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# A POTPOURRI OF SCHWARZ RELATED INEQUALITIES IN INNER PRODUCT SPACES (II)

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ABSTRACT. Further inequalities related to the Schwarz inequality in real or complex inner product spaces are given.

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## 1. INTRODUCTION

Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$. One of the most important inequalities in inner product spaces with numerous applications is the Schwarz inequality, that may be written in two forms:

$$|\langle x,y\rangle|^2 \le \|x\|^2\|y\|^2, \quad x,y\in H \quad \text{(quadratic form)} \tag{1.1}$$

or, equivalently,

$$|\langle x,y\rangle| \le \|x\|\,\|y\|, \quad x,y\in H \quad \text{(simple form).} \tag{1.2}$$

The case of equality holds in (1.1) or in (1.2) if and only if the vectors $x$ and $y$ are linearly dependent.

In the previous paper [6], several results related to Schwarz inequalities have been established. We recall few of them below:

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1. If $x, y \in H\backslash\{0\}$ and $\|x\| \geq \|y\|$ , then

$$\|x\|\,\|y\| - \operatorname{Re}\langle x, y\rangle \leq \begin{cases} \frac{1}{2}r^2\left(\frac{\|x\|}{\|y\|}\right)^{r-1}\|x-y\|^2 & \text{if } r \geq 1 \\ \\ \frac{1}{2}\left(\frac{\|x\|}{\|y\|}\right)^{1-r}\|x-y\|^2 & \text{if } r < 1. \end{cases} \tag{1.3}$$

2. If $(H;\langle\cdot,\cdot\rangle)$ is complex, $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha, \operatorname{Im}\alpha > 0$ and $x, y \in H$ are such that

$$\left\|x - \frac{\operatorname{Im}\alpha}{\operatorname{Re}\alpha}\cdot y\right\| \leq r \tag{1.4}$$

then

$$\|x\|\,\|y\| - \operatorname{Re}\langle x, y\rangle \leq \frac{1}{2}\cdot\frac{\operatorname{Re}\alpha}{\operatorname{Im}\alpha}r^2. \tag{1.5}$$

3. If $\alpha \in \mathbb{K}\backslash\{0\}$ , then for any $x, y \in H$

$$\|x\|\,\|y\| - \operatorname{Re}\left[\frac{\alpha^2}{|\alpha|^2}\langle x, y\rangle\right] \leq \frac{1}{2}\cdot\frac{\left[|\operatorname{Re}\alpha|\,\|x-y\| + |\operatorname{Im}\alpha|\,\|x+y\|\right]^2}{|\alpha|^2}. \tag{1.6}$$

4. If $p \geq 1$, then for any $x, y \in H$ one has

$$\|x\|\,\|y\| - \operatorname{Re}\langle x, y\rangle \leq \frac{1}{2}\times\begin{cases} \left[(\|x\|+\|y\|)^{2p} - \|x+y\|^{2p}\right]^{\frac{1}{p}}, \\ \\ \left[\|x-y\|^{2p} - |\|x\|-\|y\||^{2p}\right]^{\frac{1}{p}}. \end{cases} \tag{1.7}$$

5. If $\alpha, \gamma > 0$ and $\beta \in \mathbb{K}$ with $|\beta|^2 \geq \alpha\gamma$ then for $x, a \in H$ with $a \neq 0$ and

$$\left\|x - \frac{\beta}{\alpha}a\right\| \leq \frac{\left(|\beta|^2 - \alpha\gamma\right)^{\frac{1}{2}}}{\alpha}\|a\|, \tag{1.8}$$

one has

$$\begin{aligned} \|x\|\,\|a\| &\leq \frac{\operatorname{Re}\beta\operatorname{Re}\langle x, a\rangle + \operatorname{Im}\beta\operatorname{Im}\langle x, a\rangle}{\sqrt{\alpha\gamma}} \\ &\leq \frac{|\beta|\,|\langle x, a\rangle|}{\sqrt{\alpha\gamma}} \end{aligned} \tag{1.9}$$

and

$$\|x\|^2\|a\|^2 - |\langle x, a\rangle|^2 \leq \frac{|\beta|^2 - \alpha\gamma}{\alpha\gamma}|\langle x, a\rangle|^2. \tag{1.10}$$

The aim of this paper is to provide other results related to the Schwarz inequality. Applications for reversing the generalised triangle inequality are also given.

## 2. Quadratic Schwarz Related Inequalities

The following result holds.

