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FEJÉR INEQUALITIES FOR WRIGHT-CONVEX FUNCTIONS

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ABSTRACT. In this paper, we establish several inequalities of Fejér type for Wright-convex functions.

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1. INTRODUCTION

If $f : [a, b] \to \mathbb{R}$ is a convex function, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2} \tag{1.1}$$

is known as the Hermite-Hadamard inequality ([5]).

In [4], Fejér established the following weighted generalization of the inequality (1.1):

Theorem A. *If $f : [a, b] \to \mathbb{R}$ is a convex function, then the inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx \le \int_a^b f(x)\,p(x)\,dx \le \frac{f(a)+f(b)}{2}\int_a^b p(x)\,dx \tag{1.2}$$

holds, where $p : [a, b] \to \mathbb{R}$ is nonnegative, integrable, and symmetric about $x = \frac{a+b}{2}$.

In recent years there have been many extensions, generalizations, applications and similar results of the inequalities (1.1) and (1.2) see [1] – [8], [10] – [16].

In [2], Dragomir established the following theorem which is a refinement of the first inequality of (1.1).

Theorem B. *If $f : [a, b] \to \mathbb{R}$ is a convex function, and H is defined on $[0, 1]$ by*

$$H(t) = \frac{1}{b-a}\int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right)dx,$$

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then H is convex, increasing on $[0,1]$, and for all $t\in[0,1]$, we have

$$f\left(\frac{a+b}{2}\right)=H(0)\le H(t)\le H(1)=\frac{1}{b-a}\int_a^b f(x)\,dx. \tag{1.3}$$

In [11], Yang and Hong established the following theorem which is a refinement of the second inequality of (1.1):

Theorem C. *If $f:[a,b]\to\mathbb{R}$ is a convex function, and F is defined on $[0,1]$ by*

$$F(t)=\frac{1}{2(b-a)}\int_a^b\left[f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)x\right)\right.\\ \left.+f\left(\left(\frac{1+t}{2}\right)b+\left(\frac{1-t}{2}\right)x\right)\right]dx,$$

then F is convex, increasing on $[0,1]$, and for all $t\in[0,1]$, we have

$$\frac{1}{b-a}\int_a^b f(x)\,dx=F(0)\le F(t)\le F(1)=\frac{f(a)+f(b)}{2}. \tag{1.4}$$

We recall the definition of a Wright-convex function:

Definition 1.1 ([9, p. 223])**.** We say that $f:[a,b]\to\mathbb{R}$ is a Wright-convex function, if, for all $x,y+\delta\in[a,b]$ with $x<y$ and $\delta\ge 0$, we have

$$f(x+\delta)+f(y)\le f(y+\delta)+f(x). \tag{1.5}$$

Let $C([a,b])$ be the set of all convex functions on $[a,b]$ and $W([a,b])$ be the set of all Wright-convex functions on $[a,b]$. Then $C([a,b])\subsetneqq W([a,b])$. That is, a convex function must be a Wright-convex function but the converse is not true. (see [9, p. 224]).

In [10], Tseng, Yang and Dragomir established the following theorems for Wright-convex functions related to the inequality (1.1), Theorem A and Theorem B:

Theorem D. *Let $f\in W([a,b])\cap L_1[a,b]$. Then the inequality (1.1) holds.*

Theorem E. *Let $f\in W([a,b])\cap L_1[a,b]$ and let H be defined as in Theorem B. Then $H\in W([0,1])$ is increasing on $[0,1]$, and the inequality (1.3) holds for all $t\in[0,1]$.*

Theorem F. *Let $f\in W([a,b])\cap L_1[a,b]$ and let F be defined as in Theorem C. Then $F\in W([0,1])$ is increasing on $[0,1]$, and the inequality (1.4) holds for all $t\in[0,1]$.*

In [12], Yang and Tseng established the following theorem which refines the inequality (1.2):

