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STARLIKE LOG-HARMONIC MAPPINGS OF ORDER α

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ABSTRACT. In this paper, we consider univalent log-harmonic mappings of the form $f = zh\overline{g}$ defined on the unit disk U which are starlike of order α . Representation theorems and distortion theorem are obtained.

Key words and phrases: log-harmonic, Univalent, Starlike of order α.

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1. INTRODUCTION

Let $H(U)$ be the linear space of all analytic functions defined on the unit disk $U = \{z : |z| < 1\}$. A log-harmonic mapping is a solution of the nonlinear elliptic partial differential equation

$$\frac{\overline{f_{\overline{z}}}}{\overline{f}} = a\frac{f_z}{f}, \tag{1.1}$$

where the second dilation function $a \in H(U)$ is such that $|a(z)| < 1$ for all $z \in U$. It has been shown that if f is a non-vanishing log-harmonic mapping, then f can be expressed as

$$f(z) = h(z)\overline{g(z)},$$

where h and g are analytic functions in U. On the other hand, if f vanishes at $z = 0$ but is not identically zero, then f admits the following representation

$$f(z) = z|z|^{2\beta}h(z)\overline{g(z)},$$

where $\operatorname{Re}\beta > -1/2$, and h and g are analytic functions in U, $g(0) = 1$ and $h(0) \neq 0$ (see [3]). Univalent log-harmonic mappings have been studied extensively (for details see [1] – [5]).

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Let $f = z|z|^{2\beta}h\overline{g}$ be a univalent log-harmonic mapping. We say that f is a starlike log-harmonic mapping of order α if

$$\frac{\partial \arg f(re^{i\theta})}{\partial\theta} = \operatorname{Re}\ \frac{zf_z - \overline{z}f_{\overline{z}}}{f} > \alpha, \quad 0 \le \alpha < 1 \tag{1.2}$$

for all $z \in U$. Denote by $ST_{Lh}(\alpha)$ the set of all starlike log-harmonic mappings of order α. If $\alpha = 0$, we get the class of starlike log-harmonic mappings. Also, let $ST(\alpha) = \{f \in ST_{Lh}(\alpha)$ and $f \in H(U)\}$. If $f \in ST_{Lh}(0)$ then $F(\zeta) = \log(f(e^{\zeta}))$ is univalent and harmonic on the half plane $\{\zeta : \operatorname{Re}\{\zeta\} < 0\}$. It is known that F is closely related with the theory of nonparametric minimal surfaces over domains of the form $-\infty < u < u_0(v)$, $u_0(v + 2\pi) = u_0(v)$, (see [7]).

In Section 2 we include two representation theorems which establish the linkage between the classes $ST_{Lh}(\alpha)$ and $ST(\alpha)$. In Section 3 we obtain a sharp distortion theorem for the class $ST_{Lh}(\alpha)$.

2. REPRESENTATION THEOREMS

In this section, we obtain two representation theorems for functions in $ST_{Lh}(\alpha)$. In the first one we establish the connection between the classes $ST_{Lh}(\alpha)$ and $ST(\alpha)$. The second one is an integral representation theorem.

Theorem 2.1. *Let $f(z) = zh(z)\overline{g(z)}$ be a log-harmonic mapping on U, $0 \notin hg(U)$. Then $f \in ST_{Lh}(\alpha)$ if and only if $\varphi(z) = \frac{zh(z)}{g(z)} \in ST(\alpha)$.*

Proof. Let $f(z) = zh(z)\overline{g(z)} \in ST_{Lh}(\alpha)$, then it follows that

$$\begin{aligned}\frac{\partial \arg f(re^{i\theta})}{\partial\theta} &= \operatorname{Re}\frac{zf_z - \overline{z}f_{\overline{z}}}{f}\\ &= \operatorname{Re}\left(1 + \frac{zh'}{h} - \frac{\overline{z}\overline{g'}}{\overline{g}}\right)\\ &= \operatorname{Re}\left(1 + \frac{zh'}{h} - \frac{zg'}{g}\right) > \alpha.\end{aligned}$$

Setting

$$\varphi(z) = \frac{zh(z)}{g(z)},$$

we obtain

$$\operatorname{Re}\frac{zf_z - \overline{z}f_{\overline{z}}}{f} = \operatorname{Re}\frac{z\varphi'}{\varphi} > \alpha.$$

Since f is univalent, we know that $0 \notin f_z(U)$. Furthermore,

$$\varphi \circ f^{-1}(w) = q_1(w) = w|g \circ f^{-1}(w)|^{-2},$$

is locally univalent on $f(U)$.