**Theorem 2.1.** *Let $(H;\langle\cdot,\cdot\rangle)$ be a complex inner product space and $x, y \in H$, $\alpha \in [0,1]$. Then*

$$\begin{aligned} &\left[\alpha\|ty-x\|^2 + (1-\alpha)\|ity-x\|^2\right]\|y\|^2 \\ &\qquad \geq \|x\|^2\|y\|^2 - \left[(1-\alpha)\operatorname{Im}\langle x, y\rangle + \alpha\operatorname{Re}\langle x, y\rangle\right]^2 \geq 0 \end{aligned} \tag{2.1}$$

*for any $t \in \mathbb{R}$.*

*Proof.* Firstly, recall that for a quadratic polynomial $P:\mathbb{R}\to\mathbb{R}$, $P(t)=at^2+2bt+c$, $a>0$, we have that

$$\inf_{t\in\mathbb{R}} P(t) = P\left(-\frac{b}{a}\right) = \frac{ac-b^2}{a}. \tag{2.2}$$

Now, consider the polynomial $P:\mathbb{R}\to\mathbb{R}$ given by

$$P(t) := \alpha\left\|ty-x\right\|^2 + (1-\alpha)\left\|ity-x\right\|^2. \tag{2.3}$$

Since

$$\left\|ty-x\right\|^2 = t^2\left\|y\right\|^2 - 2t\operatorname{Re}\langle x,y\rangle + \left\|x\right\|^2$$

and

$$\left\|ity-x\right\|^2 = t^2\left\|y\right\|^2 - 2t\operatorname{Im}\langle x,y\rangle + \left\|x\right\|^2,$$

hence

$$P(t) = t^2\left\|y\right\|^2 - 2t\left[\alpha\operatorname{Re}\langle x,y\rangle + (1-\alpha)\operatorname{Im}\langle x,y\rangle\right] + \left\|x\right\|^2.$$

By the definition of $P$ (see (2.3)), we observe that $P(t)\geq 0$ for every $t\in\mathbb{R}$, therefore $\frac{1}{4}\Delta\leq 0$, i.e.,

$$\left[(1-\alpha)\operatorname{Im}\langle x,y\rangle + \alpha\operatorname{Re}\langle x,y\rangle\right]^2 - \left\|x\right\|^2\left\|y\right\|^2 \leq 0,$$

proving the second inequality in (2.1).

The first inequality follows by (2.2) and the theorem is proved. □

The following particular cases are of interest.

**Corollary 2.2.** *For any $x,y\in H$ one has the inequalities:*

$$\left\|ty-x\right\|^2\left\|y\right\|^2 \geq \left\|\alpha\right\|^2\left\|y\right\|^2 - \left[\operatorname{Re}\langle x,y\rangle\right]^2 \geq 0; \tag{2.4}$$

$$\left\|ity-x\right\|^2\left\|y\right\|^2 \geq \left\|\alpha\right\|^2\left\|y\right\|^2 - \left[\operatorname{Im}\langle x,y\rangle\right]^2 \geq 0; \tag{2.5}$$

*and*

$$\frac{1}{2}\left[\left\|ty-x\right\|^2 + \left\|ity-x\right\|^2\right]\left\|y\right\|^2 \geq \left\|x\right\|^2\left\|y\right\|^2 - \left(\frac{\operatorname{Im}\langle x,y\rangle + \operatorname{Re}\langle x,y\rangle}{2}\right)^2 \geq 0, \tag{2.6}$$

*for any $t\in\mathbb{R}$.*

The following corollary may be stated as well:

**Corollary 2.3.** *Let $x,y\in H$ and $M_i,m_i\in\mathbb{R}$, $i\in\{1,2\}$ such that $M_i\geq m_i>0$, $i\in\{1,2\}$. If either*

$$\operatorname{Re}\langle M_1y-x, x-m_1y\rangle \geq 0 \quad \textit{and} \quad \operatorname{Re}\langle M_2iy-x, x-im_2y\rangle \geq 0, \tag{2.7}$$

*or, equivalently,*

$$\left\|x-\frac{M_1+m_1}{2}y\right\| \leq \frac{1}{2}(M_1-m_1)\left\|y\right\| \quad \textit{and} \tag{2.8}$$
$$\left\|x-\frac{M_2+m_2}{2}iy\right\| \leq \frac{1}{2}(M_2-m_2)\left\|y\right\|$$

*hold, then*

$$\begin{aligned}(0\leq)\left\|x\right\|^2\left\|y\right\|^2 &- \left[(1-\alpha)\operatorname{Im}\langle x,y\rangle + \alpha\operatorname{Re}\langle x,y\rangle\right]^2 \\ &\leq \frac{1}{4}\left\|y\right\|^4\left[\alpha(M_1-m_1)^2 + (1-\alpha)(M_2-m_2)^2\right]\end{aligned} \tag{2.9}$$

*for any $\alpha\in[0,1]$.*

*Proof.* It is easy to see that, if $x,z,Z\in H$, then the following statements are equivalent:

(i) $\operatorname{Re}\langle Z-x, x-z\rangle \geq 0,$
(ii) $\left\|x-\frac{z+Z}{2}\right\| \leq \frac{1}{2}\|Z-z\|.$

Utilising this property one may simply realize that the statements (2.7) and (2.8) are equivalent.