Theorem G ([12, Remark 6])**.** *Let f and p be defined as in Theorem A. If P, Q are defined on $[0,1]$ by*

$$P(t)=\int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right)p(x)\,dx \qquad (t\in(0,1)) \tag{1.6}$$

and

$$Q(t)=\int_a^b\frac{1}{2}\left[f\left(\frac{1+t}{2}a+\frac{1-t}{2}x\right)p\left(\frac{x+a}{2}\right)\right.\\ \left.+f\left(\frac{1+t}{2}b+\frac{1-t}{2}x\right)p\left(\frac{x+b}{2}\right)\right]dx \qquad (t\in(0,1)), \tag{1.7}$$

then P, Q *are convex and increasing on* $[0,1]$ *and, for all* $t \in [0,1]$,

$$f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx = P(0) \le P(t) \le P(1) = \int_a^b f(x)\,p(x)\,dx \tag{1.8}$$

and

$$\int_a^b f(x)\,p(x)\,dx = Q(0) \le Q(t) \le Q(1) = \frac{f(a)+f(b)}{2}\int_a^b p(x)\,dx. \tag{1.9}$$

In this paper, we establish some results about Theorem A and Theorem G for Wright-convex functions which are weighted generalizations of Theorem D, E and F.

2. MAIN RESULTS

In order to prove our main theorems, we need the following lemma [10]:

Lemma 2.1. *If* $f : [a,b] \to \mathbb{R}$, *then the following statements are equivalent:*

(1) $f \in W([a,b])$;
(2) *for all* $s,t,u,v \in [a,b]$ *with* $s \le t \le u \le v$ *and* $t+u = s+v$, *we have*

$$f(t) + f(u) \le f(s) + f(v). \tag{2.1}$$

Theorem 2.2. *Let* $f \in W([a,b]) \cap L_1[a,b]$ *and let* $p : [a,b] \to \mathbb{R}$ *be nonnegative, integrable, and symmetric about* $x = \frac{a+b}{2}$. *Then the inequality* (1.2) *holds.*

Proof. For the inequality (2.1) and the assumptions that p is nonnegative, integrable, and symmetric about $x = \frac{a+b}{2}$, we have

$$\begin{aligned}
&f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx \\
&\qquad= \int_a^{\frac{a+b}{2}} f\left(\frac{a+b}{2}\right) p(x)\,dx + \int_a^{\frac{a+b}{2}} f\left(\frac{a+b}{2}\right) p(a+b-x)\,dx \\
&\qquad= \int_a^{\frac{a+b}{2}} \left[f\left(\frac{a+b}{2}\right) + f\left(\frac{a+b}{2}\right)\right] p(x)\,dx \\
&\qquad\le \int_a^{\frac{a+b}{2}} \left[f(x) + f(a+b-x)\right] p(x)\,dx \qquad \left(x \le \frac{a+b}{2} \le \frac{a+b}{2} \le a+b-x\right) \\
&\qquad= \int_a^{\frac{a+b}{2}} f(x)\,p(x)\,dx + \int_{\frac{a+b}{2}}^b f(x)\,p(x)\,dx \\
&\qquad= \int_a^b f(x)\,p(x)\,dx,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{f(a)+f(b)}{2}\int_a^b p(x)\,dx \\
&\qquad= \int_a^{\frac{a+b}{2}} \left[\frac{f(a)+f(b)}{2}\right] p(x)\,dx + \int_a^{\frac{a+b}{2}} \left[\frac{f(a)+f(b)}{2}\right] p(a+b-x)\,dx
\end{aligned}$$

$$\begin{aligned}&= \int_a^{\frac{a+b}{2}} [f(a)+f(b)]\,p(x)\,dx\\&\geq \int_a^{\frac{a+b}{2}} [f(x)+f(a+b-x)]\,p(x)\,dx \qquad (a\leq x\leq a+b-x\leq b)\\&= \int_a^{\frac{a+b}{2}} f(x)\,p(x)\,dx+\int_{\frac{a+b}{2}}^b f(x)\,p(x)\,dx=\int_a^b f(x)\,p(x)\,dx.\end{aligned}$$

This completes the proof. □

Remark 2.3. If we set $p(x)\equiv 1$ $(x\in[a,b])$ in Theorem 2.2, then Theorem 2.2 generalizes Theorem D.

