Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 7, Issue 2, Article 56, 2006

WALLIS INEQUALITY WITH A PARAMETER

YUEQING ZHAO AND QINGBIAO WU

DEPARTMENT OF MATHEMATICS
TAIZHOU UNIVERSITY
LINHAI 317000, ZHEJIANG
P.R. CHINA.
zhaoyq@tzc.edu.cn

DEPARTMENT OF MATHEMATICS
ZHEJIANG UNIVERSITY
HANGZHOU 310028, ZHEJIANG
P.R.CHINA.
qbwu@zju.edu.cn

Received 30 June, 2005; accepted 10 March, 2006
Communicated by S.S. Dragomir

ABSTRACT. We introduce a parameter z for the well-known Wallis' inequality, and improve results on Wallis' inequality are proposed. Recent results by other authors are also improved.

Key words and phrases: Wallis' inequality; Γ -function; Taylor formula.

2000 *Mathematics Subject Classification.* Primary 05A10, 26D20; Secondary 33B15.

1. INTRODUCTION

Wallis' inequality is a well-known and important inequality, it has wide applications in mathematics formulae, in particular, in combinatorics (see [1, 2]). It can be defined by the following expression,

$$\frac{1}{2\sqrt{n}} < \frac{1}{\sqrt{\pi(n+1/2)}} < P_n < \frac{1}{\sqrt{\pi(n+1/4)}} < \frac{1}{\sqrt{3n+1}} < \frac{1}{\sqrt{2n+1}} < \frac{1}{\sqrt{2n}}, \tag{1.1}$$

where

$$P_n = \frac{(2n-1)!!}{(2n)!!} = \frac{\left(1-\frac{1}{2}\right)\left(2-\frac{1}{2}\right)\cdot\left(n-\frac{1}{2}\right)}{n!}.$$

Improvements of the lower and upper bounds of P_n in (1.1) and some generalizations can be

ISSN (electronic): 1443-5756

This work is supported by National Natural Science Foundation of China. (No. 10471128).
199-05

found in [2] – [5]. The main results are

$$\frac{2(2n)!!}{\pi(2n+1)!!} < P_n < \frac{2(2n-2)!!}{\pi(2n-1)!!}. \tag{1.2}$$

$$\sqrt{\frac{8(n+1)}{(4n+3)(2n+1)\pi}} < P_n < \sqrt{\frac{4n+1}{(2n)(2n+1)\pi}}. \tag{1.3}$$

$$\frac{m-1}{m\sqrt[m]{n}} < \frac{(m-1)(2m-1)\cdot(nm-1)}{m(2m)\cdot(nm)} < \frac{m-1}{\sqrt[m]{n(m+1)+m-1}}. \tag{1.4}$$

$$\frac{m}{(m+1)\sqrt[m]{n}} < \frac{m(2m)\cdot(nm)}{(m+1)(2m+1)\cdot(nm+1)} < \frac{m-1}{\sqrt[m]{n(m+1)+m+4}}. \tag{1.5}$$

The largest lower bound $\frac{1}{\sqrt{\pi(n+1/2)}}$ and the smallest upper bound $\frac{1}{\sqrt{\pi(n+1/4)}}$ of P_n in (1.1) were presented by Kazarinoff in 1956 (see [1]). The following improvement of the bound in (1.1) can be found in [7]

$$\frac{1}{\sqrt{\pi n\left(1+\frac{1}{4n-1/2}\right)}} < P_n < \frac{1}{\sqrt{\pi n\left(1+\frac{1}{4n-1/3}\right)}}, \qquad (n \geq 1). \tag{1.6}$$

In [8] a new method of proof was proposed for the largest lower bound about Wallis' inequality, and in [9] the largest lower bound was improved. The result was

$$\frac{1}{\sqrt{\pi\left(n+\frac{4}{\pi}-1\right)}} < P_n < \frac{1}{\sqrt{\pi(n+1/4)}}, \qquad (n \geq 1). \tag{1.7}$$

In this paper, we introduce the parameter z, and use the $\Gamma(z)$ formula (Euler) below,

$$\Gamma(z) = \lim_{n\to\infty} \frac{(n-1)!n^z}{z(z+1)\cdot(n-1+z)} \quad \text{(see [6])}. \tag{1.8}$$

An improvement and generalization of Wallis' inequality is proposed.

