Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 7, Issue 4, Article 126, 2006

ON AN UPPER BOUND FOR THE DEVIATIONS FROM THE MEAN VALUE

AURELIA CIPU

RAILWAY HIGHSCHOOL BUCHAREST
STR. BUTUCENI, NR.10
BUCHAREST, ROMANIA
aureliacipu@netscape.net

Received 12 October, 2005; accepted 20 February, 2006
Communicated by A. Sofo

ABSTRACT. A completely elementary proof of a known upper bound for the deviations from the mean value is given. Related inequalities are also discussed. Applications to triangle inequalities provide characterizations of isosceles triangles.

Key words and phrases: Arithmetic mean, Square mean, Cauchy-Schwarz-Buniakovski inequality, Triangle inequality.

2000 *Mathematics Subject Classification.* 26D15, 51M16.

1. INTRODUCTION

For positive real numbers $x_1, x_2, \ldots, x_n$, $n \geq 2$, one denotes by a their arithmetic mean and by b the arithmetic mean of their squares (also known as the *square mean* of the given numbers). In [1], V. Nicula proves the inequalities

$$|x_k - a| \leq \sqrt{(n-1)(b-a^2)}\,, \qquad k = 1, 2, \ldots, n\,, \tag{1.1}$$

which are equivalent to

$$\max\{\,|x_k - a| \,:\, 1 \leq k \leq n\,\} \leq \sqrt{(n-1)(b-a^2)}\,. \tag{1.2}$$

The proof uses calculus and the author asks for an elementary proof. The aim of this paper is to provide such an approach. Our proof uses nothing more than the Cauchy-Schwarz-Buniakovski (CSB for short) inequality and is therefore accessible to pupils acquainted with polynomials of degree two. This approach has an additional advantage—it makes it very easy to determine the necessary and sufficient conditions under which equality holds in (1.2).

In the next section we shall prove the result below.

Theorem 1.1. *Let $n > 1$ be an integer and $x_1, x_2, \ldots, x_n$ be positive real numbers. Denote $a = (x_1 + x_2 + \cdots + x_n)/n$ and $b = (x_1^2 + x_2^2 + \cdots + x_n^2)/n$. Then*

$$\max\{\,|x_k - a| \,:\, 1 \leq k \leq n\,\} \leq \sqrt{(n-1)(b-a^2)}\,.$$

ISSN (electronic): 1443-5756

308-05

The equality holds in this relation if and only if either $n = 2$ or $n \geq 3$ and $n - 1$ of the given numbers are equal.

Section 3 contains several consequences of this result or of its proof. In the final section we give applications to triangle inequalities. In addition to being new, these results provide characterizations for isosceles triangles.

2. A SIMPLE PROOF

Let us denote by $y_k := x_k - a$, $k = 1, 2, \ldots, n$, the deviations from the mean value. Then we have

$$\sum_{k=1}^{n} y_k = \sum_{k=1}^{n} x_k - na = 0, \tag{2.1}$$

$$\sum_{k=1}^{n} y_k^2 = \sum_{k=1}^{n} x_k^2 - 2a\sum_{k=1}^{n} x_k + na^2 = n(b - a^2). \tag{2.2}$$

From equation (2.2) it follows that $b \geq a^2$, so the square root in the statement of Theorem 1.1 is real.

We have

$$\begin{aligned} y_n^2 &= \left(-\sum_{k=1}^{n-1} y_k\right)^2 && \text{by } (2.1) \\ &\leq (n-1)\sum_{k=1}^{n-1} y_k^2 && \text{by CSB} \\ &= (n-1)\left(\sum_{k=1}^{n} y_k^2 - y_n^2\right) \\ &= n(n-1)(b-a^2) - (n-1)y_n^2 && \text{by } (2.2). \end{aligned}$$

Hence,

$$y_n^2 \leq (n-1)(b-a^2). \tag{2.3}$$

Taking the square root results in relation (1.1) written for $k = n$. A similar reasoning yields the inequalities for $k = 1, 2, \ldots, n - 1$.

