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MONOTONICITY AND CONCAVITY PROPERTIES OF SOME FUNCTIONS INVOLVING THE GAMMA FUNCTION WITH APPLICATIONS

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ABSTRACT. In this article, we give the monotonicity and concavity properties of some functions involving the gamma function and some equivalence sequences to the sequence $n!$ with exact equivalence constants.

Key words and phrases: Gamma function, Monotonicity, Concavity, Equivalence.

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1. INTRODUCTION AND MAIN RESULTS

Throughout the paper, let $\mathbb{N}$ denote the set of all positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

We say $a_n \simeq b_n$ $(n \geq n_0)$ if there exist two constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 b_n \leq a_n \leq c_2 b_n \tag{1.1}$$

hold for all $n \geq n_0$. The fixed numbers c_1 and c_2 in (1.1) are called equivalence constants.

The incomplete gamma function is defined for $\operatorname{Re} z > 0$ by

$$\Gamma(z,x) = \int_x^\infty t^{z-1}e^{-t}\,\mathrm{d}t, \quad \gamma(z,x) = \int_0^x t^{z-1}e^{-t}\,\mathrm{d}t, \tag{1.2}$$

and $\Gamma(z,0) = \Gamma(z)$ is called the gamma function. The logarithmic derivative of $\Gamma(z)$, denoted by $\psi(z) = \Gamma'(z)/\Gamma(z)$, is called the psi or digamma function, and $\psi^{(k)}$ for $k \in \mathbb{N}$ are called the polygamma functions. One of the elementary properties of the gamma function is $\Gamma(x+1) = x\Gamma(x)$. In particular, $\Gamma(n+1) = n!$.

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In [13], it was proved by F. Qi that the functions

$$f(s,r) = \left[\frac{\Gamma(s)}{\Gamma(r)}\right]^{\frac{1}{s-r}}, \tag{1.3}$$

$$f(s,r,x) = \left[\frac{\Gamma(s,x)}{\Gamma(r,x)}\right]^{\frac{1}{s-r}} \tag{1.4}$$

and

$$g(s,r,x) = \left[\frac{\gamma(s,x)}{\gamma(r,x)}\right]^{\frac{1}{s-r}} \tag{1.5}$$

are increasing with respect to $r > 0$, $s > 0$, or $x > 0$.

E. A. Karatsuba [9] proved that the function

$$f_1(x) = [g(x)]^6 - (8x^3 + 4x^2 + x), \tag{1.6}$$

where

$$g(x) = \left(\frac{e}{x}\right)^x \frac{\Gamma(1+x)}{\sqrt{\pi}}, \tag{1.7}$$

is strictly increasing from $[1, \infty)$ onto $[f_1(1), f_1(\infty))$ with

$$f_1(1) = \frac{e^6}{\pi^3} - 13 \quad \text{and} \quad f_1(\infty) = \frac{1}{30}.$$

In 2003, in [1], H. Alzer proved that

$$\alpha \le f_1(x) < \frac{1}{30}, \quad x \in (0, \infty),$$

where

$$\alpha = \min_{x>0} f_1(x) = 0.0100450 \cdots = f_1(x_0)$$

for some $x_0 \in [0.6, 0.7]$. Since $f_1(x_0) < f_1(1)$ and

$$f_1(x_0) < \lim_{x\to 0+} f_1(x) = \frac{1}{\sqrt{\pi}},$$

his result shows that $f_1(x)$ is not still monotonic on $(0, 1]$.

In [3], it was shown in 1997, by G. Anderson and S. Qiu, that the function

$$f_2(x) = \frac{\ln \Gamma(x+1)}{x \ln x} \tag{1.8}$$

is strictly increasing from $(1, \infty)$ onto $(1 - \gamma, 1)$, where γ is the Euler-Mascheroni constant. H. Alzer, in 1998 in [2], proved that $f_2(x)$, with

$$f_2(1) = \lim_{x\to 1} f_2(x) = 1 - \gamma, \tag{1.9}$$

is strictly increasing on $(0, \infty)$. Also note that the function $f_2(x)$ was proved to be concave on $(1, \infty)$ in [6] in 2000 by A. Elbert and A. Laforgia.

In [5, 8, 10, 12, 14, 17], monotonicity properties of other functions related to the (di)gamma function were obtained.

In this article, we shall give some monotonicity and concavity properties of several functions involving the gamma function and, as applications, deduce some equivalence sequences to the sequence $n!$ with best equivalence constants.