Indeed, we have $\frac{z\varphi'z)}{\varphi(z)} = (1 - a(z))\frac{zf_z}{f} \neq 0$ for all $z \in U$. From Lemma 2.3 in [4] we conclude that φ is univalent on U. Hence $\varphi \in ST(\alpha)$.

Conversely, let $\varphi \in ST(\alpha)$ and $a \in H(U)$ such that $|a(z)| < 1$ for all $z \in U$ be given. We consider

$$g(z) = \exp\left(\int_0^z \frac{a(s)\varphi'(s)}{\varphi(s)(1 - a(s))}ds\right), \tag{2.1}$$

where $\frac{z\varphi'z)}{\varphi(z)} = (1 - \alpha)p(z) + \alpha$, and $p \in H(U)$ such that $p(0) = 1$ and $\operatorname{Re}(p) > 0$.

Also, let

$$h(z) = \frac{\varphi(z)g(z)}{z}$$

and

$$f(z) = zh(z)\overline{g(z)} = \varphi(z)|g(z)|^2. \tag{2.2}$$

Then h and g are non-vanishing analytic functions defined on U, normalized by $h(0) = g(0) = 1$ and f is a solution of (1.1) with respect to a.

Simple calculations give that

$$\frac{\partial \arg f(re^{i\theta})}{\partial\theta} = \operatorname{Re}\frac{zf_z - \overline{z}f_{\overline{z}}}{f} = \operatorname{Re}\frac{z\varphi' z)}{\varphi(z)} > \alpha.$$

Using the same argument we conclude that

$$f \circ \varphi^{-1}(w) = q_2(w) = w|g \circ \varphi^{-1}(w)|^2$$

is locally univalent on $\varphi(U)$ and that f is univalent from Lemma 2.3 in [4]. It follows that $f \in ST_{Lh}(\alpha)$, which completes the proof of Theorem 2.1. ☐

The next result is an integral representation for $f \in ST_{Lh}(\alpha)$ for the case $a(0) = 0$. For $\varphi \in ST(\alpha)$, we have

$$\frac{z\varphi' z)}{\varphi(z)} = (1-\alpha)p(z) + \alpha,$$

where $p \in H(U)$ is such that $p(0) = 1$ and $\operatorname{Re}(p) > 0$. Hence, there is a probability measure μ defined on the Borel $\sigma-$algebra of ∂U such that

$$\frac{z\varphi' z)}{\varphi(z)} = (1-\alpha)\int_{\partial U}\frac{1+\zeta z}{1-\zeta z}d\mu(\zeta) + \alpha, \tag{2.3}$$

and therefore,

$$\varphi(z) = z\exp\left(-2(1-\alpha)\int_{\partial U}\log(1-\zeta z)d\mu(\zeta)\right). \tag{2.4}$$

On the other hand, let $a \in H(U)$ be such that $|a(z)| < 1$ for all $z \in U$ and $a(0) = 0$. Then there is a probability measure ν defined on the Borel $\sigma-$algebra of ∂U such that

$$\frac{a(z)}{1-a(z)} = \int_{\partial U}\frac{\xi z}{1-\xi z}d\nu(\xi). \tag{2.5}$$

Substituting (2.3), (2.4), and (2.5), into (2.1) and (2.2) we get

$$f(z) = z\exp\left(-2(1-\alpha)\int_{\partial U}\log(1-\zeta z)d\mu(\zeta)\right) + L(z),$$

where

$$L(z) = \int_{\partial U x \partial U}\left[\int_0^z \frac{\xi}{1-\xi s}\left[(1-\alpha)\frac{1+\zeta s}{1-\zeta s} + \alpha\right]ds\right]d\mu(\zeta)d\nu(\xi).$$

Integrating and simplifying implies the following theorem:

Theorem 2.2. *$f = zh\overline{g} \in ST_{Lh}(\alpha)$ with $a(0) = 0$ if and only if there are two probability measures μ and ν such that*

$$f(z) = z\exp\left(\int_{\partial U x \partial U} K(z,\zeta,\xi)d\mu(\zeta)d\nu(\xi)\right),$$

where

$$K(z,\zeta,\xi)=(1-\alpha)\log\left(\frac{1+\overline{\zeta z}}{1-\zeta z}\right)+T(z,\zeta,\xi);$$