For $0 < z < 1$, $n > 1$ and n a natural number,

$$\begin{aligned}\frac{1}{\Gamma(1-z)n^z(1+\frac{1-z}{2(n-1)})^z} &< \frac{(1-z)(2-z)\cdots(n-z)}{n!}\\ &< \frac{1}{\Gamma(1-z)n^z\left(1+\frac{1-z}{2n+1-z}\right)^z}\\ &< \frac{1}{\Gamma(1-z)n^z\left(1+\frac{1-z}{2n+1}\right)^z}.\end{aligned}$$

When $0 < z < 1$ and $n \geq 22$,

$$\frac{1}{\Gamma(1-z)n^z\left(1+\frac{1-z}{2(n-1)}\right)^z} < \frac{(1-z)(2-z)\cdots(n-z)}{n!} < \frac{1}{\Gamma(1-z)n^z\left(1+\frac{1-z}{2n}\right)^z}.$$

When $z = \frac{1}{2}$, we have Wallis' inequalities. The result in C.P. Chen and F. Qi [9] also is improved, and when $z = \frac{1}{m}$ or $z = 1 - \frac{1}{m}$, the results of (1.4), (1.5) are improved. When $n \geq 1,\ z = \frac{1}{2}$, and $0 < \varepsilon < \frac{1}{2}$, we also have,

$$\frac{1}{\sqrt{n\pi\left(1+\frac{1}{4n-1/2}\right)}} < P_n < \frac{1}{\sqrt{n\pi\left(1+\frac{1}{4n-1/2+\varepsilon}\right)}}.$$

The inequality on the right holds for $n > n^*$, where n^* is the maximal root of the equation

$$32\varepsilon n^2 + 4\varepsilon^2 n + 32\varepsilon n - 17n + 4\varepsilon^2 - 1 = 0.$$

The result in [7] also is improved.

2. SOME LEMMAS

Lemma 2.1. *When* $0 < z < 1, n > k \geq 1$*,*

$$\int_0^1 \frac{t^{n-z} - t^n}{1-t} dt = \frac{z}{n} + \frac{z(z-1)}{2n(n-1)} + \cdots + \frac{z(z-1)\cdots(z-k+1)}{kn(n-1)\cdots(n-k+1)} + \frac{z(z-1)\cdots(z-k)}{(k+1)n(n-1)\cdots(n-k+1)(n+\theta)},$$

where $-k < \theta < 1 - z$*.*

Proof. We easily get

$$\int_0^1 t^{n-k-1}(1-t)^k dt = \frac{k!}{n(n-1)\cdots(n-k)},$$

$$\begin{aligned}
&\frac{1}{n(n-1)\cdots(n-k)} \\
&> \frac{1}{n(n-1)\cdots(n-k)} + \frac{z-k-1}{k!}\int_0^1 dx \int_0^x \frac{(1-x)^{k+1}}{1-t} x^{z-k-2} t^{n-z} dt \\
&> \frac{1}{n(n-1)\cdots(n-k)} + \frac{z-k-1}{k!}\int_0^1 dx \int_0^x (1-x)^k x^{z-k-2} t^{n-z} dt \\
&= \frac{1}{n(n-1)\cdots(n-k)} + \frac{z-k-1}{k!(n+1-z)}\int_0^1 x^{n-k-1}(1-x)^k dx \\
&= \frac{1}{n(n-1)\cdots(n-k+1)(n+1-z)}.
\end{aligned}$$

Hence,

$$\frac{1}{n(n-1)\cdots(n-k)} + \frac{z-k-1}{k!}\int_0^1 dx \int_0^x \frac{(1-x)^{k+1}}{1-t} x^{z-k-2} t^{n-z} dt = \frac{1}{n(n-1)\cdots(n-k+1)(n+\theta)},$$

$(-k < \theta < 1 - z)$.

