

# Inequalities for the q-Gamma Function **Toufik Mansour** vol. 9, iss. 1, art. 18, 2008 **Title Page** Contents 44 ◀ Page 1 of 9 Go Back Full Screen Close journal of inequalities

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# SOME INEQUALITIES FOR THE q -GAMMA FUNCTION

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Abstract:	Recently, Shabani [4, Theorem 2.4] established some inequalities involving the gamma function. In this paper we present the $q$ -analogues of these inequalities involving the $q$ -gamma function.
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#### 1. Introduction

The Euler gamma function  $\Gamma(x)$  is defined for x > 0 by

$$\Gamma(x) = \int_0^\infty e^{-t} e^{x-1} dt.$$

The psi or digamma function, the logarithmic derivative of the gamma function is defined by

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \quad x > 0.$$

Alsina and Tomás [1] proved that

$$\frac{1}{n!} \le \frac{\Gamma(1+x)^n}{\Gamma(1+nx)} \le 1,$$

for all  $x \in [0, 1]$  and nonnegative integers n. This inequality can be generalized to

$$\frac{1}{\Gamma(1+a)} \le \frac{\Gamma(1+x)^a}{\Gamma(1+ax)} \le 1,$$

for all  $a \ge 1$  and  $x \in [0, 1]$ , see [3]. Recently, Shabani [4] using the series representation of the function  $\psi(x)$  and the ideas used in [3] established some double inequalities involving the gamma function. In particular, Shabani [4, Theorem 2.4] proved

(1.1) 
$$\frac{\Gamma(a)^c}{\Gamma(b)^d} \le \frac{\Gamma(a+bx)^c}{\Gamma(b+ax)^d} \le \frac{\Gamma(a+b)^c}{\Gamma(a+b)^d},$$

for all  $x \in [0, 1]$ ,  $a \ge b > 0$ , c, d are positive real numbers such that  $bc \ge ad > 0$ , and  $\psi(b + ax) > 0$ .





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In this paper we give the q-inequalities of the above results by using similar techniques to those in [4]. The main ideas of Shabani's paper, as well as of the present one, are contained in paper [3] by Sándor. More precisely, we define the q-psi function as (0 < q < 1)

$$\psi_q(x) = \frac{d}{dx} \log \Gamma_q(x),$$

where the q-gamma function  $\Gamma_q(x)$  is defined by (0 < q < 1)

$$\Gamma_q(x) = (1-q)^{1-x} \prod_{i=1}^{\infty} \frac{1-q^i}{1-q^{x+i}}$$

Many properties of the q-gamma function were derived by Askey [2]. The explicit form of q-psi function  $\psi_q(x)$  is

(1.2) 
$$\psi_q(x) = -\log(1-q) + \log q \sum_{i=0}^{\infty} \frac{q^{x+i}}{1-q^{x+i}}$$

In this paper we extend (1.1) to the case of  $\Gamma_q(x)$ . In particular, by using the facts that  $\lim_{q\to 1^-} \Gamma_q(x) = \Gamma(x)$  and  $\lim_{q\to 1^-} \psi_q(x) = \psi(x)$  we obtain all the results of Shabani [4].



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#### 2. Main Results

In order to establish the proof of the theorems, we need the following lemmas.

**Lemma 2.1.** Let  $x \in [0,1]$ ,  $q \in (0,1)$ , and a, b be any two positive real numbers such that  $a \ge b$ . Then

$$\psi_q(a+bx) \ge \psi_q(b+ax).$$

*Proof.* Clearly, a + bx, b + ax > 0. The series presentation of  $\psi_q(x)$ , see (1.2), gives

$$\psi_q(a+bx) - \psi_q(b+ax) = \log q \sum_{i=0}^{\infty} \left( \frac{q^{a+bx+i}}{1-q^{a+bx+i}} - \frac{q^{b+ax+i}}{1-q^{b+ax+i}} \right)$$
$$= \log q \sum_{i=0}^{\infty} \frac{q^i(q^{a+bx}-q^{b+ax})}{(1-q^{a+bx+i})(1-q^{b+ax+i})}$$
$$= \log q \sum_{i=0}^{\infty} \frac{q^{b+bx+i}(q^{a-b}-q^{(a-b)x})}{(1-q^{a+bx+i})(1-q^{b+ax+i})}.$$

Since 0 < q < 1 we have that  $\log q < 0$ . In addition, since  $a \ge b$  we get that  $q^{a-b} \le q^{(a-b)x}$ . Hence,

$$\psi_q(a+bx) - \psi_q(b+ax) \ge 0,$$

which completes the proof.

**Lemma 2.2.** Let  $x \in [0, 1]$ ,  $q \in (0, 1)$ , a, b be any two positive real numbers such that  $a \ge b$  and  $\psi_q(b + ax) > 0$ . Let c, d be any two positive real numbers such that  $bc \ge ad > 0$ . Then  $bc\psi_q(a + bx) - ad\psi_q(b + ax) > 0$ .

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*Proof.* Lemma 2.1 together with  $\psi_q(b + ax) > 0$  give that  $\psi_q(a + bx) > 0$ . Thus Lemma 2.1 obtains

$$bc\psi_q(a+bx) \ge ad\psi_q(a+bx) \ge ad\psi_q(b+ax),$$

as required.

Now we present the q-inequality of (1.1).