(2.6) $T(z,\zeta,\xi)$

$$=\left\{\begin{array}{ll} -2(1-\alpha)\,\mathrm{Im}\left(\frac{\zeta+\xi}{\zeta-\xi}\right)\arg\left(\frac{1-\xi z}{1-\zeta z}\right)-2\alpha\log|1-\xi z|; & \textit{if } |\zeta|=|\xi|=1, \zeta\neq\xi \\ (1-\alpha)\,\mathrm{Re}\left(\frac{4\zeta z}{1-\zeta z}\right)-2\alpha\log|1-\zeta z|; & \textit{if } |\zeta|=|\xi|=1, \zeta=\xi \end{array}\right\}.$$

Remark 2.3. Theorem 2.2 can be used in order to solve extremal problems for the class $ST_{Lh}(\alpha)$ with $a(0)=0$. For example see Theorem 3.1.

3. DISTORTION THEOREM

The following is a distortion theorem for the class $ST_{Lh}(\alpha)$ with $a(0)=0$.

Theorem 3.1. *Let $f(z)=zh(z)\overline{g(z)}\in ST_{Lh}(\alpha)$ with $a(0)=0$. Then for $z\in U$ we have*

$$\frac{|z|}{(1+|z|)^{2\alpha}}\exp\left((1-\alpha)\frac{-4|z|}{1+|z|}\right)\leq|f(z)|\leq\frac{|z|}{(1-|z|)^{2\alpha}}\exp\left((1-\alpha)\frac{4|z|}{1-|z|}\right). \tag{3.1}$$

The equalities occur if and only if $f(z)=\overline{\zeta}f_0(\zeta z)$, $|\zeta|=1$, where

$$f_0(z)=z\left(\frac{1-\overline{z}}{1-z}\right)\frac{1}{(1-\overline{z})^{2\alpha}}\exp\left((1-\alpha)\,\mathrm{Re}\,\frac{4z}{1-z}\right). \tag{3.2}$$

Proof. Let $f(z)=zh(z)\overline{g(z)}\in ST_{Lh}(\alpha)$ with $a(0)=0$. It follows from (2.1) and (2.2) that f admits the representation

$$f(z)=\varphi(z)\exp\left(2\,\mathrm{Re}\int_0^z\frac{a(s)\varphi'(s)}{\varphi(s)(1-a(s))}ds\right), \tag{3.3}$$

where $\varphi\in ST(\alpha)$ and $a\in H(U)$ such that $|a(z)|<1$ for all $z\in U$.

For $|z|=r$, the well known facts

$$\left|\frac{z\varphi'(z)}{\varphi(z)}\right|\leq(1-\alpha)\,\frac{1+r}{1-r}+\alpha,$$

$$\left|\frac{a(z)}{z(1-a(z))}\right|\leq\frac{1}{1-r},$$

and

$$|\varphi(z)|\leq\frac{r}{(1-r)^{2(1-\alpha)}},$$

imply that

$$\begin{aligned}|f(z)|&\leq\frac{r}{(1-r)^{2(1-\alpha)}}\exp\left(2\int_0^r\frac{1}{1-t}\left((1-\alpha)\frac{1+t}{1-t}+\alpha\right)dt\right)\\&=\frac{r}{(1-r)^{2\alpha}}\exp\left((1-\alpha)\frac{4r}{1-r}\right).\end{aligned}$$

Equality occurs if and only if, $a(z)=\zeta z$ and $\varphi(z)=\frac{z}{(1-\zeta z)^{2-2\alpha}}$, $|\zeta|=1$, which leads to $f(z)=\overline{\zeta}f_0(\zeta z)$.

For the left-hand side, we have

$$f(z) = z \exp\left(\int_{\partial U x \partial U} K(z,\zeta,\xi)d\mu(\zeta)d\nu(\xi)\right),$$

where

$$K(z,\zeta,\xi) = (1-\alpha)\log\left(\frac{1+\overline{\zeta z}}{1-\zeta z}\right) + T(z,\zeta,\xi);$$

$$T(z,\zeta,\xi) = \left\{\begin{array}{ll} -2(1-\alpha)\,\mathrm{Im}\left(\frac{\zeta+\xi}{\zeta-\xi}\right)\arg\left(\frac{1-\xi z}{1-\zeta z}\right) - 2\alpha\log|1-\xi z|; & \text{if } |\zeta|=|\xi|=1, \zeta\neq\xi \\ (1-\alpha)\,\mathrm{Re}\left(\frac{4\zeta z}{1-\zeta z}\right) - 2\alpha\log|1-\zeta z|; & \text{if } |\zeta|=|\xi|=1, \zeta=\xi \end{array}\right\}.$$