Now, on making use of (2.8) and (2.1), one may deduce the desired inequality (2.9). □

**Remark 2.4.** If one assumes that $M_1 = M_2 = M$, $m_1 = m_2 = m$ in either (2.7) or (2.8), then

$$(0 \leq) \|x\|^2 \|y\|^2 - [(1-\alpha)\operatorname{Im}\langle x,y\rangle + \alpha \operatorname{Re}\langle x,y\rangle]^2 \leq \frac{1}{4}\|y\|^4 (M-m)^2 \tag{2.10}$$

for each $\alpha \in [0,1]$.

**Remark 2.5.** Corollary 2.3 may be seen as a potential source of some reverse results for the Schwarz inequality. For instance, if $x, y \in H$ and $M \geq m > 0$ are such that either

$$\operatorname{Re}\langle My - x, x - my\rangle \geq 0 \quad \text{or} \quad \left\|x - \frac{M+m}{2}y\right\| \leq \frac{1}{2}(M-m)\|y\| \tag{2.11}$$

hold, then

$$(0 \leq) \|x\|^2 \|y\|^2 - [\operatorname{Re}\langle x,y\rangle]^2 \leq \frac{1}{4}(M-m)^2 \|y\|^4. \tag{2.12}$$

If $x, y \in H$ and $N \geq n > 0$ are such that either

$$\operatorname{Re}\langle Niy - x, x - niy\rangle \geq 0 \quad \text{or} \quad \left\|x - \frac{N+n}{2}iy\right\| \leq \frac{1}{2}(N-n)\|y\| \tag{2.13}$$

hold, then

$$(0 \leq) \|x\|^2 \|y\|^2 - [\operatorname{Im}\langle x,y\rangle]^2 \leq \frac{1}{4}(N-n)^2 \|y\|^4. \tag{2.14}$$

We notice that (2.12) is an improvement of the inequality

$$(0 \leq) \|x\|^2 \|y\|^2 - |\langle x,y\rangle|^2 \leq \frac{1}{4}(M-m)^2 \|y\|^4$$

that has been established in [4] under the same condition (2.11) given above.

The following result may be stated as well.

**Theorem 2.6.** *Let $(H; \langle\cdot,\cdot\rangle)$ be a real or complex inner product space and $x, y \in H$, $\alpha \in [0,1]$. Then*

$$\begin{aligned}\left[\alpha\|ty-x\|^2 + (1-\alpha)\|y-tx\|^2\right]&\left[\alpha\|y\|^2 + (1-\alpha)\|x\|^2\right] \\ &\geq \left[(1-\alpha)\|x\|^2 + \alpha\|y\|^2\right]\left[\alpha\|x\|^2 + (1-\alpha)\|y\|^2\right] - [\operatorname{Re}\langle x,y\rangle]^2 \geq 0\end{aligned} \tag{2.15}$$

*for any $t \in \mathbb{R}$.*

*Proof.* Consider the polynomial $P: \mathbb{R} \to \mathbb{R}$ given by

$$P(t) := \alpha\|ty-x\|^2 + (1-\alpha)\|y-tx\|^2. \tag{2.16}$$

Since

$$\|ty-x\|^2 = t^2\|y\|^2 - 2t\operatorname{Re}\langle x,y\rangle + \|x\|^2$$

and

$$\|y-tx\|^2 = t^2\|x\|^2 - 2t\operatorname{Re}\langle x,y\rangle + \|y\|^2,$$

hence

$$P(t) = \left[\alpha\|y\|^2 + (1-\alpha)\|x\|^2\right]t^2 - 2t\operatorname{Re}\langle x,y\rangle + \left[\alpha\|x\|^2 + (1-\alpha)\|y\|^2\right]$$

for any $t \in \mathbb{R}$.

By the definition of $P$ (see (2.16)), we observe that $P(t) \geq 0$ for every $t \in \mathbb{R}$, therefore $\frac{1}{4}\Delta \leq 0$, i.e., the second inequality in (2.15) holds true.

The first inequality follows by (2.2) and the theorem is proved. □

**Remark 2.7.** We observe that if either $\alpha = 0$ or $\alpha = 1$, then (2.15) will generate the same reverse of the Schwarz inequality as (2.4) does.