**Theorem 2.3.** Let  $x \in [0, 1]$ ,  $q \in (0, 1)$ ,  $a \ge b > 0, c, d$  positive real numbers with  $bc \ge ad > 0$  and  $\psi_q(b + ax) > 0$ . Then

$$\frac{\Gamma_q(a)^c}{\Gamma_q(b)^d} \leq \frac{\Gamma_q(a+bx)^c}{\Gamma_q(b+ax)^d} \leq \frac{\Gamma_q(a+b)^c}{\Gamma_q(a+b)^d}.$$
*Proof.* Let  $f(x) = \frac{\Gamma_q(a+bx)^c}{\Gamma_q(b+ax)^d}$  and  $g(x) = \log f(x)$ . Then

$$g(x) = c \log \Gamma_q(a + bx) - d \log \Gamma_q(b + ax),$$

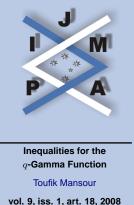
which implies that

$$g'(x) = \frac{d}{dx}g(x)$$
  
=  $bc\frac{\Gamma'_q(a+bx)}{\Gamma_q(a+bx)} - ad\frac{\Gamma'_q(b+ax)}{\Gamma(b+ax)}$   
=  $bc\psi_q(a+bx) - ad\psi_q(b+ax).$ 

Thus, Lemma 2.2 gives  $g'(x) \ge 0$ , that is, g(x) is an increasing function on [0, 1]. Therefore, f(x) is an increasing function on [0, 1]. Hence, for all  $x \in [0, 1]$  we have that  $f(0) \le f(x) \le f(1)$ , which is equivalent to

$$\frac{\Gamma_q(a)^c}{\Gamma_q(b)^d} \le \frac{\Gamma_q(a+bx)^c}{\Gamma_q(b+ax)^d} \le \frac{\Gamma_q(a+b)^c}{\Gamma_q(a+b)^d},$$

as requested.



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Similarly as in the argument proofs of Lemmas 2.1 - 2.2 and Theorem 2.3 we obtain the following results.

**Lemma 2.4.** Let  $x \ge 1$ ,  $q \in (0, 1)$ , and a, b be any two positive real numbers with b > a. Then

$$\psi_q(a+bx) \ge \psi_q(b+ax).$$

**Lemma 2.5.** Let  $x \ge 1$ ,  $q \in (0, 1)$ , a, b be any two positive real numbers with  $b \ge a$ and  $\psi_q(b+ax) > 0$ , and c, d be any two real numbers such that  $bc \ge ad > 0$ . Then

$$bc\psi_q(a+bx) - ad\psi_q(b+ax) \ge 0$$

Using similar techniques to the ones in the proof of Theorem 2.3 with Lemmas 2.4 and 2.5, instead of Lemmas 2.1 and 2.2, we can prove the following result.

**Theorem 2.6.** Let  $x \ge 1$ ,  $q \in (0,1)$ , a, b be any two positive real numbers with  $b \geq a > 0$  and  $\psi_q(b + ax) > 0$ , and c, d be any two real numbers such that  $bc \geq ad > 0$ . Then  $\frac{\Gamma_q(a+bx)^c}{\Gamma_q(b+ax)^d}$  is an increasing function on  $[1, +\infty)$ .

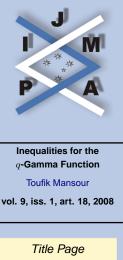
In addition, similar arguments as in the proof of Lemma 2.2 will obtain the following lemmas.

**Lemma 2.7.** Let  $x \in [0,1]$ ,  $q \in (0,1)$ , a, b be any two positive real numbers with  $a \geq b > 0$  and  $\psi_a(a + bx) < 0$ , and c, d be any two real numbers such that ad > bc > 0. Then

$$bc\psi_q(a+bx) - ad\psi_q(b+ax) \ge 0$$

**Lemma 2.8.** Let  $x \ge 1$ ,  $q \in (0, 1)$ , a, b be any two positive real numbers with  $b \ge a$ and  $\psi_a(a+bx) < 0$ , and c, d be any two real numbers such that  $ad \ge bc > 0$ . Then

 $bc\psi_a(a+bx) - ad\psi_a(b+ax) \ge 0.$ 



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Using similar techniques to the ones in the proof of Theorem 2.3, with Lemmas 2.2 and 2.7, we obtain the following.

**Theorem 2.9.** Let  $x \in [0,1]$ ,  $q \in (0,1)$ , a, b be any two positive real numbers with  $a \ge b > 0$  and  $\psi_q(a + bx) < 0$ , and c, d be any two real numbers such that  $ad \ge bc > 0$ . Then  $\frac{\Gamma_q(a+bx)^c}{\Gamma_q(b+ax)^d}$  is an increasing function on [0,1].

Using similar techniques to the ones in the proof of Theorem 2.3, with Lemmas 2.4 and 2.8, we obtain the following.

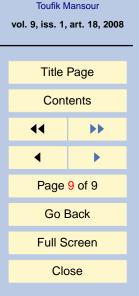
**Theorem 2.10.** Let  $x \ge 1$ ,  $q \in (0, 1)$ , a, b be any two positive real numbers with  $b \ge a > 0$  and  $\psi_q(a + bx) < 0$ , and c, d be any two real numbers such that  $ad \ge bc > 0$ . Then  $\frac{\Gamma_q(a+bx)^c}{\Gamma_q(b+ax)^d}$  is an increasing function on  $[1, +\infty)$ .



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