(x) \right) \\
& \times \prod_{p=1}^{m} \left(z^2 + (-1)^{st+1} L_{s(4p-2m-1)}(x)\, z + (-1)^s \right).
\end{aligned} \tag{15}$$

Proof. According to (10) it is sufficient to prove that

$$\begin{aligned}
& \sum_{r=0}^{2m+1} (-1)^{\frac{(sr+2(s+1))(r+1)}{2}+sr(t+1)} \binom{2m+1}{r}_{F_s(x)} H_{s(2m-r+k)+l}(x)\, z^r \\
= \; & \left((-1)^{s(t+1)+1} H_{s(k-1)+l}(x)\, z + H_{s(2m+k)+l}(x) \right) \\
& \times \sum_{r=0}^{2m} (-1)^{\frac{(sr+2(s+1))(r+1)}{2}+sr(t+1)} \binom{2m}{r}_{F_s(x)} z^r.
\end{aligned} \tag{16}$$

We have

$$\begin{aligned}
& \left((-1)^{s(t+1)+1} H_{s(k-1)+l}(x)\, z + H_{s(2m+k)+l}(x) \right) \\
& \times \sum_{r=0}^{2m} (-1)^{\frac{(sr+2(s+1))(r+1)}{2}+sr(t+1)} \binom{2m}{r}_{F_s(x)} z^r \\
= \; & H_{s(k-1)+l}(x) \sum_{r=1}^{2m+1} (-1)^{\frac{(s(r-1)+2(s+1))r}{2}+rs(t+1)+1} \binom{2m}{r-1}_{F_s(x)} z^r \\
& + H_{s(2m+k)+l}(x) \sum_{r=0}^{2m} (-1)^{\frac{(sr+2(s+1))(r+1)}{2}+sr(t+1)} \binom{2m}{r}_{F_s(x)} z^r \\
= \; & \sum_{r=0}^{2m+1} (-1)^{\frac{(sr+2(s+1))(r+1)}{2}+sr(t+1)} \binom{2m+1}{r}_{F_s(x)} \\
& \times \frac{1}{F_{s(2m+1)}(x)} \left((-1)^{s(r+1)} H_{s(k-1)+l}(x)\, F_{sr}(x) + H_{s(2m+k)+l}(x)\, F_{s(2m+1-r)}(x) \right) z^r
\end{aligned}$$

But

$$\begin{aligned}
& (-1)^{s(r+1)} H_{s(k-1)+l}(x)\, F_{sr}(x) + H_{s(2m+k)+l}(x)\, F_{s(2m+1-r)}(x) \\
= \; & F_{s(2m+1)}(x)\, H_{s(2m-r+k)+l}(x),
\end{aligned} \tag{17}$$

(see appendix). Then (16) follows. ∎

Proposition 3. *For given $t, m \in \mathbb{N}$ and $k, l \in \mathbb{Z}$ we have*

(a)

$$\begin{aligned}
&\left(\begin{array}{c}(-1)^{s+1} H_{s(4m+k)+l}(x)\, z \\ +(-1)^{st} H_{s(2m+k)+l}(x)\end{array}\right) \sum_{r=0}^{2m-1} (-1)^{\frac{(sr+2(s+1))(r+1)}{2}+st(r+1)} \binom{2m-1}{r}_{F_s(x)} z^r \\
= & \sum_{r=0}^{2m} (-1)^{\frac{(sr+2(s+1))(r+1)}{2}+sr(t+1)} \binom{2m}{r}_{F_s(x)} H_{s(2m+r+k)+l}(x)\, z^r. \qquad (18)
\end{aligned}$$

(b)

$$\begin{aligned}
&\left(z^2-(-1)^{s(t+1)} L_{2sm}(x)\, z+1\right) \sum_{r=0}^{2m-1} (-1)^{\frac{(sr+2(s+1))(r+1)}{2}+st(r+1)} \binom{2m-1}{r}_{F_s(x)} z^r \\
= & \sum_{r=0}^{2m+1} (-1)^{\frac{(sr+2(s+1))(r+1)}{2}+st(r+1)+sr} \binom{2m+1}{r}_{F_s(x)} z^r. \qquad (19)
\end{aligned}$$

Proof. (a) We have that

$$\begin{aligned}
&\left(\begin{array}{c}(-1)^{s+1} H_{s(4m+k)+l}(x)\, z \\ +(-1)^{st} H_{s(2m+k)+l}(x)\end{array}\right) \sum_{r=0}^{2m-1} (-1)^{\frac{(sr+2(s+1))(r+1)}{2}+st(r+1)} \binom{2m-1}{r}_{F_s(x)} z^r \\
= & \sum_{r=1}^{2m} (-1)^{\frac{(s(r-1)+2(s+1))r}{2}+str+s+1} \binom{2m-1}{r-1}_{F_s(x)} H_{s(4m+k)+l}(x)\, z^r \\
& + \sum_{r=0}^{2m-1} (-1)^{\frac{(sr+2(s+1))(r+1)}{2}+str} \binom{2m-1}{r}_{F_s(x)} H_{s(2m+k)+l}(x)\, z^r \\
= & \sum_{r=0}^{2m} (-1)^{\frac{(sr+2(s+1))(r+1)}{2}+sr(t+1)} \binom{2m}{r}_{F_s(x)} \frac{1}{F_{2sm}(x)} \\
& \times \left(\begin{array}{c} H_{s(4m+k)+l}(x)\, F_{sr}(x) \\ +(-1)^{sr} H_{s(2m+k)+l}(x)\, F_{s(2m-r)}(x)\end{array}\right) z^r.
\end{aligned}$$

But

$$\begin{aligned}
& H_{s(4m+k)+l}(x)\, F_{sr}(x) + (-1)^{sr} H_{s(2m+k)+l}(x)\, F_{s(2m-r)}(x) \qquad (20) \\
= & F_{2sm}(x)\, H_{s(2m+r+k)+l}(x),
\end{aligned}$$

(see appendix). Then (18) follows.

(b) We have that

$$\begin{aligned}
&\left(z^{2}-(-1)^{s(t+1)}L_{2sm}(x)\,z+1\right)\sum_{r=0}^{2m-1}(-1)^{\frac{(sr+2(s+1))(r+1)}{2}+st(r+1)}\binom{2m-1}{r}_{F_s(x)}z^{r}\\
=&\sum_{r=2}^{2m+1}(-1)^{\frac{(s(r-2)+2(s+1))(r-1)}{2}+st(r-1)}\binom{2m-1}{r-2}_{F_s(x)}z^{r}\\
&-\sum_{r=1}^{2m}(-1)^{\frac{(s(r-1)+2(s+1))r}{2}+str+s(t+1)}\binom{2m-1}{r-1}_{F_s(x)}L_{2sm}(x)\,z^{r}\\
&+\sum_{r=0}^{2m-1}(-1)^{\frac{(sr+2(s+1))(r+1)}{2}+st(r+1)}\binom{2m-1}{r}_{F_s(x)}z^{r}\\
=&\sum_{r=0}^{2m+1}(-1)^{\frac{(sr+2(s+1))(r+1)}{2}+st(r+1)+sr}\binom{2m+1}{r}_{F_s(x)}\\
&\times\frac{1}{F_{s(2m+1)}(x)\,F_{2sm}(x)}\begin{pmatrix}(-1)^{s(r+1)}F_{sr}(x)\,F_{s(r-1)}(x)\\+L_{2sm}(x)\,F_{sr}(x)\,F_{s(2m+1-r)}(x)\\+(-1)^{rs}F_{s(2m+1-r)}(x)\,F_{s(2m-r)}(x)\end{pmatrix}z^{r}.
\end{aligned}$$