**Corollary 2.8.** *If $x, y \in H$, then*

$$\frac{\|ty - x\|^2 + \|y - tx\|^2}{2} \cdot \frac{\|x\|^2 + \|y\|^2}{2} \geq \left(\frac{\|x\|^2 + \|y\|^2}{2}\right)^2 - [\operatorname{Re}\langle x, y\rangle]^2 \geq 0 \tag{2.17}$$

*for any $t \in \mathbb{R}$ and*

$$\begin{aligned} \|x \pm y\|^2 \left[\alpha \|y\|^2 + (1-\alpha)\|x\|^2\right] \\ \geq \left[(1-\alpha)\|x\|^2 + \alpha\|y\|^2\right]\left[\alpha\|x\|^2 + (1-\alpha)\|y\|^2\right] - [\operatorname{Re}\langle x, y\rangle]^2 \geq 0 \end{aligned} \tag{2.18}$$

*for any $\alpha \in [0,1]$.*

*In particular, we have*

$$\|x \pm y\|^2 \cdot \left(\frac{\|x\|^2 + \|y\|^2}{2}\right) \geq \left(\frac{\|x\|^2 + \|y\|^2}{2}\right)^2 - [\operatorname{Re}\langle x, y\rangle]^2 \geq 0. \tag{2.19}$$

In [7, p. 210], C.S. Lin has proved the following reverse of the Schwarz inequality in real or complex inner product spaces $(H; \langle\cdot,\cdot\rangle)$ :

$$(0 \leq)\, \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \leq \frac{1}{r^2} \|x\|^2 \|x - ry\|^2$$

for any $r \in \mathbb{R}$, $r \neq 0$ and $x, y \in H$.

The following slightly more general result may be stated:

**Theorem 2.9.** *Let $(H; \langle\cdot,\cdot\rangle)$ be a real or complex inner product space. Then for any $x, y \in H$ and $\alpha \in \mathbb{K}\,(\mathbb{C}, \mathbb{R})$ with $\alpha \neq 0$ we have*

$$(0 \leq)\, \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \leq \frac{1}{|\alpha|^2} \|x\|^2 \|x - \alpha y\|^2. \tag{2.20}$$

*The case of equality holds in (2.20) if and only if*

$$\operatorname{Re}\langle x, \alpha y\rangle = \|x\|^2. \tag{2.21}$$

*Proof.* Observe that

$$\begin{aligned} I(\alpha) &:= \|x\|^2 \|x - \alpha y\|^2 - |\alpha|^2 \left[\|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2\right] \\ &= \|x\|^2 \left[\|x\|^2 - 2\operatorname{Re}\left[\bar{\alpha}\langle x, y\rangle\right] + |\alpha|^2 \|y\|^2\right] \\ &\qquad - |\alpha|^2 \|x\|^2 \|y\|^2 + |\alpha|^2 |\langle x, y\rangle|^2 \\ &= \|x\|^4 - 2\|x\|^2 \operatorname{Re}\left[\bar{\alpha}\langle x, y\rangle\right] + |\alpha|^2 |\langle x, y\rangle|^2. \end{aligned}$$

Since

$$\operatorname{Re}\left[\bar{\alpha}\langle x, y\rangle\right] \leq |\alpha|\,|\langle x, y\rangle|, \tag{2.22}$$

hence

$$\begin{aligned} I(\alpha) &\geq \|x\|^4 - 2\|x\|^2 |\alpha|\,|\langle x, y\rangle| + |\alpha|^2 |\langle x, y\rangle|^2 \\ &= \left(\|x\|^2 - |\alpha|\,|\langle x, y\rangle|\right)^2 \geq 0. \end{aligned} \tag{2.23}$$

Conversely, if (2.21) holds true, then $I(\alpha) = 0$, showing that the equality case holds in (2.20).

Now, if the equality case holds in (2.20), then we must have equality in (2.22) and in (2.23) implying that

$$\operatorname{Re}[\langle x, \alpha y\rangle] = |\alpha| |\langle x, y\rangle| \quad \text{and} \quad |\alpha| |\langle x, y\rangle| = \|x\|^2$$

which imply (2.21). □

The following result may be stated.

**Corollary 2.10.** *Let* $(H; \langle\cdot,\cdot\rangle)$ *be as above and* $x, y \in H$, $\alpha \in \mathbb{K}\backslash\{0\}$ *and* $r > 0$ *such that* $|\alpha| \geq r$. *If*

$$\|x - \alpha y\| \leq r \|y\|, \tag{2.24}$$

*then*

$$\frac{\sqrt{|\alpha|^2 - r^2}}{|\alpha|} \leq \frac{|\langle x, y\rangle|}{\|x\| \|y\|} (\leq 1). \tag{2.25}$$