Denote $h(x) = x^z$, and let $0 < z < 1, n > k$, then, by Taylor's formula we have

$$\begin{aligned}
1 - t^z &= h(1) - h(t) \\
&= h'(t)(1-t) + \frac{h''(t)}{2}(1-t)^2 + \cdots \\
&\quad + \frac{h^{(k+1)}(t)}{(k+1)!}(1-t)^{k+1} + \frac{1}{(k+1)!}\int_t^1 h^{(k+2)}(x)(1-x)^{k+1} dx
\end{aligned}$$

$$= zt^{z-1}(1-t) + \frac{z(z-1)}{2!}t^{z-2}(1-t)^2 + \cdots + \frac{z(z-1)\cdots(z-k)}{(k+1)!}t^{z-k-1}(1-t)^{k+1}$$
$$+ \frac{z(z-1)\cdots(z-k-1)}{(k+1)!}\int_t^1 x^{z-k-2}(1-x)^{k+1}dx.$$

Hence,

$$\int_0^1 \frac{t^{n-z}-t^n}{1-t}dt = \int_0^1 zt^{n-1}dt + \frac{z(z-1)}{2}\int_0^1 t^{n-2}(1-t)dt + \cdots$$
$$+ \frac{z(z-1)\cdots(z-k+1)}{k!}\int_0^1 t^{n-k}(1-t)^{k-1}dt$$
$$+ \frac{z(z-1)\cdots(z-k)}{(k+1)!}\int_0^1 t^{n-k-1}(1-t)^k dt$$
$$+ \frac{z(z-1)\cdots(z-k-1)}{(k+1)!}\int_0^1 dt \int_t^1 \frac{(1-x)^{k+1}}{1-t}x^{z-k-2}t^{n-z}dx$$
$$= \frac{z}{n} + \frac{z(z-1)}{2n(n-1)} + \cdots + \frac{z(z-1)\cdots(z-k+1)}{kn(n-1)\cdots(n-k+1)}$$
$$+ \frac{z(z-1)\cdots(z-k)}{(k+1)}\left[\frac{1}{n(n-1)\cdots(n-k)}\right.$$
$$\left.+\frac{z-k-1}{k!}\int_0^1 dx\int_0^x \frac{(1-x)^{k+1}}{1-t}x^{z-k-2}t^{n-z}dt\right]$$
$$= \frac{z}{n} + \frac{z(z-1)}{2n(n-1)} + \cdots + \frac{z(z-1)\cdots(z-k+1)}{kn(n-1)\cdots(n-k+1)}$$
$$+ \frac{z(z-1)\cdots(z-k)}{(k+1)n(n-1)\cdots(n-k+1)(n+\theta)}.$$

Hence, the proof of Lemma 2.1 is completed. □

Let $k = 1$ or $k = 2$, we then have:

$$\frac{z}{n} - \frac{z(1-z)}{2n(n-1)} < \int_0^1 \frac{t^{n-z}-t^n}{1-t}dt < \frac{z}{n} - \frac{z(1-z)}{2n(n+1-z)}$$

and

$$\frac{z}{n} - \frac{z(1-z)}{2n(n-1)} < \int_0^1 \frac{t^{n-z}-t^n}{1-t}dt < \frac{z}{n} - \frac{z(1-z)}{2n(n-1)} + \frac{z(z-1)(z-2)}{3n(n-1)(n-2)}.$$

Lemma 2.2. *For $0 < z < 1$ and $n > 1$, let*

$$r_n(z) = \sum_{k=1}^{\infty}\left(\frac{1}{n+k-z} - \frac{1}{n+k}\right),$$

then

$$r_n(z) = \int_0^1 \frac{t^{n-z}-t^n}{1-t}dt.$$

Moreover,

$$\frac{z}{n} - \frac{z(1-z)}{2n(n-1)} < r_n(z) < \frac{z}{n} - \frac{z(1-z)}{2n(n+1-z)}, \tag{2.1}$$

$$\frac{z}{n} - \frac{z(1-z)}{2n(n-1)} < r_n(z) < \frac{z}{n} - \frac{z(1-z)}{2n(n-1)} + \frac{z(z-1)(z-2)}{3n(n-1)(n-2)}. \tag{2.2}$$

Proof. Let $g(t) = \sum_{k=1}^{\infty} \left(\frac{t^{n+k-z}}{n+k-z} - \frac{t^{n+k}}{n+k} \right)$, then $g(t)$ is convergent on $[0,1]$. Hence, $g(t)$ is continuous on $[0,1]$. Moreover, because

$$\sum_{k=1}^{\infty} \left(\frac{t^{n+k-z}}{n+k-z} - \frac{t^{n+k}}{n+k} \right)' = \sum_{k=1}^{\infty} (t^{n+k-z-1} - t^{n+k-1})$$

is continuous on the closed region of $[0,1)$, then, for $0 < t < 1$,

$$g'(t) = \sum_{k=1}^{\infty} (t^{n+k-z-1} - t^{n+k-1}) = \frac{t^{n-z} - t^n}{1-t}.$$