Now we use the identity

$$\begin{aligned}
&(-1)^{s(r+1)}F_{sr}(x)\,F_{s(r-1)}(x)+L_{2sm}(x)\,F_{sr}(x)\,F_{s(2m+1-r)}(x)\\
&+(-1)^{rs}F_{s(2m+1-r)}(x)\,F_{s(2m-r)}(x)\\
=&\ F_{s(2m+1)}(x)\,F_{2sm}(x),
\end{aligned}\tag{21}$$

(see appendix) to obtain (19). ∎

Proposition 4. *For given $t,m\in\mathbb{N}$ and $k,l\in\mathbb{Z}$ we have that*

$$\begin{aligned}
&\sum_{r=0}^{2m}(-1)^{\frac{(sr+2(s+1))(r+1)}{2}+str}\binom{2m}{r}_{F_s(x)}H_{s(2m+r+k)+l}(x)\,z^{r}\\
=&\ (-1)^{sm+s+1}\left(z-(-1)^{s(t+m)}\right)\left(H_{s(4m+k)+l}(x)\,z-(-1)^{st}H_{s(2m+k)+l}(x)\right)\\
&\times\prod_{p=1}^{m-1}\left(z^{2}-(-1)^{s(p+m+t)}L_{2sp}(x)\,z+1\right).
\end{aligned}\tag{22}$$

Proof. We proceed by induction on m. To see that (22) is true for $m=1$ we use the identity

$$H_{s(2+k)+l}(x)+(-1)^{s}H_{s(4+k)+l}(x)=(-1)^{s}L_{s}(x)\,H_{s(3+k)+l}(x),$$

(easy to check). Let us suppose the result is valid for a given $m\in\mathbb{N}$, and let us see it is also valid for $m+1$. We begin by writing

$$
\begin{aligned}
& (-1)^{s(m+1)+s+1}\left(z-(-1)^{s(t+m+1)}\right)\left(H_{s(4m+4+k)+l}(x)\,z-(-1)^{st}H_{s(2m+2+k)+l}(x)\right)\\
& \times\prod_{p=1}^{m}\left(z^{2}-(-1)^{s(p+m+1+t)}L_{2sp}(x)\,z+1\right)\\
= \;& (-1)^{s}\frac{H_{s(4m+4+k)+l}(x)\,z-(-1)^{st}H_{s(2m+2+k)+l}(x)}{H_{s(4m+k)+l}(x)\,z-(-1)^{s(t+1)}H_{s(2m+k)+l}(x)}\left(z^{2}-(-1)^{s(t+1)}L_{2sm}(x)\,z+1\right)\\
& \times(-1)^{sm+s+1}\left(z-(-1)^{s(t+m+1)}\right)\left(H_{s(4m+k)+l}(x)\,z-(-1)^{s(t+1)}H_{s(2m+k)+l}(x)\right)\\
& \times\prod_{p=1}^{m-1}\left(z^{2}-(-1)^{s(p+m+1+t)}L_{2sp}(x)\,z+1\right)\\
= \;& -\frac{H_{s(4m+4+k)+l}(x)\,z-(-1)^{st}H_{s(2m+2+k)+l}(x)}{(-1)^{s+1}H_{s(4m+k)+l}(x)\,z+(-1)^{st}H_{s(2m+k)+l}(x)}\left(z^{2}-(-1)^{s(t+1)}L_{2sm}(x)\,z+1\right)\\
& \times\sum_{r=0}^{2m}(-1)^{\frac{(sr+2(s+1))(r+1)}{2}+s(t+1)r}\binom{2m}{r}_{F_s(x)}H_{s(2m+r+k)+l}(x)\,z^{r}.
\end{aligned}
$$

By using proposition (3), first (a), then (b), and finally again (a) (this second time with m replaced by $m+1$ and t replaced by $t+1$), we obtain that

$$
\begin{aligned}
& -\frac{H_{s(4m+4+k)+l}(x)\,z-(-1)^{st}H_{s(2m+2+k)+l}(x)}{(-1)^{s+1}H_{s(4m+k)+l}(x)\,z+(-1)^{st}H_{s(2m+k)+l}(x)}\left(z^{2}-(-1)^{s(t+1)}L_{2sm}(x)\,z+1\right)\\
& \times\sum_{r=0}^{2m}(-1)^{\frac{(sr+2(s+1))(r+1)}{2}+s(t+1)r}\binom{2m}{r}_{F_s(x)}H_{s(2m+r+k)+l}(x)\,z^{r}\\
= \;& -\left(H_{s(4m+4+k)+l}(x)\,z-(-1)^{st}H_{s(2m+2+k)+l}(x)\right)\left(z^{2}-(-1)^{s(t+1)}L_{2sm}(x)\,z+1\right)\\
& \times\sum_{r=0}^{2m-1}(-1)^{\frac{(sr+2(s+1))(r+1)}{2}+st(r+1)}\binom{2m-1}{r}_{F_s(x)}z^{r}\\
= \;& -\left(H_{s(4m+4+k)+l}(x)\,z-(-1)^{st}H_{s(2m+2+k)+l}(x)\right)\\
& \times\sum_{r=0}^{2m+1}(-1)^{\frac{(sr+2(s+1))(r+1)}{2}+st(r+1)+sr}\binom{2m+1}{r}_{F_s(x)}z^{r}\\
= \;& \sum_{r=0}^{2m+2}(-1)^{\frac{(sr+2(s+1))(r+1)}{2}+srt}\binom{2m+2}{r}_{F_s(x)}H_{s(2m+r+k)+l}(x)\,z^{r},
\end{aligned}
$$

as wanted. $\square$

The following identity is a corollary from (22):

$$
\sum_{r=0}^{2m}\binom{2m}{r}_{F_{2s}(x)}H_{2s(2m+r+k)+l}(x)=2\left(H_{2s(4m+k)+l}(x)+H_{2s(2m+k)+l}(x)\right)\prod_{p=1}^{m-1}L_{2sp}^{2}(x). \tag{23}
$$

Another immediate corollary from (22) is the identity:

$$\sum_{r=0}^{2m-1}(-1)^{\frac{(sr+2(s+1))(r+1)}{2}+sr(m+1)}\binom{2m-1}{r}_{F_s(x)}=0. \tag{24}$$

Observe that (24) is of the same type of (6). In (24) we have a non-trivial distribution of signs in the list of s-polyfibonomials $\binom{2m-1}{r}_{F_s(x)}$, $r=0,1,\ldots,2m-1$, that makes they cancel out in the sum, as happens in (6) (where we have the trivial distribution of signs given by $(-1)^r$).

Proposition 5. *Let $m\in\mathbb{N}$, $k,l\in\mathbb{Z}$, and $t=1,2,\ldots,2m$ be given. For $u=\alpha(x)$ or $u=\beta(x)$ we have*

$$\sum_{r=0}^{2m+1}(-1)^{\frac{(sr+2(s+1))(r+1)}{2}+sr(t+1)}\binom{2m+1}{r}_{F_s(x)}H_{s(2m-r+k)+l}(x)\,u^{sr(2t-2m-1)}=0. \tag{25}$$

Proof. First of all observe that for given $1\leq t\leq 2m$ there exists p, $1\leq p\leq m$, such that

$$u^{2s(2t-2m-1)}+(-1)^{st+1}L_{s(4p-2m-1)}(x)\,u^{s(2t-2m-1)}+(-1)^s=0, \tag{26}$$

where $u=\alpha(x)$ or $u=\beta(x)$. Indeed, observe we can write (26) as

$$L_{s(4p-2m-1)}(x)=(-1)^{st}L_{s(2t-2m-1)}(x).$$

Thus, we see that if $t=2k$, $k=1,2,\ldots,m$, we can take $p=k$, and if $t=2k-1$, $k=1,2,\ldots,m$, we can take $p=m-k+1$. This fact, together with (15) give us the desired conclusion. □