*Proof.* From (2.24) and (2.20) we have

$$\|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \leq \frac{r^2}{|\alpha|^2} \|x\|^2 \|y\|^2,$$

that is,

$$\frac{\left(|\alpha|^2 - r^2\right)}{|\alpha|^2} \|x\|^2 \|y\|^2 \leq |\langle x, y\rangle|^2,$$

which is clearly equivalent to (2.25). □

**Remark 2.11.** Since for $\Gamma, \gamma \in \mathbb{K}$ the following statements are equivalent

(i) $\operatorname{Re}\langle \Gamma y - x, x - \gamma y\rangle \geq 0$,
(ii) $\left\|x - \frac{\gamma+\Gamma}{2} \cdot y\right\| \leq \frac{1}{2} |\Gamma - \gamma| \|y\|$,

hence by Corollary 2.10 we deduce

$$\frac{2\left[\operatorname{Re}(\Gamma\bar{\gamma})\right]^{\frac{1}{2}}}{|\Gamma + \gamma|} \leq \frac{|\langle x, y\rangle|}{\|x\| \|y\|}, \tag{2.26}$$

provided $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$, an inequality that has been obtained in a different way in [3].

**Corollary 2.12.** *If* $x, y \in H$, $\alpha \in \mathbb{K}\backslash\{0\}$ *and* $\rho > 0$ *such that* $\|x - \alpha y\| \leq \rho$, *then*

$$(0 \leq) \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \leq \frac{\rho^2}{|\alpha|^2} \|x\|^2. \tag{2.27}$$

## 3. Other Inequalities

The following result holds.

**Proposition 3.1.** *Let* $x, y \in H \setminus \{0\}$ *and* $\varepsilon \in (0, \frac{1}{2}]$. *If*

$$(0 \leq) 1 - \varepsilon - \sqrt{1 - 2\varepsilon} \leq \frac{\|x\|}{\|y\|} \leq 1 - \varepsilon + \sqrt{1 - 2\varepsilon}, \tag{3.1}$$

*then*

$$(0 \leq) \|x\| \|y\| - \operatorname{Re}\langle x, y\rangle \leq \varepsilon \|x - y\|^2. \tag{3.2}$$

*Proof.* If $x = y$, then (3.2) is trivial.

Suppose $x \neq y$. Utilising the inequality (2.5) from [6], we can state that

$$\frac{\|x\| \|y\| - \operatorname{Re}\langle x, y\rangle}{\|x - y\|^2} \leq \frac{2\|x\| \|y\|}{(\|x\| + \|y\|)^2}$$

for any $x, y \in H \setminus \{0\}, x \neq y$.

Now, if we assume that

$$\frac{2\|x\| \|y\|}{(\|x\| + \|y\|)^2} \leq \varepsilon,$$

then, after some manipulation, we get that

$$\varepsilon \|x\|^2 + 2(\varepsilon - 1)\|x\| \|y\| + \varepsilon \|y\|^2 \geq 0,$$

which, for $\varepsilon \in (0, \frac{1}{2}]$ and $y \neq 0$, is clearly equivalent to (3.1).

The proof is complete. □

The following result may be stated:

**Proposition 3.2.** *Let $(H; \langle \cdot, \cdot \rangle)$ be a real or complex inner product space. Then for any $x, y \in H$ and $\varphi \in \mathbb{R}$ one has:*

$$\|x\| \|y\| - [\cos 2\varphi \cdot \operatorname{Re}\langle x, y\rangle + \sin 2\varphi \cdot \operatorname{Im}\langle x, y\rangle] \leq \frac{1}{2}[|\cos\varphi| \|x - y\| + |\sin\varphi| \|x + y\|]^2.$$

*Proof.* For $\varphi \in \mathbb{R}$, consider the complex number $\alpha = \cos\varphi - i\sin\varphi$. Then $\alpha^2 = \cos 2\varphi - i\sin 2\varphi$, $|\alpha| = 1$ and by the inequality (1.6) we deduce the desired result. □

From a different perspective, we may consider the following results as well:

**Theorem 3.3.** *Let $(H; \langle \cdot, \cdot \rangle)$ be a real or complex inner product space, $\alpha \in \mathbb{K}$ with $|\alpha - 1| = 1$. Then for any $e \in H$ with $\|e\| = 1$ and $x, y \in H$, we have*

$$|\langle x, y\rangle - \alpha \langle x, e\rangle \langle e, y\rangle| \leq \|x\| \|y\|. \tag{3.3}$$

*The equality holds in (3.3) if and only if there exists a $\lambda \in \mathbb{K}$ such that*

$$\alpha \langle x, e\rangle e = x + \lambda y. \tag{3.4}$$

*Proof.* It is known that for $u, v \in H$, we have equality in the Schwarz inequality

$$|\langle u, v\rangle| \leq \|u\| \|v\| \tag{3.5}$$

if and only if there exists a $\lambda \in \mathbb{K}$ such that $u = \lambda v$.