Remark 2.4. From $C([a,b])\subsetneqq W([a,b])$, Theorem 2.2 generalizes Theorem A.

Theorem 2.5. *Let f and p be defined as in Theorem 2.2 and let P be defined as in* (1.6). *Then $P\in W([0,1])$ is increasing on $[0,1]$, and the inequality (1.8) holds for all $t\in[0,1]$.*

Proof. If $s,t,u,v\in[0,1]$ and $s\leq t\leq u\leq v$, $t+u=s+v$, then for $x\in\left[a,\frac{a+b}{2}\right]$ we have

$$\begin{aligned}b&\geq sx+(1-s)\frac{a+b}{2}\geq tx+(1-t)\frac{a+b}{2}\\&\geq ux+(1-u)\frac{a+b}{2}\geq vx+(1-v)\frac{a+b}{2}\geq a\end{aligned}$$

and if $x\in\left[\frac{a+b}{2},b\right]$, then

$$\begin{aligned}a&\leq sx+(1-s)\frac{a+b}{2}\leq tx+(1-t)\frac{a+b}{2}\\&\leq ux+(1-u)\frac{a+b}{2}\leq vx+(1-v)\frac{a+b}{2}\leq b,\end{aligned}$$

where

$$\begin{aligned}&\left[tx+(1-t)\frac{a+b}{2}\right]+\left[ux+(1-u)\frac{a+b}{2}\right]\\&\qquad=\left[sx+(1-s)\frac{a+b}{2}\right]+\left[vx+(1-v)\frac{a+b}{2}\right].\end{aligned}$$

By the inequality (2.1), we have

$$\begin{aligned}&f\left(tx+(1-t)\frac{a+b}{2}\right)+f\left(ux+(1-u)\frac{a+b}{2}\right)\\&\qquad\leq f\left(sx+(1-s)\frac{a+b}{2}\right)+f\left(vx+(1-v)\frac{a+b}{2}\right)\end{aligned}\tag{2.2}$$

for all $x\in[a,b]$. Now, using the inequality (2.2) and p is nonnegative on $[a,b]$, we have

$$\begin{aligned}&\left[f\left(tx+(1-t)\frac{a+b}{2}\right)+f\left(ux+(1-u)\frac{a+b}{2}\right)\right]p(x)\\&\qquad\leq\left[f\left(sx+(1-s)\frac{a+b}{2}\right)+f\left(vx+(1-v)\frac{a+b}{2}\right)\right]p(x)\end{aligned}\tag{2.3}$$

for all $x\in[a,b]$. Integrating the inequality (2.3) over x on $[a,b]$, we have

$$P(t)+P(u)\leq P(s)+P(v).$$

Hence $P \in W([0,1])$.

Next, if $0 \le s \le t \le 1$ and $x \in \left[a, \frac{a+b}{2}\right]$, then

$$\begin{aligned} tx + (1-t)\frac{a+b}{2} &\le sx + (1-s)\frac{a+b}{2} \\ &\le s(a+b-x) + (1-s)\frac{a+b}{2} \\ &\le t(a+b-x) + (1-t)\frac{a+b}{2}, \end{aligned}$$

where

$$\begin{aligned} &\left[sx + (1-s)\frac{a+b}{2}\right] + \left[s(a+b-x) + (1-s)\frac{a+b}{2}\right] \\ &\qquad = \left[tx + (1-t)\frac{a+b}{2}\right] + \left[t(a+b-x) + (1-t)\frac{a+b}{2}\right]. \end{aligned}$$