Moreover, $g(0) = 0$. So $g(x) = \int_0^x \frac{t^{n-z} - t^n}{1-t} dt$,

$$r_n(z) = \sum_{k=1}^{\infty} \left(\frac{1}{n+k-z} - \frac{1}{n+k} \right) = g(1),$$

Hence, $r_n(z) = \int_0^1 \frac{t^{n-z} - t^n}{1-t} dt$. The proof of Lemma 2.2 is completed. □

Lemma 2.3.

$$\frac{(1-z)(2-z)\cdots(n-z)n^z\Gamma(1-z)}{n!} = \frac{n-z}{n} \prod_{k=0}^{\infty} \left(1 - \frac{z}{n+k}\right)^{-1} \left(1 + \frac{1}{n+k}\right)^{-z}.$$

Proof. By (1.8), we know

$$\begin{aligned}
\Gamma(1-z) &= \lim_{n\to\infty} \frac{(n-1)!n^{1-z}}{(1-z)(2-z)\cdots(n-z)} \\
&= \lim_{n\to\infty} \frac{n!}{(1-z)(2-z)\cdots(n-z)n^z} \\
&= \lim_{n\to\infty} \prod_{k=1}^{n} \left(1 - \frac{z}{k}\right)^{-1} \prod_{k=1}^{n-1} \left(1 + \frac{1}{k}\right)^{-z} \\
&= \prod_{k=1}^{\infty} \left(1 - \frac{z}{k}\right)^{-1} \left(1 + \frac{1}{k}\right)^{-z} \\
&= \prod_{k=1}^{n-1} \left(1 - \frac{z}{k}\right)^{-1} \left(1 + \frac{1}{k}\right)^{-z} \cdot \prod_{k=n}^{\infty} \left(1 - \frac{z}{k}\right)^{-1} \left(1 + \frac{1}{k}\right)^{-z} \\
&= \frac{(n-1)!}{(1-z)(2-z)\cdots(n-1-z)n^z} \prod_{k=0}^{\infty} \left(1 - \frac{z}{n+k}\right)^{-1} \left(1 + \frac{1}{n+k}\right)^{-z}.
\end{aligned}$$

Hence,

$$\frac{(1-z)(2-z)\cdots(n-z)n^z\Gamma(1-z)}{n!} = \frac{n-z}{n} \prod_{k=0}^{\infty} \left(1 - \frac{z}{n+k}\right)^{-1} \left(1 + \frac{1}{n+k}\right)^{-z}.$$

The proof of Lemma 2.3 is completed. □

Lemma 2.4. *When* $n \geq 1$,

$$\frac{1}{2n} - \frac{1}{8n^2} < r_n\left(\frac{1}{2}\right) = \int_0^1 \frac{t^{n-1/2} - t^n}{1-t} dt < \frac{1}{2n+1} + \frac{1}{2(2n+1)^2}.$$

Proof. When $k > 1$,

$$\int_0^1 \frac{t^k}{1+t}dt = \frac{1}{k+1}\left(\frac{t^{k+1}}{1+t}\bigg|_0^1 + \int_0^1 \frac{t^k \cdot t}{(1+t)^2}dt \right) \tag{2.3}$$
$$< \frac{1}{2(k+1)} + \int_0^1 \frac{t^k}{4(k+1)}dt$$
$$= \frac{1}{2(k+1)} + \frac{1}{4(k+1)^2}.$$

By (2.3), we obtain

$$\int_0^1 \frac{t^k}{1+t}dt = \int_0^1 t^{k-1}dt - \int_0^1 \frac{t^{k-1}}{1+t}dt > \frac{1}{k} - \frac{1}{2k} - \frac{1}{4k^2} = \frac{1}{2k} - \frac{1}{4k^2}.$$

Moreover,

$$\int_0^1 \frac{t^{n-1/2} - t^n}{1-t}dt = \int_0^1 t^{n-1/2}\frac{1}{1+\sqrt{t}}dt = 2\int_0^1 \frac{x^{2n}}{1+x}dx.$$

Hence,

$$\frac{1}{2n} - \frac{1}{8n^2} < r_n\left(\frac{1}{2}\right) = \int_0^1 \frac{t^{n-1/2} - t^n}{1-t}dt < \frac{1}{2n+1} + \frac{1}{2(2n+1)^2}.$$

The proof of Lemma 2.4 is completed. □

3. MAIN THEOREMS

Denote $P_n(z) = \frac{(1-z)(2-z)\cdots(n-z)}{n!}$.