For $|z|=r$ we have

$$\begin{aligned} &\log\left|\frac{f(z)}{z}\right| \\ &= \mathrm{Re}\left(\int_{\partial U x\partial U} K(z,\zeta,\xi)d\mu(\zeta)d\nu(\xi)\right) \\ &\geq \min_{\mu,\nu}\left[\min_{|z|=r}\mathrm{Re}\left(\int_{\partial U x\partial U} K(z,\zeta,\xi)d\mu(\zeta)d\nu(\xi)\right)\right] \\ &\geq \log\frac{1}{|1+r|^{2\alpha}} + \min_{\mu,\nu}\left[\min_{|z|=r}\left(\int_{\partial U x\partial U} -2(1-\alpha)\,\mathrm{Im}\left(\frac{\zeta+\xi}{\zeta-\xi}\right)\arg\left(\frac{1-\xi z}{1-\zeta z}\right)d\mu(\zeta)d\nu(\xi)\right)\right] \\ &= \log\frac{1}{|1+r|^{2\alpha}} \\ &\qquad + \min\left[\min_{0<|l|\leq\frac{\pi}{2}}\left[\min_{|z|=r}\left(-2(1-\alpha)\,\mathrm{Im}\left(\frac{1+e^{2il}}{1-e^{2il}}\right)\arg\left(\frac{1-e^{2il}z}{1-z}\right);\ (1-\alpha)\frac{-4r}{1+r}\right)\right]\right], \end{aligned}$$

where $e^{2il} = \overline{\zeta}\xi$.

Let

$$\Phi_r(l) = \left\{\begin{array}{ll} \min_{|z|=r}\left(-2(1-\alpha)\,\mathrm{Im}\left(\frac{1+e^{2il}}{1-e^{2il}}\right)\arg\left(\frac{1-e^{2il}z}{1-z}\right)\right), & \text{if } 0<|l|<\frac{\pi}{2} \\ (1-\alpha)\frac{-4r}{1+r} & \text{if } l=0 \end{array}\right\}.$$

Then $\Phi_r(l)$ is a continuous and even function on $|l|<\frac{\pi}{2}$. Hence

$$\log\left|\frac{f(z)}{z}\right| \geq \log\frac{1}{|1+r|^{2\alpha}} + \min_{0<|l|\leq\frac{\pi}{2}}\Phi_r(l) = \log\frac{1}{|1+r|^{2\alpha}} + \inf_{0<l<\frac{\pi}{2}}\Phi_r(l).$$

Since

$$\max_{|z|=r}\arg\left(\frac{1-e^{2il}z}{1-z}\right) = 2\arctan\left(\frac{r\sin(l)}{1+r\cos(l)}\right),$$

we get

$$\log\left|\frac{f(z)}{z}\right| \geq \log\frac{1}{|1+r|^{2\alpha}} + \inf_{0<l<\frac{\pi}{2}}\left[-4(1-\alpha)\cot(l)\arctan\left(\frac{r\sin(l)}{1+r\cos(l)}\right)\right],$$

and using the fact that $|\arctan(x)| \leq |x|$, we have

$$\begin{aligned}\log\left|\frac{f(z)}{z}\right| &\geq \log\frac{1}{|1+r|^{2\alpha}} + \inf_{0<l<\frac{\pi}{2}}\left[-4(1-\alpha)\frac{r\cos(l)}{1+r\cos(l)}\right] \\ &\geq \log\frac{1}{|1+r|^{2\alpha}} + \inf_{0<l<\frac{\pi}{2}}\left[-4(1-\alpha)\frac{r}{1+r}\right].\end{aligned}$$

The case of equality is attained by the functions $f(z) = \overline{\zeta} f_0(\zeta z)$, $|\zeta| = 1$. □

The next application is a consequence of Theorem 2.1.

Theorem 3.2. *Let* $f(z) = zh(z)\overline{g(z)} \in ST_{Lh}(\alpha)$. *Then*

$$\left|\arg\frac{f(z)}{z}\right| \leq 2(1-\alpha)\arcsin(|z|).$$

Proof. Let $f(z) = zh(z)\overline{g(z)} \in ST_{Lh}(\alpha)$. Then $\varphi(z) = \frac{zh(z)}{g(z)} \in ST(\alpha)$ by Theorem 2.1. The result follows immediately from $\arg\frac{f(z)}{z} = \arg\frac{\varphi(z)}{z}$ and from [6, p. 142]. □

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