By the inequality (2.1) and the assumptions that p is nonnegative, integrable, and symmetric about $x = \frac{a+b}{2}$, we have

$$\begin{aligned} P(s) &= \int_a^b f\left(sx + (1-s)\frac{a+b}{2}\right) p(x)\,dx \\ &= \int_a^{\frac{a+b}{2}} f\left(sx + (1-s)\frac{a+b}{2}\right) p(x)\,dx \\ &\qquad + \int_a^{\frac{a+b}{2}} f\left(s(a+b-x) + (1-s)\frac{a+b}{2}\right) p(a+b-x)\,dx \\ &= \int_a^{\frac{a+b}{2}} \left[f\left(sx + (1-s)\frac{a+b}{2}\right) + f\left(s(a+b-x) + (1-s)\frac{a+b}{2}\right)\right] p(x)\,dx \\ &\le \int_a^{\frac{a+b}{2}} \left[f\left(tx + (1-t)\frac{a+b}{2}\right) + f\left(t(a+b-x) + (1-t)\frac{a+b}{2}\right)\right] p(x)\,dx \\ &= \int_a^{\frac{a+b}{2}} f\left(tx + (1-t)\frac{a+b}{2}\right) p(x)\,dx \\ &\qquad + \int_a^{\frac{a+b}{2}} f\left(t(a+b-x) + (1-t)\frac{a+b}{2}\right) p(a+b-x)\,dx \\ &= \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) p(x)\,dx = P(t). \end{aligned}$$

Thus, P is increasing on $[0,1]$, and the inequality (1.8) holds for all $t \in [0,1]$.

This completes the proof. □

Remark 2.6. If we set $p(x) \equiv 1$ $(x \in [a,b])$ in Theorem 2.5, then Theorem 2.2 generalizes Theorem E.

Theorem 2.7. *Let f and p be defined as in Theorem 2.2 and let Q be defined as in* (1.7). *Then $Q \in W([0,1])$ is increasing on $[0,1]$, and the inequality (1.9) holds for all $t \in [0,1]$.*

Proof. If $s,t,u,v\in[0,1]$ and $s\le t\le u\le v$, $t+u=s+v$, then for all $x\in[a,b]$ we have

$$\begin{aligned} a &\le \left(\frac{1+v}{2}\right)a+\left(\frac{1-v}{2}\right)x \le \left(\frac{1+u}{2}\right)a+\left(\frac{1-u}{2}\right)x \\ &\le \left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)x \le \left(\frac{1+s}{2}\right)a+\left(\frac{1-s}{2}\right)x \le b \end{aligned}$$

and

$$\begin{aligned} a &\le \left(\frac{1+s}{2}\right)b+\left(\frac{1-s}{2}\right)x \le \left(\frac{1+t}{2}\right)b+\left(\frac{1-t}{2}\right)x \\ &\le \left(\frac{1+u}{2}\right)b+\left(\frac{1-u}{2}\right)x \le \left(\frac{1+v}{2}\right)b+\left(\frac{1-v}{2}\right)x \le b, \end{aligned}$$

where

$$\begin{aligned} &\left[\left(\frac{1+u}{2}\right)a+\left(\frac{1-u}{2}\right)x\right]+\left[\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)x\right] \\ &\qquad = \left[\left(\frac{1+v}{2}\right)a+\left(\frac{1-v}{2}\right)x\right]+\left[\left(\frac{1+s}{2}\right)a+\left(\frac{1-s}{2}\right)x\right] \end{aligned}$$

and

$$\begin{aligned} &\left[\left(\frac{1+t}{2}\right)b+\left(\frac{1-t}{2}\right)x\right]+\left[\left(\frac{1+u}{2}\right)b+\left(\frac{1-u}{2}\right)x\right] \\ &\qquad = \left[\left(\frac{1+s}{2}\right)b+\left(\frac{1-s}{2}\right)x\right]+\left[\left(\frac{1+v}{2}\right)b+\left(\frac{1-v}{2}\right)x\right]. \end{aligned}$$