Theorem 3.1. *When* $0 < z < 1,\ n > 1$,

$$\frac{1}{n^z\Gamma(1-z)\left(1+\frac{1-z}{2(n-1)}\right)^z} < P_n(z)$$
$$< \frac{1}{n^z\Gamma(1-z)\left(1+\frac{1-z}{2n+1-z}\right)^z}$$
$$< \frac{1}{n^z\Gamma(1-z)\left(1+\frac{1-z}{2n+1}\right)^z}.$$

Proof. Let

$$F(n,z,\alpha) = \frac{(1-z)(2-z)\cdots(n-z)n^z\Gamma(1-z)}{n!}\left(1+\frac{1-z}{2n+\alpha}\right)^z.$$

By Lemma 2.3, we have, for $0 < z < 1,\ n > 1$,

$$\ln F(n,z,\alpha) = \ln\frac{n-z}{n} - \sum_{k=0}^{\infty}\left[\ln\left(1-\frac{z}{n+k}\right) + z\ln\left(1+\frac{1}{n+k}\right)\right] + z\ln\left(1+\frac{1-z}{2n+\alpha}\right).$$

We also know when $k \geq 1$,

$$\left|\ln\left(1-\frac{z}{n+k}\right) + z\ln\left(1+\frac{1}{n+k}\right)\right| \sim \left|\frac{z+z^2}{2(n+k)^2}\right| \leq \frac{1}{k^2},$$

because $\ln F(+\infty, z, \alpha) = 0$,

$$\text{(3.1)}\qquad \frac{\partial \ln F(n,z,\alpha)}{\partial n} = \frac{1}{n-z} - \frac{1}{n} - \sum_{k=0}^{\infty}\left(\frac{1}{n+k-z} - \frac{1}{n+k} + \frac{z}{n+k+1} - \frac{z}{n+k}\right) + \frac{2z}{2n+\alpha+1-z} - \frac{2z}{2n+\alpha}$$
$$= -\sum_{k=1}^{\infty}\left(\frac{1}{n+k-z} - \frac{1}{n+k}\right) + \frac{z}{n} - \frac{2z(1-z)}{(2n+\alpha+1-z)(2n+\alpha)}.$$

By Lemma 2.2 and (2.1), we can get

$$\frac{\partial \ln F(n,z,1-z)}{\partial n} = -\sum_{k=1}^{\infty}\left(\frac{1}{n+k-z} - \frac{1}{n+k}\right) + \frac{z}{n} - \frac{z(1-z)}{(n+1-z)(2n+1-z)}$$
$$> \frac{z(1-z)}{2n(n+1-z)} - \frac{z(1-z)}{(n+1-z)(2n+1-z)}$$
$$= \frac{z(1-z)^2}{2n(n+1-z)(2n+1-z)} > 0.$$

Hence, $\ln F(n, z, 1-z) < 0$. Moreover, we have $F(n, z, 1-z) < 1$, hence, the right inequality of Theorem 3.1 holds.

Since

$$\frac{\partial \ln F(n,z,-2)}{\partial n} = -\sum_{k=1}^{\infty}\left(\frac{1}{n+k-z} - \frac{1}{n+k}\right) + \frac{z}{n} - \frac{z(1-z)}{(2n-1-z)(n-1)}$$
$$< \frac{z(1-z)}{2n(n-1)} - \frac{z(1-z)}{(2n-1-z)(n-1)}$$
$$= -\frac{z(1-z)^2}{2n(n-1)(2n-1-z)} < 0.$$

Moreover, we pay attention to $\ln F(+\infty, z, \alpha) = 0$, hence, $\ln F(n, z, -2) > 0$. Thus we have $F(n, z, -2) > 1$ and the left inequality of Theorem 3.1 holds.