Let us determine when equality holds in relation (1.2). It is easily seen that for $n = 2$ we have

$$\mid x_1 - a \mid = \mid x_2 - a \mid = \frac{1}{2} \mid x_1 - x_2 \mid = \sqrt{b - a^2}.$$

For $n \geq 3$, the equality holds in relation (2.3) if and only if the CSB inequality for $y_1, y_2, \ldots, y_{n-1}$ turns into an equality. It is well-known that this is the case if and only if all the involved numbers coincide. In terms of the given numbers, the necessary and sufficient condition that

$$x_n - a = (n-1)(b - a^2)$$

is $x_1 = x_2 = \cdots = x_{n-1}$.

Theorem 1.1 is proved.

3. CONSEQUENCES

The reasoning used to characterize the equality in relation (1.2) immediately yields:

Corollary 3.1. *If equality holds in relation* (1.1) *for two values of the index k, then all numbers are equal.*

Inequality (1.2) has a companion inequality, in which the maximal deviation from the mean value is bounded from **below** in terms of a and b.

Corollary 3.2. *Using the same hypothesis and notation we have*

$$\max\{\,|x_k - a| \,:\, 1 \le k \le n\,\} \ge \sqrt{b - a^2}\,.$$

The equality holds in this relation if and only if $b = a^2$, that is, when all x_k coincide.

A natural question is whether there are similar results for the **smallest** deviation from the mean value. From relation (2.2) one easily gets an upper bound.

Corollary 3.3. $\min\{\,|x_k - a| \,:\, 1 \le k \le n\,\} \le \sqrt{b - a^2}\,.$

However, in general there are no lower bounds for the smallest deviation from the mean value better than the trivial one (being fulfilled by the absolute value of any expression)

$$\min\{\,|x_k - a| \,:\, 1 \le k \le n\,\} \ge 0.$$

Indeed, the claim is clear if one considers n numbers, one of which is equal to the arithmetic mean of the others.

A final remark compares inequality (1.2) to inequalities obtained by other methods. When $x_k = k$, $k = 1, 2, \dots, n$, we have $a = (n+1)/2$ and $b = (n+1)(2n+1)/6$, and Theorem 1.1 yields

$$\max\left\{\left|k - \frac{n+1}{2}\right| \,:\, 1 \le k \le n\right\} \le \frac{n-1}{2}\sqrt{\frac{n+1}{3}}\,. \tag{3.1}$$

On the other hand, since x_k are contained in an interval of length $n-1$, from the geometric interpretation of the modulus one obtains the stronger inequality

$$\max\left\{\left|k - \frac{n+1}{2}\right| \,:\, 1 \le k \le n\right\} \le \frac{n-1}{2}.$$

4. APPLICATIONS

The previous results allow us to characterize isosceles triangles. The idea is to conveniently specialize the x_k in Theorem 1.1. The computations needed in each case are simple and therefore omitted. Applying Theorem 1.1 successively for $n = 3$ and x_k the side lengths, altitudes, and radii of excircles, one obtains

Proposition 4.1. *In any triangle with sides of length a, b, c, one denotes by $s = (a+b+c)/2$ its semiperimeter, by F its area, and by r and R the radius of the in- and circumcircle, respectively. Then*

$$\max|b + c - 2a| \le 2\sqrt{a^2 + b^2 + c^2 - ab - bc - ca}\,,$$
$$\max|ab + bc - ca| \le 2\sqrt{a^2b^2 + b^2c^2 + c^2a^2 - 4sabc}\,,$$
$$\max\left|r + 4R - \frac{3F}{s-a}\right| \le 2\sqrt{(r+4R)^2 - s^2}\,.$$

In each relation the equality holds if and only if the triangle is isosceles.

These results have an additional interest due to the fact that there are comparatively few symmetrical inequalities in which equality holds not only when all variables are equal.

REFERENCES

[1] V. NICULA, On a classical extremum problem (Romanian), *Gaz. Matem.* (Bucharest), **100** (1995), 498–501.