By the inequality (2.1), we have

$$\begin{aligned} f\left(\left(\frac{1+u}{2}\right)a+\left(\frac{1-u}{2}\right)x\right)+f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)x\right) \\ \le f\left(\left(\frac{1+v}{2}\right)a+\left(\frac{1-v}{2}\right)x\right)+f\left(\left(\frac{1+s}{2}\right)a+\left(\frac{1-s}{2}\right)x\right) \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} f\left(\left(\frac{1+t}{2}\right)b+\left(\frac{1-t}{2}\right)x\right)+f\left(\left(\frac{1+u}{2}\right)b+\left(\frac{1-u}{2}\right)x\right) \\ \le f\left(\left(\frac{1+s}{2}\right)b+\left(\frac{1-s}{2}\right)x\right)+f\left(\left(\frac{1+v}{2}\right)b+\left(\frac{1-v}{2}\right)x\right) \end{aligned} \tag{2.5}$$

for all $x\in[a,b]$. Now, using the inequality (2.4), (2.5) and the assumptions that p is nonnegative on $[a,b]$, we have

$$\begin{aligned} &\frac{1}{2}f\left(\left(\frac{1+u}{2}\right)a+\left(\frac{1-u}{2}\right)x\right)p\left(\frac{x+a}{2}\right) \\ &\quad+\frac{1}{2}f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)x\right)p\left(\frac{x+a}{2}\right) \\ &\quad+\frac{1}{2}f\left(\left(\frac{1+t}{2}\right)b+\left(\frac{1-t}{2}\right)x\right)p\left(\frac{x+b}{2}\right) \\ &\quad+\frac{1}{2}f\left(\left(\frac{1+u}{2}\right)b+\left(\frac{1-u}{2}\right)x\right)p\left(\frac{x+b}{2}\right) \end{aligned} \tag{2.6}$$

$$\leq \frac{1}{2} f\left(\left(\frac{1+v}{2}\right) a+\left(\frac{1-v}{2}\right) x\right) p\left(\frac{x+a}{2}\right)$$
$$+\frac{1}{2} f\left(\left(\frac{1+s}{2}\right) a+\left(\frac{1-s}{2}\right) x\right) p\left(\frac{x+a}{2}\right)$$
$$+\frac{1}{2} f\left(\left(\frac{1+s}{2}\right) b+\left(\frac{1-s}{2}\right) x\right) p\left(\frac{x+b}{2}\right)$$
$$+\frac{1}{2} f\left(\left(\frac{1+v}{2}\right) b+\left(\frac{1-v}{2}\right) x\right) p\left(\frac{x+b}{2}\right)$$

Integrating the inequality (2.6) over x on $[a, b]$, we have

$$Q(t)+Q(u) \leq Q(s)+Q(v) .$$

Hence $Q \in W([0,1])$.

Next, if $0 \leq s \leq t \leq 1$ and $x \in[a, b]$, then

$$\begin{aligned}
\left(\frac{1+t}{2}\right) a+\left(\frac{1-t}{2}\right) x & \leq\left(\frac{1+s}{2}\right) a+\left(\frac{1-s}{2}\right) x \\
& \leq\left(\frac{1+s}{2}\right) b+\left(\frac{1-s}{2}\right)(a+b-x) \\
& \leq\left(\frac{1+t}{2}\right) b+\left(\frac{1-t}{2}\right)(a+b-x)
\end{aligned}$$

and

$$\begin{aligned}
\left(\frac{1+t}{2}\right) a+\left(\frac{1-t}{2}\right)(a+b-x) & \leq\left(\frac{1+s}{2}\right) a+\left(\frac{1-s}{2}\right)(a+b-x) \\
& \leq\left(\frac{1+s}{2}\right) b+\left(\frac{1-s}{2}\right) x \\
& \leq\left(\frac{1+t}{2}\right) b+\left(\frac{1-t}{2}\right) x,
\end{aligned}$$

where

$$\begin{aligned}
& {\left[\left(\frac{1+s}{2} a\right)+\left(\frac{1-s}{2}\right) x\right]+\left[\left(\frac{1+s}{2}\right) b+\left(\frac{1-s}{2}\right)(a+b-x)\right] } \\
& \quad=\left[\left(\frac{1+t}{2}\right) a+\left(\frac{1-t}{2}\right) x\right]+\left[\left(\frac{1+t}{2}\right) b+\left(\frac{1-t}{2}\right)(a+b-x)\right],
\end{aligned}$$

