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On the Number of Subsets Relatively Prime to an Integer

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Abstract

Fix a positive integer and a finite set whose elements are in arithmetic progression. We give a formula for the number of nonempty subsets of this set that are coprime to the given integer. A similar formula is given when we restrict our attention to the subsets having the same fixed cardinality. These formulas generalize previous results of El Bachraoui.

1 Introduction

A nonempty subset A of $\{1, 2, \ldots, n\}$ is said to be relatively prime if $\gcd(A) = 1$. Nathanson [4] defined $f(n)$ to be the number of relatively prime subsets of $\{1, 2, \ldots, n\}$ and, for $k \geq 1$, $f_k(n)$ to be the number of relatively prime subsets of $\{1, 2, \ldots, n\}$ of cardinality k. Nathanson [4] defined $\Phi(n)$ to be the number of nonempty subsets A of the set $\{1, 2, \ldots, n\}$ such that $\gcd(A)$ is relatively prime to n and, for integer $k \geq 1$, $\Phi_k(n)$ to be the number of subsets A of the set $\{1, 2, \ldots, n\}$ such that $\gcd(A)$ is relatively prime to n and $\text{card}(A) = k$.

He obtained explicit formulas for these functions and deduced asymptotic estimates. These functions have been generalized by El Bachraoui [3] to subsets $A \in \{m+1, m+2, \ldots, n\}$ where m is any nonnegative integer, and then by Ayad and Kihel [1] to subsets of the set $\{a, a+b, \ldots, a+(n-1)b\}$ where a and b are any integers.

El Bachraoui [2] defined for any given positive integers $l \le m \le n$, $\Phi([l,m],n)$ to be the number of nonempty subsets of $\{l, l+1, \ldots, m\}$ which are relatively prime to n and $\Phi_k([l,m],n)$ to be the number of such subsets of cardinality k. He found formulas for these functions when $l = 1$ [2]. In this paper, we generalize these functions and obtain El Bachraoui's result as a particular case.

2 Phi functions for $\{1, 2, \ldots, m\}$

Let k and $l \le m \le n$ be positive integers. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x, and $\mu(n)$ the Möbius function. El Bachraoui [2] defined $\Phi([l,m],n)$ to be the number of nonempty subsets of $\{l, l+1, \ldots, m\}$ which are relatively prime to n and $\Phi_k([l,m],n)$ to be the number of such subsets of cardinality k. He proved the following formulas [2]:

$$\Phi([1,m],n) = \sum_{d|n} \mu(d) 2^{[\frac{m}{d}]} \tag{1}$$

and

$$\Phi_k([1,m],n) = \sum_{d|n} \mu(d) \left(\begin{array}{c} [\frac{m}{d}] \\ k \end{array} \right). \tag{2}$$

In his proof of Eqs. (1) and (2), El Bachraoui [2] used the Möbius inversion formula and its extension to functions of several variables. The case $m = n$ in (1), was proved by Nathanson [4].

3 Phi functions for $\{a, a+b, \ldots, a+(m-1)b\}$

It is natural to ask whether one can generalize the formulas obtained by El Bachraoui [2] to a set $A = \{a, a+b, \ldots, a+(m-1)b\}$, where a, b, and m are positive integers. Let $\Phi^{(a,b)}(m,n)$ be the number of nonempty subsets of $\{a, a+b, \ldots, a+(m-1)b\}$ which are relatively prime to n and $\Phi_k([l,m],n)$ to be the number of such subsets of cardinality k. To state our main theorem, we need the following lemma, which is proved in [1]:

Lemma 1. *For an integer $d \ge 1$, and for nonzero integers a and b such that* $\gcd(a,b) = 1$, *let $A_d = \{x = a + ib \text{ for } i = 0, \ldots, (m-1) \Big| \, d \mid x\}$. Then*

(i) If gcd $(b,d) \neq 1$, then $|A_d| = 0$.

(ii) If gcd $(b,d)=1$*, then* $|A_d| = \lfloor \frac{m}{d} \rfloor + \varepsilon_d$*, where*

$$\varepsilon_d = \begin{cases} 0, & \text{if } d \mid m; \\ 1, & \text{if } d \nmid m \text{ and } (-ab^{-1}) \bmod d \in \left\{0, \ldots, m - \lfloor \frac{m}{d} \rfloor d - 1\right\}; \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

Theorem 2.

$$\Phi^{(a,b)}(m,n) = \sum_{\substack{d \mid n \\ \gcd(b,d)=1}} \mu(d) \left(2^{\lfloor \frac{m}{d} \rfloor + \epsilon_d} - 1\right) \tag{4}$$

and

$$\Phi_k^{(a,b)}(m,n) = \sum_{\substack{d \mid n \\ \gcd(b,d)=1}} \mu(d) \binom{\lfloor \frac{m}{d} \rfloor + \epsilon_d}{k}, \tag{5}$$

where ϵ_d *is the function defined in Lemma 1.*

Proof. Let $A_d = \{x = a+ib \text{ for } i = 0, \ldots, (m-1) \,\big|\, d \mid x\}$, and $\mathcal{P}(A_d) = \{\text{the nonempty subsets of } A_d\}$. It is easy to see that $\Phi^{(a,b)}(m,n) = (2^m - 1) - \left| \bigcup_{\substack{p \text{ prime} \\ p \mid n}} \mathcal{P}(A_p) \right|$. Clearly, if $p_1, \ldots, p_t$ are distinct primes, then

$$\left| \bigcap_{i=1}^{t} \mathcal{P}(A_{p_i}) \right| = \left| \mathcal{P}(A_{\prod_{i=1}^{t} p_i}) \right|.$$

Thus, using the principle of inclusion-exclusion, one obtains from above that

$$\Phi^{(a,b)}(m,n) = \sum_{d \mid n} \mu(d) |\mathcal{P}(A_d)|.$$

It was proved in Lemma 1, that if $\gcd(b,d) \neq 1$, then $|A_d| = 0$ and if $\gcd(b,d) = 1$, then $|A_d| = \left(\lfloor \dfrac{m}{d} \rfloor + \varepsilon_d\right)$. Hence

$$\Phi^{(a,b)}(m,n) = \sum_{\substack{d \mid n \\ \gcd(b,d)=1}} \mu(d) \left(2^{[\frac{m}{d}] + \epsilon_d} - 1\right).$$

The proof for Eq. (5) is similar. □

Theorem 3 in [2] can be deduced from Theorem 2 above as the particular case where $a = b = 1$. We prove the following.

Corollary 3. *(a)* $\Phi([1,m],n) = \Phi^{(1,1)}(m,n)$
and
(b) $\Phi_k([l,m],n) = \Phi_k^{(1,1)}(m,n)$.

Proof. It is not difficult to prove that when $a = b = 1$ in Lemma 1, $\epsilon_d = 0$. Using Theorem 2, and the well-known equality $\sum_{d|n} \mu(d) = 0$, one obtains that

$$\Phi^{(1,1)}(m,n) = \sum_{d|n} \mu(d) \left(2^{\lfloor \frac{m}{d} \rfloor} - 1\right) = \sum_{d|n} \mu(d) 2^{\lfloor \frac{m}{d} \rfloor} = \Phi([1,m],n) \tag{6}$$

and

$$\Phi_k^{(1,1)}(n) = \sum_{d|n} \mu(d) \binom{\lfloor \frac{m}{d} \rfloor}{k} = \Phi_k([1,m],n). \tag{7}$$

□

Example 4. Using Theorem 2, one can obtain asymptotic estimates and generalize Corollary 4 proved by El Bachraoui [2].

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