If we apply (3.5) for $u = \alpha \langle x, e\rangle e - x$, $v = y$, we get

$$|\langle \alpha \langle x, e\rangle e - x, y\rangle| \leq \|\alpha \langle x, e\rangle e - x\| \|y\| \tag{3.6}$$

with equality iff there exists a $\lambda \in \mathbb{K}$ such that

$$\alpha \langle x, e\rangle e = x + \lambda y.$$

Since

$$\begin{aligned}
\|\alpha \langle x, e\rangle e - x\|^2 &= |\alpha|^2 |\langle x, e\rangle|^2 - 2\operatorname{Re}[\alpha] |\langle x, e\rangle|^2 + \|x\|^2 \\
&= \left(|\alpha|^2 - 2\operatorname{Re}[\alpha]\right) |\langle x, e\rangle|^2 + \|x\|^2 \\
&= \left(|\alpha - 1|^2 - 1\right) |\langle x, e\rangle|^2 + \|x\|^2 \\
&= \|x\|^2
\end{aligned}$$

and

$$\langle \alpha \langle x, e \rangle e - x, y \rangle = \alpha \langle x, e \rangle \langle e, y \rangle - \langle x, y \rangle$$

hence by (3.6) we deduce the desired inequality (3.3). □

**Remark 3.4.** If $\alpha = 0$ in (3.3), then it reduces to the Schwarz inequality.

**Remark 3.5.** If $\alpha \neq 0$, then (3.3) is equivalent to

$$\left| \langle x, e \rangle \langle e, y \rangle - \frac{1}{\alpha} \langle x, y \rangle \right| \leq \frac{1}{|\alpha|} \|x\| \|y\| . \tag{3.7}$$

Utilising the continuity property of modulus for complex numbers, i.e., $|z - w| \geq ||z| - |w||$ we then obtain

$$\left| |\langle x, e \rangle \langle e, y \rangle| - \frac{1}{|\alpha|} |\langle x, y \rangle| \right| \leq \frac{1}{|\alpha|} \|x\| \|y\| ,$$

which implies that

$$|\langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{|\alpha|} [|\langle x, y \rangle| + \|x\| \|y\|] . \tag{3.8}$$

For $e = \frac{z}{\|z\|}$, $z \neq 0$, we get from (3.8) that

$$|\langle x, z \rangle \langle z, y \rangle| \leq \frac{1}{|\alpha|} [|\langle x, y \rangle| + \|x\| \|y\|] \|z\|^2 \tag{3.9}$$

for any $\alpha \in \mathbb{K} \backslash \{0\}$ with $|\alpha - 1| = 1$ and $x, y, z \in H$.

For $\alpha = 2$, we get from (3.9) the *Buzano inequality* [1]

$$|\langle x, z \rangle \langle z, y \rangle| \leq \frac{1}{2} [|\langle x, y \rangle| + \|x\| \|y\|] \|z\|^2 \tag{3.10}$$

for any $x, y, z \in H$.

**Remark 3.6.** In the case of real spaces the condition $|\alpha - 1| = 1$ is equivalent to either $\alpha = 0$ or $\alpha = 2$. For $\alpha = 2$ we deduce from (3.7) that

$$\frac{1}{2} [\langle x, y \rangle - \|x\| \|y\|] \leq \langle x, e \rangle \langle e, y \rangle \leq \frac{1}{2} [\langle x, y \rangle + \|x\| \|y\|] \tag{3.11}$$

for any $x, y \in H$ and $e \in H$ with $\|e\| = 1$, which implies Richard's inequality [8]:

$$\frac{1}{2} [\langle x, y \rangle - \|x\| \|y\|] \|z\|^2 \leq \langle x, z \rangle \langle z, y \rangle \leq \frac{1}{2} [\langle x, y \rangle + \|x\| \|y\|] \|z\|^2 , \tag{3.12}$$

for any $x, y, z \in H$.

The following result concerning a generalisation for orthornormal families of the inequality (3.3) may be stated.