Proposition 6. *Let $m\in\mathbb{N}$, $k,l\in\mathbb{Z}$, and $t=1,2,\ldots,2m-1$ be given. For $u=\alpha(x)$ or $u=\beta(x)$ we have*

$$\sum_{r=0}^{2m}(-1)^{\frac{(sr+2(s+1))(r+1)}{2}+str}\binom{2m}{r}_{F_s(x)}H_{s(2m+r+k)+l}(x)\,u^{sr(2t-2m)}=0. \tag{27}$$

Proof. We begin our proof by claiming that for given $1\leq t\leq 2m-1$, $t\neq m$, there exists p, $1\leq p\leq m-1$, such that

$$u^{2s(2t-2m)}-(-1)^{s(p+m+t)}L_{2sp}(x)\,u^{s(2t-2m)}+1=0. \tag{28}$$

In fact, observe that we can write (28) as

$$L_{2sp}(x)=(-1)^{s(p+m+t)}L_{s(2t-2m)}(x).$$

Thus, if $t=k$ or $t=2m-k$, $k=1,2,\ldots,m-1$, take $p=m-k$ to see our claim is true. On the other hand observe that the case $m=1$ of (27) is the identity

$$(-1)^{s+1}H_{s(2+k)+l}(x)+L_s(x)\,H_{s(3+k)+l}(x)-H_{s(4+k)+l}(x)=0,$$

which can be verified easily. So let us take $m\geq 2$. Consider (22) with z replaced by $u^{s(2t-2m)}$. Observe that if $t=m$ the conclusion follows from the fact that $u^{s(2t-2m)}-(-1)^{s(t+m)}=0$. For the remaining cases ($t\neq m$) use the previous claim to obtain the desired conclusion. □

Proposition 7. *Let $m \in \mathbb{N}'$ and $k, l \in \mathbb{Z}$ be given. For $u = \alpha(x)$ or $u = \beta(x)$ we have the following identity*

$$\begin{aligned} & \sum_{r=0}^{2m+1} (-1)^{\frac{(sr+2(s+1))(r+1)}{2}} \binom{2m+1}{r}_{F_s(x)} H_{s(2m-r+k)+l}(x)\, u^{sr(2m+1)} \\ = \; & (-1)^{s(k+m)+l+1} \left(x^2+4\right)^m (F_{2m+1}(x)!)_s \left(H_1(x) - H_0(x)\, u\right) u^{s(m(2m+1)-k+1)-l}. \end{aligned} \tag{29}$$

Proof. By using (15) with $t = 1$ and $z = u^{s(2m+1)}$ we can write

$$\begin{aligned} & \sum_{r=0}^{2m+1} (-1)^{\frac{(sr+2(s+1))(r+1)}{2}} \binom{2m+1}{r}_{F_s(x)} H_{s(2m-r+k)+l}(x)\, u^{sr(2m+1)} \\ = \; & (-1)^{sm+1} \left((-1)^{s+1} H_{s(k-1)+l}(x)\, u^{s(2m+1)} + (-1)^s H_{s(2m+k)+l}(x)\right) \\ & \times \prod_{p=1}^{m} \left(u^{2s(2m+1)} + (-1)^{s+1} L_{s(4p-2m-1)}(x)\, u^{s(2m+1)} + (-1)^s\right). \end{aligned}$$

Use the identities (straightforward proofs)

$$\begin{aligned} & u^{2s(2m+1)} + (-1)^{s+1} L_{s(4p-2m-1)}(x)\, u^{s(2m+1)} + (-1)^s \\ = \; & \left(x^2+4\right) F_{2sp}(x)\, F_{s(2m-2p+1)}(x)\, u^{s(2m+1)}, \end{aligned}$$

and

$$\begin{aligned} & (-1)^{sk+l+s+1} F_{s(2m+1)}(x) \left(H_1(x) - H_0(x)\, u\right) u^{s(-k+1)-l} \\ = \; & H_{s(k-1)+l}(x)\, u^{s(2m+1)} - H_{s(2m+k)+l}(x), \end{aligned}$$

to obtain

$$\begin{aligned} & (-1)^{sm+1} \sum_{r=0}^{2m+1} (-1)^{\frac{(sr+2(s+1))(r+1)}{2}} \binom{2m+1}{r}_{F_s(x)} H_{s(2m-r+k)+l}(x)\, u^{sr(2m+1)} \\ = \; & (-1)^{sm+1} (-1)^{sk+l} F_{s(2m+1)}(x) \left(H_1(x) - H_0(x)\, u\right) u^{s(-k+1)-l} \\ & \times \prod_{p=1}^{m} \left(\left(x^2+4\right) F_{2sp}(x)\, F_{s(2m-2p+1)}(x)\, u^{s(2m+1)}\right) \\ = \; & (-1)^{s(k+m)+l+1} F_{s(2m+1)}(x) \left(H_1(x) - H_0(x)\, u\right) u^{s(-k+1)-l} \left(x^2+4\right)^m \\ & \times (F_{2m}(x)!)_s\, u^{sm(2m+1)} \\ = \; & (-1)^{s(k+m)+l+1} \left(x^2+4\right)^m (F_{2m+1}(x)!)_s \left(H_1(x) - H_0(x)\, u\right) u^{s(m(2m+1)-k+1)-l}, \end{aligned}$$

as expected. ∎

Proposition 8. *Let $m \in \mathbb{N}$ be given. For $u = \alpha(x)$ and $u = \beta(x)$ we have*

$$\begin{aligned} & \pm \sum_{r=0}^{2m} (-1)^{\frac{(sr+2(s+1))(r+1)}{2}} \binom{2m}{r}_{F_s(x)} H_{s(2m+r+k)+l}(x)\, u^{2msr} \\ = \; & (-1)^{sm+s+1} \left(x^2+4\right)^{m-\frac{1}{2}} (F_{2m}(x)!)_s \left(H_1(x) - H_0(x)\, \overline{u}\right) u^{s\left(2m^2+3m+k\right)+l}, \end{aligned} \tag{30}$$

where the plus sign of the left-hand side corresponds to $u=\alpha(x)$ and $\overline{u}=\beta(x)$, and the minus sign corresponds to $u=\beta(x)$ and $\overline{u}=\alpha(x)$.

Proof. We use (22) with $t=2$ and $z=u^{2ms}$ to write

$$\begin{aligned}
& \sum_{r=0}^{2m}(-1)^{\frac{(sr+2(s+1))(r+1)}{2}}\binom{2m}{r}_{F_s(x)} H_{s(2m+r+k)+l}(x)\, u^{2msr} \\
= \ & (-1)^{sm+s+1}\left(u^{2ms}-(-1)^{sm}\right)\left(H_{s(4m+k)+l}(x)\, u^{2ms}-H_{s(2m+k)+l}(x)\right) \\
& \times \prod_{p=1}^{m-1}\left(u^{4ms}-(-1)^{s(p+m)} L_{2sp}(x)\, u^{2ms}+1\right).
\end{aligned}$$

Use now the identities (straightforward proofs)

$$\begin{aligned}
u^{4ms}-(-1)^{s(p+m)} L_{2sp}(x)\, u^{2ms}+1 &= \left(x^2+4\right) u^{2sm} F_{s(m+p)}(x)\, F_{s(m-p)}(x), \\
H_{s(4m+k)+l}(x)\, u^{2ms}-H_{s(2m+k)+l}(x) &= \left(H_1(x)-H_0(x)\,\overline{u}\right) F_{2sm}(x)\, u^{s(4m+k)+l}, \\
u^{2ms}-(-1)^{sm} &= \pm\left(x^2+4\right)^{\frac{1}{2}} F_{sm}(x)\, u^{sm},
\end{aligned}$$