The proof of Theorem 3.1 is completed. □

Theorem 3.2. *For* $0 < z < 1$ *and* $n \geq 22$,

$$\frac{1}{\Gamma(1-z)n^z\left(1+\frac{1-z}{2(n-1)}\right)^z} < P_n(z) < \frac{1}{\Gamma(1-z)n^z\left(1+\frac{1-z}{2n}\right)^z}.$$

Proof. By (3.1), Lemma 2.2 and (2.2), we have

$$\frac{\partial \ln F(n,z,0)}{\partial n} = -\sum_{k=1}^{\infty}\left(\frac{1}{n+k-z} - \frac{1}{n+k}\right) + \frac{z}{n} - \frac{z(1-z)}{n(2n+1-z)}$$
$$> \frac{z(1-z)}{2n(n-1)} - \frac{z(1-z)(2-z)}{3n(n-1)(n-2)} - \frac{z(1-z)}{n(2n+1-z)}$$
$$= z(1-z)\left[\frac{1}{2n(n-1)} - \frac{2-z}{3n(n-1)(n-2)} - \frac{1}{n(2n+1-z)}\right]$$
$$= z(1-z)\frac{n-22+nz+z(12-2z)}{6n(n-1)(n-2)(2n+1-z)} > 0 \qquad (n \geq 22).$$

Since $\ln F(+\infty, z, \alpha) = 0$, then, $\ln F(n, z, 0) < 0$. Moreover, $F(n, z, 0) < 1$. So, the right inequality of Theorem 3.2 holds. The proof of Theorem 3.2 is completed. □

Theorem 3.3. *When* $n \geq 1,\ \ 0 < \varepsilon < \frac{1}{2}$,

$$\frac{1}{\sqrt{n\pi\left(1+\frac{1}{4n-1/2}\right)}} < \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)} < \frac{1}{\sqrt{n\pi\left(1+\frac{1}{4n-1/2+\varepsilon}\right)}}.$$

The right-hand inequality holds for $n > n^*$, *where* n^* *is the maximal root on*

$$32\varepsilon n^2 + 4\varepsilon^2 n + 32\varepsilon n - 17n + 4\varepsilon^2 - 1 = 0.$$

Proof. By Lemma 2.4 and (3.1)

$$\begin{aligned}\frac{\partial \ln F\left(n, \frac{1}{2}, -\frac{1}{4}\right)}{\partial n} &= -r_n\left(\frac{1}{2}\right) + \frac{1}{2n} - \frac{1}{2\left(2n-\frac{1}{4}\right)\left(2n+\frac{1}{4}\right)} \\ &< -\frac{1}{2n} + \frac{1}{8n^2} + \frac{1}{2n} - \frac{1}{2\left(4n^2 - \frac{1}{16}\right)} < 0.\end{aligned}$$

Using a similar line of proof as in Theorem 3.1,we obtain the left-hand inequality.

Now, for $0 < \varepsilon < \frac{1}{2}$, we have

$$\begin{aligned}&\frac{\partial \ln F\left(n, \frac{1}{2}, -\frac{1}{4}+\frac{\varepsilon}{2}\right)}{\partial n} \\ &= -r_n\left(\frac{1}{2}\right) + \frac{1}{2n} - \frac{1}{2\left(2n-\frac{1}{4}+\frac{\varepsilon}{2}\right)\left(2n+\frac{1}{4}+\frac{\varepsilon}{2}\right)} \\ &> -\frac{1}{2n+1} - \frac{1}{2(2n+1)^2} + \frac{1}{2n} - \frac{1}{2\left(2n-\frac{1}{4}+\frac{\varepsilon}{2}\right)\left(2n+\frac{1}{4}+\frac{\varepsilon}{2}\right)} \\ &= \frac{32\varepsilon n^2 + 4\varepsilon^2 n + 32\varepsilon n - 17n + 4\varepsilon^2 - 1}{32n(2n+1)^2\left(2n-\frac{1}{4}+\frac{\varepsilon}{2}\right)\left(2n+\frac{1}{4}+\frac{\varepsilon}{2}\right)} > 0 \qquad (n > n^*).\end{aligned}$$

n^* is the maximal root of the equation $32\varepsilon n^2 + 4\varepsilon^2 n + 32\varepsilon n - 17n + 4\varepsilon^2 - 1 = 0$.
Using a similar line of proof as in Theorem 3.1, we obtain the right-hand inequality. The proof of Theorem 3.3 is completed. □

By Theorem 3.1 and Theorem 3.2, we obtain the following corollaries.

Corollary 3.4. *When* $0 < z < 1$,

$$P_n(z) = \frac{1}{n^z\Gamma(1-z)\left(1+\frac{1-z}{2(n-\theta)}\right)^z}, \quad -\frac{1}{2} < \theta < 1 \qquad (n > 1),$$

$$P_n(z) = \frac{1}{n^z\Gamma(1-z)\left(1+\frac{1-z}{2(n-\theta)}\right)^z}, \quad 0 < \theta < 1 \qquad (n \geq 22).$$

Remark 3.5. Letting $z = 1/m$ or $z = 1 - 1/m$, we can obtain better results than (1.4) and (1.5).