Our main results are as follows.

Theorem 1.1. *The functions*

$$f(x) = \frac{x^{x+\frac{1}{2}}}{e^x\Gamma(x+1)} \tag{1.10}$$

and

$$F(x) = \frac{e^x\Gamma(x+1)}{x^x} \tag{1.11}$$

are strictly logarithmically concave and strictly increasing from $(0,\infty)$*, respectively, onto* $\big(0, 1/\sqrt{2\pi}\big)$ *and onto* $(1,\infty)$*.*

Theorem 1.2. *The function*

$$g(x) = \frac{e^x\Gamma(x+1)}{\left(x+\frac{1}{2}\right)^{x+\frac{1}{2}}} \tag{1.12}$$

is strictly logarithmically concave and strictly increasing from $\left(-\frac{1}{2},\infty\right)$ *onto* $\left(\sqrt{\pi/e}, \sqrt{2\pi/e}\right)$*.*

Theorem 1.3. *The function*

$$h(x) = \frac{e^x\Gamma(x+1)\sqrt{x-1}}{x^{x+1}} \tag{1.13}$$

is strictly logarithmically concave and strictly increasing from $(1,\infty)$ *onto* $\big(0,\sqrt{2\pi}\big)$*.*

As applications of these theorems, we have the following corollaries.

Corollary 1.4. *For* $n \in \mathbb{N}$*,*

$$n! \simeq e^{-n}n^{n+1/2}. \tag{1.14}$$

Moreover, for all $n \in \mathbb{N}$*,*

$$\sqrt{2\pi}\cdot e^{-n}n^{n+1/2} < n! \le e\cdot e^{-n}n^{n+1/2}. \tag{1.15}$$

The equivalence constants $\sqrt{2\pi}$ *and* e *in* (1.15) *are best possible.*

Corollary 1.5. *For* $n \in \mathbb{N}_0$*,*

$$n! \simeq e^{-n}\left(n+\frac{1}{2}\right)^{n+\frac{1}{2}}. \tag{1.16}$$

Moreover, for all $n \in \mathbb{N}_0$*,*

$$\sqrt{2}\,e^{-n}\left(n+\frac{1}{2}\right)^{n+\frac{1}{2}} \le n! < \sqrt{\frac{2\pi}{e}}\,e^{-n}\left(n+\frac{1}{2}\right)^{n+\frac{1}{2}}. \tag{1.17}$$

The equivalence constants $\sqrt{2}$ *and* $\sqrt{2\pi/e}$ *in* (1.17) *are best possible.*

Corollary 1.6. *For* $n \ge 2$*,*

$$n! \simeq \sqrt{\frac{n}{n-1}}\,e^{-n}n^{n+1/2}. \tag{1.18}$$

Furthermore, for all $n \ge 2$*,*

$$\left(\frac{e}{2}\right)^2\sqrt{\frac{n}{n-1}}\,e^{-n}n^{n+1/2} \le n! < \sqrt{2\pi}\sqrt{\frac{n}{n-1}}\,e^{-n}n^{n+1/2}. \tag{1.19}$$

The equivalence constants $(e/2)^2$ *and* $\sqrt{2\pi}$ *in* (1.19) *are best possible.*

Remark 1.7. In [16, Theorem 5], it was proved that for $n \geq 2$,

$$\sqrt{2\pi}\, e^{-n} n^{n+1/2} < n! < \left(\frac{n}{n-1}\right)^{\frac{1}{2}} \sqrt{2\pi}\, e^{-n} n^{n+1/2}, \tag{1.20}$$

which can be directly deduced from (1.15) and (1.19).

2. LEMMAS

We need the following lemmas to prove our results.

Lemma 2.1 ([4, p. 20])**.** *As* $x \to \infty$*,*

$$\ln \Gamma(x) = \left(x - \frac{1}{2}\right) \ln x - x + \ln \sqrt{2\pi} + O\left(\frac{1}{x}\right). \tag{2.1}$$

Lemma 2.2 ([7, p. 892] and [11, p. 17])**.** *For* $x > 0$*,*

$$\psi(x) = \ln x - \frac{1}{2x} - 2\int_0^\infty \frac{t\,\mathrm{d}t}{(t^2+x^2)(e^{2\pi t}-1)}, \tag{2.2}$$

$$\psi\left(x + \frac{1}{2}\right) = \ln x + 2\int_0^\infty \frac{t\,\mathrm{d}t}{(t^2+4x^2)(e^{\pi t}+1)}. \tag{2.3}$$