(in the last identity we have $+$ if $u=\alpha(x)$ and $-$ if $u=\beta(x)$) to obtain

$$\begin{aligned}
& \pm\sum_{r=0}^{2m}(-1)^{\frac{(sr+2(s+1))(r+1)}{2}}\binom{2m}{r}_{F_s(x)} H_{s(2m+r+k)+l}(x)\, u^{2msr} \\
= \ & (-1)^{sm+s+1}\left(x^2+4\right)^{\frac{1}{2}} F_{sm}(x)\, u^{sm}\left(H_1(x)-H_0(x)\,\overline{u}\right) F_{2sm}(x)\, u^{s(4m+k)+l} \\
& \times \prod_{p=1}^{m-1}\left(\left(x^2+4\right) u^{2sm} F_{s(m+p)}(x)\, F_{s(m-p)}(x)\right) \\
= \ & (-1)^{sm+s+1}\left(x^2+4\right)^{m-\frac{1}{2}}\left(F_{2m}(x)!\right)_s\left(H_1(x)-H_0(x)\,\overline{u}\right) u^{s\left(2m^2+3m+k\right)+l},
\end{aligned}$$

as wanted. □

3 The main results

Let $m\in\mathbb{N}$, $p_1,\ldots,p_m\in\mathbb{Z}$ be given. For every $t=0,1,\ldots,m$, let $J_{t,m}$ be the family of subsets of $\{1,2,\ldots,m\}$ containing t elements. (Thus $J_{t,m}$ contains $\binom{m}{t}$ subsets.) For each subset A_t belonging to $J_{t,m}$, define p_{A_t} as $p_{A_t}=\sum_{\omega\in A_t} p_\omega$.

Let us consider the s-Gibonacci polynomial sequences

$$G_{s(n+p_i)}(x)=\left(x^2+4\right)^{-\frac{1}{2}}\left(c_1(x)\,\alpha^{s(n+p_i)}(x)-c_2(x)\,\beta^{s(n+p_i)}(x)\right),$$

where $i=1,2,\ldots,M$, and

$$\begin{aligned}
c_1(x) &= G_1(x)-G_0(x)\,\beta(x), \\
c_2(x) &= G_1(x)-G_0(x)\,\alpha(x).
\end{aligned}$$

It is not difficult to establish the following formulas for the products $\prod_{i=1}^{2m+1} G_{s(n+p_i)}(x)$ and $\prod_{i=1}^{2m} G_{s(n+p_i)}(x)$

$$\begin{aligned}
&\prod_{i=1}^{2m+1} G_{s(n+p_i)}(x) \qquad (31)\\
=\;& \left(x^2+4\right)^{-m-\frac{1}{2}} \sum_{t=0}^{2m+1} (-1)^{t(sn+1)}\, c_1^{2m+1-t}(x)\, c_2^t(x)\\
&\times \sum_{J_{t,2m+1}} \alpha^{s\left((2m+1-2t)n+p_1+\cdots+p_{2m+1}-p_{A_t}\right)}(x)\, \beta^{sp_{A_t}}(x).
\end{aligned}$$

$$\begin{aligned}
&\prod_{i=1}^{2m} G_{s(n+p_i)}(x) \qquad (32)\\
=\;& \left(x^2+4\right)^{-m} \sum_{t=0}^{2m} (-1)^{t(sn+1)}\, c_1^{2m-t}(x)\, c_2^t(x)\\
&\times \sum_{J_{t,2m}} \alpha^{s\left((2m-2t)n+p_1+\cdots+p_{2m}-p_{A_t}\right)}(x)\, \beta^{sp_{A_t}}(x).
\end{aligned}$$

(The summation $\sum_{J_{t,m}}$ in (31) and (32) is over all the subsets contained in $J_{t,m}$.)

Now we are ready to establish the two main results of this article.

Theorem 9. *Let $m \in \mathbb{N}'$ and $k, l, p_1, \ldots, p_{2m+1} \in \mathbb{Z}$ be given. We have that*

$$\begin{aligned}
&\sum_{r=0}^{2m+1} (-1)^{\frac{(sr+2(s+1))(r+1)}{2}} \binom{2m+1}{r}_{F_s(x)} H_{s(2m-r+k)+l}(x) \prod_{i=1}^{2m+1} G_{s(n+r+p_i)}(x) \qquad (33)\\
=\;& (-1)^{s(m+k)+l+1} \left(x^2+4\right)^{-\frac{1}{2}} \left(F_{2m+1}(x)!\right)_s\\
&\times \left(\begin{array}{c}
\left(H_1(x)-H_0(x)\,\alpha(x)\right)\left(G_1(x)-G_0(x)\,\beta(x)\right)^{2m+1}\\
\times \alpha^{s((2m+1)(n+m)+p_1+\cdots+p_{2m+1}-k+1)-l}(x)\\
-\left(H_1(x)-H_0(x)\,\beta(x)\right)\left(G_1(x)-G_0(x)\,\alpha(x)\right)^{2m+1}\\
\times \beta^{s((2m+1)(n+m)+p_1+\cdots+p_{2m+1}-k+1)-l}(x)
\end{array}\right).
\end{aligned}$$

Proof. After inserting (31) in the left-hand side of (33), we obtain an expression of the form

$$\begin{aligned}
&\sum_{r=0}^{2m+1} (-1)^{\frac{(sr+2(s+1))(r+1)}{2}} \binom{2m+1}{r}_{F_s(x)} H_{s(2m-r+k)+l}(x) \prod_{i=1}^{2m+1} G_{s(n+r+p_i)}(x) \qquad (34)\\
=\;& \sum_{t=0}^{2m+1} \Psi(x,t,m,s),
\end{aligned}$$

where

$$\begin{aligned}
&\Psi\left(x,t,m,s\right) &&(35)\\
= \ &\left(x^{2}+4\right)^{-m-\frac{1}{2}}\left(-1\right)^{t(sn+1)}c_{1}^{2m+1-t}\left(x\right)c_{2}^{t}\left(x\right)\\
&\times\sum_{J_{t,2m+1}}\alpha^{s\left((2m+1-2t)n-p_{A_t}+p_{1}+\cdots+p_{2m+1}\right)}\left(x\right)\beta^{sp_{A_t}}\left(x\right)\\
&\times\sum_{r=0}^{2m+1}\left(-1\right)^{\frac{(sr+2(s+1))(r+1)}{2}+tsr}\binom{2m+1}{r}_{F_s(x)}H_{s(2m-r+k)+l}\left(x\right)\alpha^{sr(2m+1-2t)}\left(x\right),
\end{aligned}$$

(with $c_1(x) = G_1(x) - G_0(x)\beta(x)$ and $c_2(x) = G_1(x) - G_0(x)\alpha(x)$). By writing the right-hand side of (34) as

$$\sum_{t=0}^{2m+1}\Psi\left(x,t,m,s\right)=\Psi\left(x,0,m,s\right)+\Psi\left(x,2m+1,m,s\right)+\sum_{t=1}^{2m}\Psi\left(x,t,m,s\right),$$

the proof ends if we show that

$$\sum_{t=1}^{2m}\Psi\left(x,t,m,s\right)=0,$$

and that the remaining sum

$$\Psi\left(x,0,m,s\right)+\Psi\left(x,2m+1,m,s\right),$$

is equal to the right-hand side of (33).