**Theorem 3.7.** *Let $(H; \langle \cdot, \cdot \rangle)$ be a real or complex inner product space, $\{e_i\}_{i \in F}$ a finite orthonormal family, i.e., $\langle e_i, e_j \rangle = \delta_{ij}$ for $i, j \in F$, where $\delta_{ij}$ is Kronecker's delta and $\alpha_i \in \mathbb{K}$, $i \in F$ such that $|\alpha_i - 1| = 1$ for each $i \in F$. Then*

$$\left| \langle x, y \rangle - \sum_{i \in F} \alpha_i \langle x, e_i \rangle \langle e_i, y \rangle \right| \leq \|x\| \|y\| . \tag{3.13}$$

*The equality holds in (3.13) if and only if there exists a constant $\lambda \in \mathbb{K}$ such that*

$$\sum_{i \in F} \alpha_i \langle x, e_i \rangle e_i = x + \lambda y. \tag{3.14}$$

*Proof.* As above, by Schwarz's inequality, we have

$$\left|\left\langle \sum_{i\in F} \alpha_i \langle x, e_i\rangle e_i - x, y\right\rangle\right| \leq \left\|\sum_{i\in F} \alpha_i \langle x, e_i\rangle e_i - x\right\| \|y\| \tag{3.15}$$

with equality if and only if there exists a $\lambda \in \mathbb{K}$ such that $\sum_{i\in F} \alpha_i \langle x, e_i\rangle e_i = x + \lambda y$.

Since

$$\begin{aligned}\left\|\sum_{i\in F} \alpha_i \langle x, e_i\rangle e_i - x\right\|^2 &= \|x\|^2 - 2\operatorname{Re}\left\langle x, \sum_{i\in F} \alpha_i \langle x, e_i\rangle e_i\right\rangle + \left\|\sum_{i\in F} \alpha_i \langle x, e_i\rangle e_i\right\|^2 \\ &= \|x\|^2 - 2\sum_{i\in F} \overline{\alpha_i} \langle x, e_i\rangle \overline{\langle x, e_i\rangle} + \sum_{i\in F} |\alpha_i|^2 |\langle x, e_i\rangle|^2 \\ &= \|x\|^2 + \sum_{i\in F} |\langle x, e_i\rangle|^2 \left(|\alpha_i|^2 - 2\operatorname{Re}\alpha_i\right) \\ &= \|x\|^2 + \sum_{i\in F} |\langle x, e_i\rangle|^2 \left(|\alpha_i - 1|^2 - 1\right) \\ &= \|x\|^2,\end{aligned}$$

hence the inequality (3.13) is obtained. □

**Remark 3.8.** If the space is real, then the nontrivial case one can get from (3.13) is for all $\alpha_i = 2$, obtaining the inequality

$$\frac{1}{2}\left[\langle x, y\rangle - \|x\| \|y\|\right] \leq \sum_{i\in F} \langle x, e_i\rangle \langle e_i, y\rangle \leq \frac{1}{2}\left[\langle x, y\rangle + \|x\| \|y\|\right] \tag{3.16}$$

that has been obtained in [5].

**Corollary 3.9.** *With the above assumptions, we have*

$$\begin{aligned}\left|\sum_{i\in F} \alpha_i \langle x, e_i\rangle \langle e_i, y\rangle\right| &\leq |\langle x, y\rangle| + \left|\langle x, y\rangle - \sum_{i\in F} \alpha_i \langle x, e_i\rangle \langle e_i, y\rangle\right| \\ &\leq |\langle x, y\rangle| + \|x\| \|y\|, \qquad x, y \in H,\end{aligned} \tag{3.17}$$

*where* $|\alpha_i - 1| = 1$ *for each* $i \in F$ *and* $\{e_i\}_{i\in F}$ *is an orthonormal family in* $H$.

## 4. APPLICATIONS FOR THE TRIANGLE INEQUALITY

In 1966, Diaz and Metcalf [2] proved the following reverse of the triangle inequality:

$$\left\|\sum_{i=1}^{n} x_i\right\| \geq r \sum_{i=1}^{n} \|x_i\|, \tag{4.1}$$

provided the vectors $x_i$ in the inner product space $(H; \langle\cdot,\cdot\rangle)$ over the real or complex number field $\mathbb{K}$ are nonzero and

$$0 \leq r \leq \frac{\operatorname{Re}\langle x_i, a\rangle}{\|x_i\|} \quad \text{for each } i \in \{1, \ldots, n\}, \tag{4.2}$$

where $a \in H$, $\|a\| = 1$. The equality holds in (4.2) if and only if

$$\sum_{i=1}^{n} x_i = r\left(\sum_{i=1}^{n} \|x_i\|\right) a. \tag{4.3}$$

The following result may be stated:

**Proposition 4.1.** *Let* $e \in H$ *with* $\|e\| = 1$, $\varepsilon \in \left(0, \frac{1}{2}\right]$ *and* $x_i \in H$, $i \in \{1, \dots, n\}$ *with the property that*

$$(0 \leq)\, 1 - \varepsilon - \sqrt{1 - 2\varepsilon} \leq \|x_i\| \leq 1 - \varepsilon + \sqrt{1 - 2\varepsilon} \tag{4.4}$$

*for each* $i \in \{1, \dots, n\}$ *. Then*

$$\sum_{i=1}^{n} \|x_i\| \leq \left\| \sum_{i=1}^{n} x_i \right\| + \varepsilon \sum_{i=1}^{n} \|x_i - e\|^2 . \tag{4.5}$$