When $z = 1/2$ and $n \geq 22$, by Theorem 3.2, we have

$$\frac{1}{n^{1/2}\Gamma\left(\frac{1}{2}\right)\left(1+\frac{1}{4(n-1)}\right)^{1/2}} < P_n\left(\frac{1}{2}\right) = P_n < \frac{1}{n^{1/2}\Gamma\left(\frac{1}{2}\right)\left(1+\frac{1}{4n}\right)^{1/2}},$$

that is,

$$\frac{1}{\sqrt{n\pi\left(1+\frac{1}{4(n-1)}\right)}} < P_n < \frac{1}{\sqrt{n\pi\left(1+\frac{1}{4n}\right)}}.$$

It can also be shown that the inequality holds when $n = 2, 3, \dots, 21$.

Corollary 3.6. *When* $n > 1$,

$$\frac{1}{\sqrt{n\pi\left(1+\frac{1}{4(n-1)}\right)}} < P_n < \frac{1}{\sqrt{n\pi\left(1+\frac{1}{4n}\right)}}.$$

Remark 3.7. Corollary 3.6 improves the result of C.P. Chen and F. Qi in [9].

By Theorem 3.3, when $\varepsilon = \frac{1}{6}$,

$$32\varepsilon n^2 + 4\varepsilon^2 n + 32\varepsilon n - 17n + 4\varepsilon^2 - 1 = \frac{8}{9}(6n^2 - 13n - 1).$$

Hence, when $n > 3$,

$$\frac{1}{\sqrt{\pi n\left(1+\frac{1}{4n-1/2}\right)}} < \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)} = P_n\left(\frac{1}{2}\right) < \frac{1}{\sqrt{\pi n\left(1+\frac{1}{4n-1/3}\right)}}.$$

It can also be shown that the inequality holds when $n = 1, 2, 3$.

The following corollary holds.

Corollary 3.8. *For* $n \geq 1$

$$\frac{1}{\sqrt{\pi n\left(1+\frac{1}{4n-1/2}\right)}} < \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)} = P_n\left(\frac{1}{2}\right) < \frac{1}{\sqrt{\pi n\left(1+\frac{1}{4n-1/3}\right)}}.$$

Remark 3.9. When ε is a positive number, we obtain other inequalities. For example, when $\varepsilon = 1/10$, we have, for $n > 1$,

$$\frac{1}{\sqrt{\pi n\left(1+\frac{1}{4n-1/2}\right)}} < \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)} = P_n\left(\frac{1}{2}\right) < \frac{1}{\sqrt{\pi n\left(1+\frac{1}{4n-2/5}\right)}} .$$

REFERENCES

[1] D.K. KAZARINOFF, On Wallis' formula, *Edinburgh Math. Notes*, **40** (1956) 19–21.

[2] N.D. KAZARINOFF, *Analytic Inequalities*, Holt, Rhinehart and Winston, New York, 1961.

[3] M. ABRAMOWITZ AND I.A. STEGUN (Eds.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series 55, 9th printing, Dover, New York, 1972.

[4] H. ALZER, On some inequalities for the gamma and psi functions, *Math. Comp.*, **66** (1997), 373–389.

[5] J.C. KUANG, *Applied Inequalities* (3^{rd} Edition), Shandong Science & Technology Press, (2004), 96–97.

[6] ZHUXI WANG AND DUNREN GUO, *Introduction to Special Functions* (in Chinese), Beijing, Peking University Press, (2000) 93.

[7] ZHAO DE JUN, On a Two-sided Inequality Involving Wallis's Formula, *Mathematics in Practice and Theory* (in Chinese), **34** (2004) 166–168.

[8] C.P. CHEN AND F. QI, A new proof of the best bounds in Wallis' inequality, *RGMIA Res. Rep. Coll.*, 2003 (6), No. 2, Art. 2, [ONLINE: `http://rgmia.vu.edu.au/v6n2.html`]

[9] C.P. CHEN AND F. QI, The best bounds In Wallis' inequality, *Proc. Amer. Math. Soc.*, **133** (2005) 397–401.