According to proposition 5, for $t = 1, 2, \ldots, 2m$ we have

$$\begin{aligned}
&\sum_{r=0}^{2m+1}\left(-1\right)^{\frac{(sr+2(s+1))(r+1)}{2}+tsr}\binom{2m+1}{r}_{F_s(x)}H_{s(2m-r+k)+l}\left(x\right)\alpha^{sr(2m+1-2t)}\left(x\right)\\
= \ &\sum_{r=0}^{2m+1}\left(-1\right)^{\frac{(sr+2(s+1))(r+1)}{2}+sr(t+1)}\binom{2m+1}{r}_{F_s(x)}H_{s(2m-r+k)+l}\left(x\right)\beta^{sr(2t-2m-1)}\left(x\right)\\
= \ &0,
\end{aligned}$$

and then $\sum_{t=1}^{2m}\Psi(x,t,m,s)=0$. On the other hand, by using proposition 7 we have

$$\begin{aligned}
&\Psi(x,0,m,s)+\Psi(x,2m+1,m,s)\\
=\;&\left(x^2+4\right)^{-m-\frac{1}{2}}c_1^{2m+1}(x)\,\alpha^{s((2m+1)n+p_1+\cdots+p_{2m+1})}(x)\\
&\times\sum_{r=0}^{2m+1}(-1)^{\frac{(sr+2(s+1))(r+1)}{2}}\binom{2m+1}{r}_{F_s(x)}H_{s(2m-r+k)+l}(x)\,\alpha^{sr(2m+1)}(x)\\
&-\left(x^2+4\right)^{-m-\frac{1}{2}}c_2^{2m+1}(x)\,\beta^{s((2m+1)n+p_1+\cdots+p_{2m+1})}(x)\\
&\times\sum_{r=0}^{2m+1}(-1)^{\frac{(sr+2(s+1))(r+1)}{2}}\binom{2m+1}{r}_{F_s(x)}H_{s(2m-r+k)+l}(x)\,\beta^{sr(2m+1)}(x)\\
=\;&\left(x^2+4\right)^{-m-\frac{1}{2}}c_1^{2m+1}(x)\,\alpha^{s((2m+1)n+p_1+\cdots+p_{2m+1})}(x)\,(-1)^{s(k+m)+l+1}\left(x^2+4\right)^{m}\\
&\times\left(F_{2m+1}(x)!\right)_s\left(H_1(x)-H_0(x)\,\alpha(x)\right)\alpha^{s(m(2m+1)+(-k+1))-l}(x)\\
&-\left(x^2+4\right)^{-m-\frac{1}{2}}c_2^{2m+1}(x)\,\beta^{s((2m+1)n+p_1+\cdots+p_{2m+1})}(x)\,(-1)^{s(k+m)+l+1}\left(x^2+4\right)^{m}\\
&\times\left(F_{2m+1}(x)!\right)_s\left(H_1(x)-H_0(x)\,\beta(x)\right)\beta^{s(m(2m+1)+(-k+1))-l}(x)\\
=\;&(-1)^{s(m+k)+l+1}\left(x^2+4\right)^{-\frac{1}{2}}\left(F_{2m+1}(x)!\right)_s\\
&\times\begin{pmatrix}\left(H_1(x)-H_0(x)\,\alpha(x)\right)c_1^{2m+1}(x)\,\alpha^{s((2m+1)(n+m)+p_1+\cdots+p_{2m+1}-k+1)-l}(x)\\ -\left(H_1(x)-H_0(x)\,\beta(x)\right)c_2^{2m+1}(x)\,\beta^{s((2m+1)(n+m)+p_1+\cdots+p_{2m+1}-k+1)-l}(x)\end{pmatrix},
\end{aligned}$$

as wanted. □

Theorem 10. *Let $m\in\mathbb{N}$ and $k,l,p_1,\ldots,p_{2m+1}\in\mathbb{Z}$ be given. We have that*

$$\begin{aligned}
&\sum_{r=0}^{2m}(-1)^{\frac{(sr+2(s+1))(r+1)}{2}}\binom{2m}{r}_{F_s(x)}H_{s(2m+r+k)+l}(x)\prod_{i=1}^{2m}G_{s(n+r+p_i)}(x) \qquad (36)\\
=\;&(-1)^{sm+s+1}\left(x^2+4\right)^{-\frac{1}{2}}\left(F_{2m}(x)!\right)_s\\
&\times\begin{pmatrix}\left(H_1(x)-H_0(x)\,\beta(x)\right)\left(G_1(x)-G_0(x)\,\beta(x)\right)^{2m}\\ \times\alpha^{s(2m(m+n)+p_1+\cdots+p_{2m}+3m+k)+l}(x)\\ -\left(H_1(x)-H_0(x)\,\alpha(x)\right)\left(G_1(x)-G_0(x)\,\alpha(x)\right)^{2m}\\ \times\beta^{s(2m(m+n)+p_1+\cdots+p_{2m}+3m+k)+l}(x)\end{pmatrix}.
\end{aligned}$$

Proof. We follow the same strategy of the proof of theorem 9. Substituting (32) in the left-hand side of (36) we can write

$$\begin{aligned}
&\sum_{r=0}^{2m}(-1)^{\frac{(sr+2(s+1))(r+1)}{2}}\binom{2m}{r}_{F_s(x)}H_{s(2m+r+k)+l}(x)\prod_{i=1}^{2m}G_{s(n+r+p_i)}(x)\\
=\;&\Phi(x,0,m,s)+\Phi(x,2m,m,s)+\sum_{t=1}^{2m-1}\Phi(x,t,m,s),
\end{aligned}$$

where

$$\begin{aligned}\Phi(x,t,m,s) &= \left(x^2+4\right)^{-m}(-1)^{t(sn+1)}c_1^{2m-t}(x)\,c_2^t(x)\\ &\times\sum_{J_{t,2m}}\alpha^{s\left((2m-2t)n+p_1+\cdots+p_{2m}-p_{A_t}\right)}(x)\,\beta^{sp_{A_t}}(x)\\ &\times\sum_{r=0}^{2m}(-1)^{\frac{(sr+2(s+1))(r+1)}{2}}\binom{2m}{r}_{F_s(x)}H_{s(2m+r+k)+l}(x)\,\alpha^{sr(2m-2t)}(x),\end{aligned}$$

(with $c_1(x)=G_1(x)-G_0(x)\beta(x)$ and $c_2(x)=G_1(x)-G_0(x)\alpha(x)$). According to proposition 6, for $t=1,2,\ldots,2m-1$ we have

$$\begin{aligned}&\sum_{r=0}^{2m}(-1)^{\frac{(sr+2(s+1))(r+1)}{2}+tsr}\binom{2m}{r}_{F_s(x)}H_{s(2m+r+k)+l}(x)\,\alpha^{sr(2m-2t)}(x)\\ =&\sum_{r=0}^{2m}(-1)^{\frac{(sr+2(s+1))(r+1)}{2}+str}\binom{2m}{r}_{F_s(x)}H_{s(2m+r+k)+l}(x)\,\alpha^{sr(2t-2m)}(x)\\ =&\ 0,\end{aligned}$$

and then

$$\sum_{t=1}^{2m-1}\Phi(x,t,m,s)=0.$$

On the other hand, by using proposition 8 we get

$$\begin{aligned}&\Phi(x,0,m,s)+\Phi(x,2m,m,s)\\ =&\left(x^2+4\right)^{-m}c_1^{2m}(x)\,\alpha^{s(2mn+p_1+\cdots+p_{2m})}(x)\\ &\times\sum_{r=0}^{2m}(-1)^{\frac{(sr+2(s+1))(r+1)}{2}}\binom{2m}{r}_{F_s(x)}G_{s(2m+r+k)+l}(x)\,\alpha^{2smr}(x)\\ &+\left(x^2+4\right)^{-m}c_2^{2m}(x)\,\beta^{s(2mn+p_1+\cdots+p_{2m})}(x)\\ &\times\sum_{r=0}^{2m}(-1)^{\frac{(sr+2(s+1))(r+1)}{2}}\binom{2m}{r}_{F_s(x)}G_{s(2m+r+k)+l}(x)\,\beta^{2msr}(x)\\ =&\left(x^2+4\right)^{-m}\alpha^{s(2mn+p_1+\cdots+p_{2m})}(x)\,(-1)^{sm+s+1}\left(x^2+4\right)^{m-\frac{1}{2}}\left(F_{2m}(x)!\right)_s\\ &\times\left(H_1(x)-H_0(x)\beta(x)\right)c_1^{2m}(x)\,\alpha^{s\left(2m^2+3m+k\right)+l}(x)\\ &-\left(x^2+4\right)^{-m}\beta^{s(2mn+p_1+\cdots+p_{2m})}(x)\,(-1)^{sm+s+1}\left(x^2+4\right)^{m-\frac{1}{2}}\left(F_{2m}(x)!\right)_s\\ &\times\left(H_1(x)-H_0(x)\alpha(x)\right)c_2^{2m}(x)\,\beta^{s\left(2m^2+3m+k\right)+l}(x)\\ =&\ (-1)^{sm+s+1}\left(x^2+4\right)^{-\frac{1}{2}}\left(F_{2m}(x)!\right)_s\\ &\times\begin{pmatrix}\left(H_1(x)-H_0(x)\beta(x)\right)c_1^{2m}(x)\,\alpha^{s(2m(m+n)+p_1+\cdots+p_{2m}+3m+k)+l}(x)\\ -\left(H_1(x)-H_0(x)\alpha(x)\right)c_2^{2m}(x)\,\beta^{s(2m(m+n)+p_1+\cdots+p_{2m}+3m+k)+l}(x)\end{pmatrix}\end{aligned}$$