Lemma 2.3. *The function*

$$\varphi(x) = \ln \frac{x+1}{x+\frac{1}{2}} - \frac{1}{2x} \tag{2.4}$$

is strictly increasing from $(0, \infty)$ *onto* $(-\infty, 0)$*.*

Proof. We omit the proof of this lemma due to its simplicity. □

3. PROOF OF MAIN RESULTS

Proof of Theorem 1.1. Taking the logarithm of $f(x)$ defined by (1.10) and differentiating directly yields

$$\ln f(x) = \left(x - \frac{1}{2}\right) \ln x - x - \ln \Gamma(x), \tag{3.1}$$

$$[\ln f(x)]' = \ln x - \frac{1}{2x} - \psi(x). \tag{3.2}$$

Then by formula (2.2) of Lemma 2.2,

$$[\ln f(x)]' = 2\int_0^\infty \frac{t\,\mathrm{d}t}{(t^2+x^2)(e^{2\pi t}-1)}, \quad x > 0. \tag{3.3}$$

Hence, $[\ln f(x)]' > 0$ for $x \in (0, \infty)$, which means that $\ln f(x)$, and then $f(x)$, is strictly increasing on $(0, \infty)$.

It is easy to see that $\lim_{x\to 0+} f(x) = 0$. By (3.1) and Lemma 2.1, we have

$$\ln f(x) = -\ln \sqrt{2\pi} + O\left(\frac{1}{x}\right) \to \ln \frac{1}{\sqrt{2\pi}}, \quad x \to \infty, \tag{3.4}$$

which implies $\lim_{x\to\infty} f(x) = 1/\sqrt{2\pi}$.

Taking the logarithm of $F(x)$ defined by (1.11) and differentiating easily gives

$$\ln F(x) = x + \ln \Gamma(x+1) - x \ln x, \tag{3.5}$$

$$[\ln F(x)]' = \psi(x+1) - \ln x. \tag{3.6}$$

Then by (2.3) of Lemma 2.2, for all $x > 0$,

$$[\ln F(x)]' = \ln\left(1 + \frac{1}{2x}\right) + 2\int_0^\infty \frac{t\,\mathrm{d}t}{\left[t^2 + 4\left(x + \frac{1}{2}\right)^2\right](e^{\pi t} + 1)} > 0. \tag{3.7}$$

Hence, $\ln F(x)$, and then $F(x)$, is strictly increasing on $(0, \infty)$.

It is easy to see that $\lim_{x\to 0+} F(x) = 1$. By using Lemma 2.1, from (3.5),

$$\ln F(x) = \frac{1}{2}\ln x + \ln\sqrt{2\pi} + O\left(\frac{1}{x}\right), \quad x \to \infty. \tag{3.8}$$

Therefore, $\ln F(x)$, and then $F(x)$ tends to ∞ as $x \to \infty$.

Formulas (3.3) and (3.7) tell us that $[\ln f(x)]'$ and $[\ln F(x)]'$ are both strictly decreasing. Therefore, $\ln f(x)$ and $\ln F(x)$ are strictly concave, that is, the function $f(x)$ and $F(x)$ are both logarithmically concave. □

Proof of Theorem 1.2. Taking the logarithm of $g(x)$ defined by (1.12) and differentiating shows

$$\ln g(x) = x + \ln\Gamma(x+1) - \left(x + \frac{1}{2}\right)\ln\left(x + \frac{1}{2}\right), \tag{3.9}$$

$$[\ln g(x)]' = \psi(x+1) - \ln\left(x + \frac{1}{2}\right). \tag{3.10}$$

Then, by formula (2.3) of Lemma 2.2, we have

$$[\ln g(x)]' = 2\int_0^\infty \frac{t\,\mathrm{d}t}{[t^2 + (2x+1)^2](e^{\pi t} + 1)}, \quad x > -\frac{1}{2}. \tag{3.11}$$

So

$$[\ln g(x)]' > 0, \quad x \in \left(-\frac{1}{2}, \infty\right), \tag{3.12}$$

which means that $\ln g(x)$, then $g(x)$, is strictly increasing on $\left(-\frac{1}{2}, \infty\right)$.

Since $\Gamma(1/2) = \sqrt{\pi}$, it is easy to verify that $\lim\limits_{x\to -1/2} g(x) = \sqrt{\pi/e}$.