and

$$\begin{aligned}
& {\left[\left(\frac{1+s}{2}\right) a+\left(\frac{1-s}{2}\right)(a+b-x)\right]+\left[\left(\frac{1+s}{2}\right) b+\left(\frac{1-s}{2}\right) x\right] } \\
& \quad=\left[\left(\frac{1+t}{2}\right) a+\left(\frac{1-t}{2}\right)(a+b-x)\right]+\left[\left(\frac{1+t}{2}\right) b+\left(\frac{1-t}{2}\right) x\right] .
\end{aligned}$$

By the inequality (2.1) and the assumptions that p is nonnegative and symmetric about $x=\frac{a+b}{2}$, we have

$$
\begin{aligned}
(2.7)\qquad & f\left(\left(\frac{1+s}{2}\right)a+\left(\frac{1-s}{2}\right)x\right)p\left(\frac{x+a}{2}\right)\\
&\quad + f\left(\left(\frac{1+s}{2}\right)b+\left(\frac{1-s}{2}\right)(a+b-x)\right)p\left(\frac{2a+b-x}{2}\right)\\
&\quad + f\left(\left(\frac{1+s}{2}\right)a+\left(\frac{1-s}{2}\right)(a+b-x)\right)p\left(\frac{a+2b-x}{2}\right)\\
&\quad + f\left(\left(\frac{1+s}{2}\right)b+\left(\frac{1-s}{2}\right)x\right)p\left(\frac{x+b}{2}\right)\\
&= \left[f\left(\left(\frac{1+s}{2}\right)a+\left(\frac{1-s}{2}\right)x\right)\right.\\
&\quad \left.+ f\left(\left(\frac{1+s}{2}\right)a+\left(\frac{1-s}{2}\right)(a+b-x)\right)\right]p\left(\frac{x+a}{2}\right)\\
&\quad + \left[f\left(\left(\frac{1+s}{2}\right)b+\left(\frac{1-s}{2}\right)(a+b-x)\right)\right.\\
&\quad \left.+ f\left(\left(\frac{1+s}{2}\right)b+\left(\frac{1-s}{2}\right)x\right)\right]p\left(\frac{x+b}{2}\right)\\
&\le \left[f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)x\right)\right.\\
&\quad \left.+ f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)(a+b-x)\right)\right]p\left(\frac{x+a}{2}\right)\\
&\quad + \left[f\left(\left(\frac{1+t}{2}\right)b+\left(\frac{1-t}{2}\right)(a+b-x)\right)\right.\\
&\quad \left.+ f\left(\left(\frac{1+t}{2}\right)b+\left(\frac{1-t}{2}\right)x\right)\right]p\left(\frac{x+b}{2}\right)\\
&= f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)x\right)p\left(\frac{x+a}{2}\right)\\
&\quad + f\left(\left(\frac{1+t}{2}\right)b+\left(\frac{1-t}{2}\right)(a+b-x)\right)p\left(\frac{2a+b-x}{2}\right)\\
&\quad + f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)(a+b-x)\right)p\left(\frac{a+2b-x}{2}\right)\\
&\quad + f\left(\left(\frac{1+t}{2}\right)b+\left(\frac{1-t}{2}\right)x\right)p\left(\frac{x+b}{2}\right).
\end{aligned}
$$

Integrating the inequality (2.7) over x on $[a,b]$, we have

$$4Q(s)\le 4Q(t)$$

Hence Q is increasing on $[0,1]$, and the inequality (1.9) holds for all $t\in[0,1]$.

This completes the proof. □

Remark 2.8. If we set $p(x)\equiv 1$ $(x\in[a,b])$ in Theorem 2.7, then Theorem 2.2 generalizes Theorem F.

Remark 2.9. From $C([a,b]) \subsetneqq W([a,b])$, Theorem 2.5 and Theorem 2.7 generalize Theorem C.

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