as wanted. $\square$

4 Some examples

If in (33) we take $H_{sn}(x) = cL_{sn}(x) + dF_{sn}(x)$ and $G_{sn}(x) = aL_{sn}(x) + bF_{sn}(x)$ (with $a, b, c, d \in \mathbb{R}$), we obtain

$$\begin{aligned}
&\sum_{r=0}^{2m+1} (-1)^{\frac{(sr+2(s+1))(r+1)}{2}} \binom{2m+1}{r}_{F_s(x)} \left(cL_{s(2m-r+k)+l}(x) + dF_{s(2m-r+k)+l}(x)\right) \\
&\times \prod_{i=1}^{2m+1} \left(aL_{s(n+r+p_i)}(x) + bF_{s(n+r+p_i)}(x)\right) \\
= \ & (-1)^{s(m+k)+l+1} \left(x^2+4\right)^{-\frac{1}{2}} \left(F_{2m+1}(x)!\right)_s \\
&\times \left(\begin{array}{c} \left(d - c\sqrt{x^2+4}\right)\left(b + a\sqrt{x^2+4}\right)^{2m+1} \alpha^{s((2m+1)(n+m)+p_1+\cdots+p_{2m+1}-k+1)-l}(x) \\ -\left(d + c\sqrt{x^2+4}\right)\left(b - a\sqrt{x^2+4}\right)^{2m+1} \beta^{s((2m+1)(n+m)+p_1+\cdots+p_{2m+1}-k+1)-l}(x) \end{array} \right).
\end{aligned}$$

By expanding the binomials $\left(b \pm a\sqrt{x^2+4}\right)^{2m+1}$ we can write this expression as

$$\begin{aligned}
&\sum_{r=0}^{2m+1} (-1)^{\frac{(sr+2(s+1))(r+1)}{2}} \binom{2m+1}{r}_{F_s(x)} H_{s(2m-r+k)+l}(x) \prod_{i=1}^{2m+1} G_{s(n+r+p_i)}(x) \\
= \ & (-1)^{s(m+k)+l+1} \left(F_{2m+1}(x)!\right)_s \\
&\times \sum_{j=0}^{m} a^{2j} \left(x^2+4\right)^j b^{2m-2j} \\
&\times \left(\begin{array}{c} \alpha^{s((2m+1)(n+m)+p_1+\cdots+p_{2m+1}-k+1)-l}(x) \left(\begin{array}{c} \binom{2m+1}{2j} bd\left(x^2+4\right)^{-\frac{1}{2}} + \binom{2m+1}{2j+1} ad \\ -\binom{2m+1}{2j} bc - \binom{2m+1}{2j+1} ac\left(x^2+4\right)^{\frac{1}{2}} \end{array} \right) \\ -\beta^{s((2m+1)(n+m)+p_1+\cdots+p_{2m+1}-k+1)-l}(x) \left(\begin{array}{c} \binom{2m+1}{2j} bd\left(x^2+4\right)^{-\frac{1}{2}} - \binom{2m+1}{2j+1} ad \\ +\binom{2m+1}{2j} bc - \binom{2m+1}{2j+1} ac\left(x^2+4\right)^{\frac{1}{2}} \end{array} \right) \end{array} \right).
\end{aligned}$$

A simple further simplification gives us finally

$$\begin{aligned}
&\sum_{r=0}^{2m+1} (-1)^{\frac{(sr+2(s+1))(r+1)}{2}} \binom{2m+1}{r}_{F_s(x)} \left(cL_{s(2m-r+k)+l}(x) + dF_{s(2m-r+k)+l}(x)\right) \\
&\times \prod_{i=1}^{2m+1} \left(aL_{s(n+r+p_i)}(x) + bF_{s(n+r+p_i)}(x)\right) \\
= \ & (-1)^{s(m+k)+l+1} \left(F_{2m+1}(x)!\right)_s \sum_{j=0}^{m} a^{2j} b^{2m-2j} \left(x^2+4\right)^j \\
&\times \left(\begin{array}{c} \left(\binom{2m+1}{2j+1} ad - \binom{2m+1}{2j} bc\right) L_{s((2m+1)(n+m)+p_1+\cdots+p_{2m+1}-k+1)-l}(x) \\ +\left(\binom{2m+1}{2j} bd - \binom{2m+1}{2j+1} ac\left(x^2+4\right)\right) F_{s((2m+1)(n+m)+p_1+\cdots+p_{2m+1}-k+1)-l}(x) \end{array} \right),
\end{aligned} \tag{37}$$

Similarly, beginning with (36) and taking $H_{sn}(x) = cL_{sn}(x) + dF_{sn}(x)$ and $G_{sn}(x) = aL_{sn}(x) + bF_{sn}(x)$, we can obtain

$$\begin{aligned}
&\sum_{r=0}^{2m} (-1)^{\frac{(sr+2(s+1))(r+1)}{2}} \binom{2m}{r}_{F_s(x)} \left(cL_{s(2m+r+k)+l}(x) + dF_{s(2m+r+k)+l}(x)\right) \\
&\times \prod_{i=1}^{2m} \left(aL_{s(n+r+p_i)}(x) + bF_{s(n+r+p_i)}(x)\right) \\
= \ & (-1)^{sm+s+1} \left(F_{2m}(x)!\right)_s \\
&\times \sum_{j=0}^{m-1} a^{2j} b^{2m-2j-1} \left(x^2+4\right)^j \begin{pmatrix} \left(\binom{2m}{2j+1} ad + \binom{2m}{2j} bc\right) L_{s(2m(m+n)+p_1+\cdots+p_{2m}+3m+k)+l}(x) \\ + \left(\binom{2m}{2j} bd + \binom{2m}{2j+1} ac\right) F_{s(2m(m+n)+p_1+\cdots+p_{2m}+3m+k)+l}(x) \end{pmatrix} \\
&+ a^{2m} \left(x^2+4\right)^m \begin{pmatrix} cL_{s(2m(m+n)+p_1+\cdots+p_{2m}+3m+k)+l}(x) \\ +dF_{s(2m(m+n)+p_1+\cdots+p_{2m}+3m+k)+l}(x) \end{pmatrix}.
\end{aligned} \tag{38}$$