From (3.9) and Lemma 2.1, it is obtained that

$$\ln g(x) = \left(x + \frac{1}{2}\right)\ln\frac{x+1}{x + \frac{1}{2}} + \ln\sqrt{2\pi} - 1 + O\left(\frac{1}{x}\right), \quad x \to \infty. \tag{3.13}$$

Hence $\ln g(x) \to \ln\sqrt{2\pi/e}$ as $x \to \infty$, and then $\lim\limits_{x\to\infty} g(x) = \sqrt{2\pi/e}$.

Formula (3.11) shows that $[\ln g(x)]'$ is strictly decreasing. Therefore, $\ln g(x)$ is strictly concave, that is, the function $g(x)$ is logarithmically concave. □

Proof of Theorem 1.3. Taking the logarithm of $h(x)$ defined by (1.13) and differentiating straightforwardly reveals

$$\ln h(x) = \ln\Gamma(x) + x + \frac{1}{2}\ln(x-1) - x\ln x, \tag{3.14}$$

$$[\ln h(x)]' = \psi(x) + \frac{1}{2(x-1)} - \ln x. \tag{3.15}$$

By setting $x = u + 1$ with $u > 0$, we have

$$[\ln h(x)]' = \psi(u+1) + \frac{1}{2u} - \ln(u+1) = [\ln g(u)]' - \varphi(u), \tag{3.16}$$

where $g(u)$ and $\varphi(u)$ are respectively defined by (1.12) and (2.4). From (3.12) and Lemma 2.3, it is deduced that $[\ln h(x)]' > 0$ for $x > 1$. Therefore, $\ln h(x)$, and then $h(x)$, is strictly increasing on $(1, \infty)$.

It is obvious that $\lim_{x\to 1+} h(x) = 0$. From (3.14) and Lemma 2.1, we see

$$\ln h(x) = \frac{1}{2} \ln \frac{x-1}{x} + \ln \sqrt{2\pi} + O\left(\frac{1}{x}\right) \to \ln \sqrt{2\pi}, \quad x \to \infty. \tag{3.17}$$

So $\lim_{x\to\infty} h(x) = \sqrt{2\pi}$.

Considering the logarithmic concavity of $g(x)$ and the increasing monotonicity of $\varphi(x)$ in (3.16) reveals that $[\ln h(x)]'$ is strictly decreasing. Therefore, $\ln h(x)$ is strictly concave, that is, the function $h(x)$ is logarithmically concave. □

Proof of Corollary 1.4. By Theorem 1.1, we know that the function $f(x)$ is strictly increasing from $(0, \infty)$ onto $\left(0, \frac{1}{\sqrt{2\pi}}\right)$, hence

$$\frac{1}{e} = f(1) \le f(n) = \frac{n^{n+1/2}}{e^n n!} < \frac{1}{\sqrt{2\pi}} \tag{3.18}$$

for $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} \frac{n^{n+1/2}}{e^n n!} = \frac{1}{\sqrt{2\pi}}. \tag{3.19}$$

From (3.18) and (3.19), we see that Corollary 1.4 is true. □

Proof of Corollary 1.5. By Theorem 1.2, we see that the function $g(x)$ is strictly increasing from $\left(-\frac{1}{2}, \infty\right)$ onto $\left(\sqrt{\pi/e}, \sqrt{2\pi/e}\right)$. So

$$\sqrt{2} = g(0) \le g(n) = \frac{e^n n!}{\left(n + \frac{1}{2}\right)^{n+1/2}} < \sqrt{\frac{2\pi}{e}}, \quad n \in \mathbb{N}_0 \tag{3.20}$$

and

$$\lim_{n\to\infty} \frac{e^n n!}{\left(n + \frac{1}{2}\right)^{n+1/2}} = \sqrt{\frac{2\pi}{e}}. \tag{3.21}$$

Inequality (3.20) is equivalent to (1.17). Since the constants $\sqrt{2}$ and $\sqrt{2\pi/e}$ are best possible in (3.20), they are also best possible in (1.17). □

Proof of Corollary 1.6. The monotonicity of $h(x)$ by Theorem 1.3 implies

$$\left(\frac{e}{2}\right)^2 = h(2) \le h(n) = \frac{e^n n! \sqrt{n-1}}{n^{n+1}} < \sqrt{2\pi}, \quad n \ge 2 \tag{3.22}$$

and

$$\lim_{n\to\infty} \frac{e^n n! \sqrt{n-1}}{n^{n+1}} = \sqrt{2\pi}. \tag{3.23}$$

From (3.22) and (3.23), we see that Corollary 1.6 is valid. □

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