*Proof.* Utilising Proposition 3.1 for $x = x_i$ and $y = e$, we can state that

$$\|x_i\| - \operatorname{Re}\langle x_i, e\rangle \leq \varepsilon \|x_i - e\|^2$$

for each $i \in \{1, \dots, n\}$ . Summing over $i$ from 1 to $n$, we deduce that

$$\sum_{i=1}^{n} \|x_i\| \leq \operatorname{Re}\left\langle \sum_{i=1}^{n} x_i, e \right\rangle + \varepsilon \sum_{i=1}^{n} \|x_i - e\|^2 . \tag{4.6}$$

By Schwarz's inequality in $(H; \langle\cdot,\cdot\rangle)$ , we also have

$$\begin{aligned} \operatorname{Re}\left\langle \sum_{i=1}^{n} x_i, e \right\rangle &\leq \left| \operatorname{Re}\left\langle \sum_{i=1}^{n} x_i, e \right\rangle \right| \leq \left| \left\langle \sum_{i=1}^{n} x_i, e \right\rangle \right| \\ &\leq \left\| \sum_{i=1}^{n} x_i \right\| \|e\| = \left\| \sum_{i=1}^{n} x_i \right\| . \end{aligned} \tag{4.7}$$

Therefore, by (4.6) and (4.7) we deduce (4.5). □

In the same spirit, we can prove the following result as well:

**Proposition 4.2.** *Let* $(H; \langle\cdot,\cdot\rangle)$ *be a real or complex inner product space and* $e \in H$ *with* $\|e\| = 1$. *Then for any* $\varphi \in \mathbb{R}$ *one has the inequality:*

$$\sum_{i=1}^{n} \|x_i\| \leq \left\| \sum_{i=1}^{n} x_i \right\| + \frac{1}{2} \sum_{i=1}^{n} \left[ |\cos\varphi| \|x_i - e\| + |\sin\varphi| \|x_i + e\| \right]^2 . \tag{4.8}$$

*Proof.* Applying Proposition 3.2 for $x = x_i$ and $y = e$, we have:

$$\begin{aligned} \|x_i\| &\leq \cos 2\varphi \cdot \operatorname{Re}\langle x_i, e\rangle + \sin 2\varphi \cdot \operatorname{Im}\langle x_i, e\rangle \\ &\quad + \frac{1}{2} \left[ |\cos\varphi| \|x_i - e\| + |\sin\varphi| \|x_i + e\| \right]^2 \end{aligned} \tag{4.9}$$

for any $i \in \{1, \dots, n\}$ .

Summing in (4.5) over $i$ from 1 to $n$, we have:

$$\begin{aligned} \sum_{i=1}^{n} \|x_i\| &\leq \cos 2\varphi \cdot \operatorname{Re}\left\langle \sum_{i=1}^{n} x_i, e \right\rangle + \sin 2\varphi \cdot \operatorname{Im}\left\langle \sum_{i=1}^{n} x_i, e \right\rangle \\ &\quad + \frac{1}{2} \sum_{i=1}^{n} \left[ |\cos\varphi| \|x_i - e\| + |\sin\varphi| \|x_i + e\| \right]^2 . \end{aligned} \tag{4.10}$$

Now, by the elementary inequality for the real numbers $m, p, M$ and $P$,

$$mM + pP \leq \left(m^2 + p^2\right)^{\frac{1}{2}} \left(M^2 + P^2\right)^{\frac{1}{2}} ,$$

we have

$$\begin{aligned}(4.11)\quad \cos 2\varphi \cdot \operatorname{Re}\left\langle \sum_{i=1}^{n} x_i, e\right\rangle + \sin 2\varphi \cdot \operatorname{Im}\left\langle \sum_{i=1}^{n} x_i, e\right\rangle \\ \leq \left(\cos^2 2\varphi + \sin^2 2\varphi\right)^{\frac{1}{2}} \left(\left[\operatorname{Re}\left\langle \sum_{i=1}^{n} x_i, e\right\rangle\right]^2 + \left[\operatorname{Im}\left\langle \sum_{i=1}^{n} x_i, e\right\rangle\right]^2\right)^{\frac{1}{2}} \\ = \left|\left\langle \sum_{i=1}^{n} x_i, e\right\rangle\right| \leq \left\|\sum_{i=1}^{n} x_i\right\| \|e\| = \left\|\sum_{i=1}^{n} x_i\right\|,\end{aligned}$$

where for the last inequality we used Schwarz's inequality in $(H; \langle\cdot,\cdot\rangle)$.

Finally, by (4.10) and (4.11) we deduce the desired result (4.8). □

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