More concretely, if in (37) we set $c = 0$, $p_i = q_i - n - m$, $k = 1$ and $l = 0$, we can write the resulting expression as an addition identity, namely

$$\begin{aligned}
&\sum_{r=0}^{2m} \frac{(-1)^{\frac{(sr+2(s+1))(r+1)}{2}+s(m+1)+1}}{\left(F_r(x)!\right)_s \left(F_{2m-r}(x)!\right)_s} \prod_{i=1}^{2m+1} \left(aL_{s(q_i+r-m)}(x) + bF_{s(q_i+r-m)}(x)\right) \\
= \ & \sum_{j=0}^{m} a^{2j} b^{2m-2j} \left(x^2+4\right)^j \left(\binom{2m+1}{2j+1} aL_{s(i_1+\cdots+i_{2m+1})}(x) + \binom{2m+1}{2j} bF_{s(i_1+\cdots+i_{2m+1})}(x)\right).
\end{aligned} \tag{39}$$

Similarly, if in (38) we set $c = 0$, $p_i = q_i - n - m + 1$, $k = -4m$ and $l = 0$, we obtain

$$\begin{aligned}
&\sum_{r=0}^{2m-1} \frac{(-1)^{\frac{(sr+2(s+1))(r+1)}{2}+s(r+m+1)}}{\left(F_r(x)!\right)_s \left(F_{2m-1-r}(x)!\right)_s} \prod_{i=1}^{2m} \left(aL_{s(q_i+r+1-m)}(x) + bF_{s(q_i+r+1-m)}(x)\right) \\
= \ & \sum_{j=0}^{m-1} a^{2j} b^{2m-2j-1} \left(x^2+4\right)^k \left(\binom{2m}{2j+1} aL_{s(i_1+\cdots+i_{2m}+m)}(x) + \binom{2m}{2j} bF_{s(i_1+\cdots+i_{2m}+m)}(x)\right) \\
&+ a^{2m} \left(x^2+4\right)^m F_{s(i_1+\cdots+i_{2m}+m)}(x).
\end{aligned} \tag{40}$$

Some examples are the following:

$$\begin{aligned}
&\frac{(-1)^{s+1}}{F_s(x)} \left(aL_{sq_1}(x) + bF_{sq_1}(x)\right) \left(aL_{sq_2}(x) + bF_{sq_2}(x)\right) \\
&+ \frac{1}{F_s(x)} \left(aL_{s(q_1+1)}(x) + bF_{s(q_1+1)}(x)\right) \left(aL_{s(q_2+1)}(x) + bF_{s(q_2+1)}(x)\right) \\
= \ & 2abL_{s(q_1+q_2+1)}(x) + \left(b^2 + a^2\left(x^2+4\right)\right) F_{s(q_1+q_2+1)}(x).
\end{aligned} \tag{41}$$

$$
\begin{aligned}
&\frac{(-1)^s \begin{pmatrix} aL_{s(q_1-1)}(x) \\ +bF_{s(q_1-1)}(x) \end{pmatrix} \begin{pmatrix} aL_{s(q_2-1)}(x) \\ +bF_{s(q_2-1)}(x) \end{pmatrix} \begin{pmatrix} aL_{s(q_3-1)}(x) \\ +bF_{s(q_3-1)}(x) \end{pmatrix}}{F_s(x) F_{2s}(x)} \qquad (42) \\
&+\frac{(-1)^{s+1} \begin{pmatrix} aL_{sq_1}(x) \\ +bF_{sq_1}(x) \end{pmatrix} \begin{pmatrix} aL_{sq_2}(x) \\ +bF_{sq_2}(x) \end{pmatrix} \begin{pmatrix} aL_{sq_3}(x) \\ +bF_{sq_3}(x) \end{pmatrix}}{F_s^2(x)} \\
&+\frac{\begin{pmatrix} aL_{s(q_1+1)}(x) \\ +bF_{s(q_1+1)}(x) \end{pmatrix} \begin{pmatrix} aL_{s(q_2+1)}(x) \\ +bF_{s(q_2+1)}(x) \end{pmatrix} \begin{pmatrix} aL_{s(q_3+1)}(x) \\ +bF_{s(q_3+1)}(x) \end{pmatrix}}{F_s(x) F_{2s}(x)} \\
= \ &\left(3b^2 + a^2\left(x^2+4\right)\right) aL_{s(q_1+q_2+q_3)}(x) + \left(b^2 + 3a^2\left(x^2+4\right)\right) bF_{s(q_1+q_2+q_3)}(x).
\end{aligned}
$$

Formula (41) includes $(-1)^{s+1} F_{sn}^2(x) + F_{s(n+1)}^2(x) = F_s(x) F_{s(2n+1)}(x)$ (the case $s = x = 1$ is a famous identity). Formula (42) includes

$$
\begin{aligned}
F_s^2(x) F_{s(a+b+c)}(x) &= \frac{1}{L_s(x)} F_{s(a+1)}(x) F_{s(b+1)}(x) F_{s(c+1)}(x) + (-1)^{s+1} F_{sa}(x) F_{sb}(x) F_{sc}(x) \\
&\quad + \frac{(-1)^s}{L_s(x)} F_{s(a-1)}(x) F_{s(b-1)}(x) F_{s(c-1)}(x).
\end{aligned}
$$

(The case $s = x = 1$ is also a known identity. See [3, identity 45, p. 89] and [2, p. 5].)

Finally, we want to call the attention to the fact that (37) and (38) include identities that express s-Fibonacci and s-Lucas polynomial sequences as linear combinations of s-polyfibonomial sequences.

Consider (37) with $a = 0$, $k = 1$, $l = 0$, $p_i = i$, $i = 1, 2, \ldots, 2m+1$, and n shifted to $n - 2m - 1$. If $c = 0$ and $bd \neq 0$ we can write the identity as

$$
\begin{aligned}
&F_{(2m+1)sn}(x) \qquad (43) \\
= \ &(-1)^{s(m+1)+1} F_{(2m+1)s}(x) \sum_{r=0}^{2m} (-1)^{\frac{(sr+2(s+1))(r+1)}{2}} \binom{2m}{r}_{F_s(x)} \binom{n+r}{2m+1}_{F_s(x)},
\end{aligned}
$$

and if $d = 0$ and $bc \neq 0$ we can write

$$
\begin{aligned}
&L_{(2m+1)sn}(x) \qquad (44) \\
= \ &(-1)^{s(m+1)} \sum_{r=0}^{2m+1} (-1)^{\frac{(sr+2(s+1))(r+1)}{2}} L_{s(2m+1-r)}(x) \binom{2m+1}{r}_{F_s(x)} \binom{n+r}{2m+1}_{F_s(x)}.
\end{aligned}
$$

Similarly, if in (38) we set $a = 0$, $k = -4m$, $l = 0$, $p_i = i$, $i = 1, 2, \ldots, 2m$, and shift n to $n - 2m$, we obtain if $c = 0$ and $bd \neq 0$

$$
F_{2smn}(x) = (-1)^{s(m+1)} F_{2ms}(x) \sum_{r=0}^{2m-1} (-1)^{\frac{(sr+2(s+1))(r+1)}{2}+sr} \binom{2m-1}{r}_{F_s(x)} \binom{n+r}{2m}_{F_s(x)}, \quad (45)
$$

and if $d=0$ and $bc\neq 0$

$$L_{2smn}(x)=(-1)^{s(m+1)+1}\sum_{r=0}^{2m}(-1)^{\frac{(sr+2(s+1))(r+1)}{2}+sr}L_{s(2m-r)}(x)\binom{2m}{r}_{F_s(x)}\binom{n+r}{2m}_{F_s(x)}. \tag{46}$$

(Formulas (43), (44), (45) and (46) when $x=1$, were obtained in [7] by using different methods.)

Some examples are

$$F_{2sn}(x)=F_{2s}(x)\left(\binom{n+1}{2}_{F_s(x)}+(-1)^{s+1}\binom{n}{2}_{F_s(x)}\right).$$

$$F_{3sn}(x)=F_{3s}(x)\left(\binom{n+2}{3}_{F_s(x)}+(-1)^{s+1}L_s(x)\binom{n+1}{3}_{F_s(x)}+(-1)^{s}\binom{n}{3}_{F_s(x)}\right).$$

$$\begin{aligned}L_{3sn}(x) &= 2\binom{n+3}{3}_{F_s(x)}-\frac{F_{3s}(x)}{F_s(x)}L_s(x)\binom{n+2}{3}_{F_s(x)}\\ &\quad+(-1)^s\frac{F_{3s}(x)}{F_s(x)}L_{2s}(x)\binom{n+1}{3}_{F_s(x)}+(-1)^{s+1}L_{3s}(x)\binom{n}{3}_{F_s(x)}.\end{aligned}$$

$$\begin{aligned}L_{4sn}(x) &= 2\binom{n+4}{4}_{F_s(x)}-L_{2s}(x)L_s^2(x)\binom{n+3}{4}_{F_s(x)}\\ &\quad+(-1)^s\frac{F_{3s}(x)}{F_s(x)}L_{2s}^2(x)\binom{n+2}{4}_{F_s(x)}\\ &\quad+(-1)^{s+1}L_s(x)L_{2s}(x)L_{3s}(x)\binom{n+1}{4}_{F_s(x)}+L_{4s}(x)\binom{n}{4}_{F_s(x)}.\end{aligned}$$

5 Appendix. Proofs of some identities

In this appendix we present proofs of identities (11) and (17) used in propositions (1) and (2), respectively. We will use some identities related to the so-called *index-reduction formula*

$$F_a(x)F_b(x)-F_c(x)F_d(x)=(-1)^r\left(F_{a-r}(x)F_{b-r}(x)-F_{c-r}(x)F_{d-r}(x)\right), \tag{47}$$

where $a,b,c,d,r\in\mathbb{Z}$ and $a+b=c+d$. Johnson [2] proves (47) in the case $x=1$, but the argument can be adapted to our case (of Fibonacci polynomials) by replacing the matrix

$$Q=\begin{bmatrix}1&1\\1&0\end{bmatrix},$$

by the matrix

$$Q(x) = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix},$$

for which we have

$$\begin{bmatrix} F_k(x) \\ F_{k-1}(x) \end{bmatrix} = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}^{k-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

We will use also the identity

$$F_{a+b}(x) - (-1)^b F_{a-b}(x) = L_a(x) F_b(x), \tag{48}$$

which proof is trivial (by using Binet's formulas).

Proof of (11). We can write (11) as

$$\begin{aligned} & (-1)^{sr} F_{sr}(x) F_{s(r-1)}(x) + F_{sr}(x) F_{s(2m+2-r)}(x) L_{s(2m+1)}(x) \\ & + (-1)^{sr} F_{s(2m+2-r)}(x) F_{s(2m+1-r)}(x) \\ = \; & F_{s(2m+2)}(x) F_{s(2m+1)}(x). \end{aligned} \tag{49}$$

Let us consider the left-hand side of (49)

$$\begin{aligned} LHS(s,r,m,x) \; = \; & (-1)^{sr} F_{sr}(x) F_{s(r-1)}(x) + F_{sr}(x) F_{s(2m+2-r)}(x) L_{s(2m+1)}(x) \\ & + (-1)^{sr} F_{s(2m+2-r)}(x) F_{s(2m+1-r)}(x). \end{aligned}$$

Observe that the right-hand side of (49) does not depend on r, and that (49) is trivial for $r = 0$ and $r = 2m+2$. Thus, to prove this identity it is sufficient to prove that $LHS(s,r,m,x) = LHS(s,r+1,m,x)$.

We have

$$\begin{aligned} & LHS(s,r,m,x) - LHS(s,r+1,m,x) \\ = \; & (-1)^{sr} F_{sr}(x) F_{s(r-1)}(x) + F_{sr}(x) F_{s(2m+2-r)}(x) L_{s(2m+1)}(x) \\ & + (-1)^{sr} F_{s(2m+2-r)}(x) F_{s(2m+1-r)}(x) \\ & - (-1)^{s(r+1)} F_{s(r+1)}(x) F_{sr}(x) - F_{s(r+1)}(x) F_{s(2m+1-r)}(x) L_{s(2m+1)}(x) \\ & - (-1)^{s(r+1)} F_{s(2m+1-r)}(x) F_{s(2m-r)}(x) \\ = \; & -(-1)^{s(r+1)} F_{sr}(x) \left(F_{s(r+1)}(x) - (-1)^s F_{s(r-1)}(x) \right) \\ & + (-1)^{sr} F_{s(2m+1-r)}(x) \left(F_{s(2m+2-r)}(x) - (-1)^s F_{s(2m-r)}(x) \right) \\ & - L_{s(2m+1)}(x) \left(F_{s(r+1)}(x) F_{s(2m+1-r)}(x) - F_{sr}(x) F_{s(2m+2-r)}(x) \right). \end{aligned}$$

Now use (48) to write

$$\begin{aligned} F_{s(r+1)}(x) - (-1)^s F_{s(r-1)}(x) \; &= \; L_{sr}(x) F_s(x), \\ F_{s(2m+2-r)}(x) - (-1)^s F_{s(2m-r)}(x) \; &= \; L_{s(2m+1-r)}(x) F_s(x), \end{aligned}$$

and use (47) to write

$$F_{s(r+1)}(x) F_{s(2m+1-r)}(x) - F_{sr}(x) F_{s(2m+2-r)}(x) = (-1)^{sr} F_s(x) F_{s(2m+1-2r)}(x).$$

Then, by using again (48) we have

$$
\begin{aligned}
& LHS\left(s,r,m,x\right)-LHS\left(s,r+1,m,x\right)\\
= & -\left(-1\right)^{s(r+1)}F_{sr}\left(x\right)F_{s}\left(x\right)L_{sr}\left(x\right)+\left(-1\right)^{sr}F_{s(2m+1-r)}\left(x\right)F_{s}\left(x\right)L_{s(2m+1-r)}\left(x\right)\\
& -L_{s(2m+1)}\left(x\right)\left(-1\right)^{sr}F_{s}\left(x\right)F_{s(2m+1-2r)}\left(x\right)\\
= & \left(-1\right)^{sr}F_{s}\left(x\right)\left(\left(-1\right)^{s+1}F_{2sr}\left(x\right)+F_{2s(2m+1-r)}-L_{s(2m+1)}\left(x\right)F_{s(2m+1-2r)}\left(x\right)\right)\\
= & 0,
\end{aligned}
$$

as wanted.

The proof of (21) is similar.

Proof of (17). We want to prove the identity

$$
\begin{aligned}
& \left(-1\right)^{s(r+1)}H_{s(k-1)+l}\left(x\right)F_{sr}\left(x\right)+H_{s(2m+k)+l}\left(x\right)F_{s(2m+1-r)}\left(x\right)\\
= & F_{s(2m+1)}\left(x\right)H_{s(2m-r+k)+l}\left(x\right).
\end{aligned}
$$

By using that $H_{n}\left(x\right)=H_{0}\left(x\right)F_{n-1}\left(x\right)+H_{1}\left(x\right)F_{n}\left(x\right)$, we can write (17) as the two following identities

$$
\begin{aligned}
& \left(-1\right)^{s(r+1)}F_{s(k-1)+l-1}\left(x\right)F_{sr}\left(x\right)+F_{s(2m+k)+l-1}\left(x\right)F_{s(2m+1-r)}\left(x\right)\\
= & F_{s(2m+1)}\left(x\right)F_{s(2m-r+k)+l-1}\left(x\right),\\
& \left(-1\right)^{s(r+1)}F_{s(k-1)+l}\left(x\right)F_{sr}\left(x\right)+F_{s(2m+k)+l}\left(x\right)F_{s(2m+1-r)}\left(x\right)\\
= & F_{s(2m+1)}\left(x\right)F_{s(2m-r+k)+l}\left(x\right),
\end{aligned}
$$

and we notice that these are true by (47).

The proof of (20) is similar.

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