

# Journal of Inequalities in Pure and Applied Mathematics

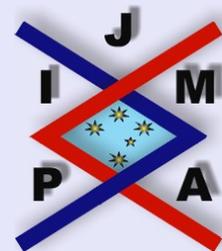
## ANOTHER REFINEMENT OF BERNSTEIN'S INEQUALITY

CLÉMENT FRAPPIER

Département de Mathématiques et de Génie Industriel  
École Polytechnique de Montréal  
C.P. 6079, succ. Centre-ville  
Montréal (Québec) H3C 3A7  
Canada

*EMail:* [clement.frappier@polymtl.ca](mailto:clement.frappier@polymtl.ca)

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## Abstract

Given a polynomial  $p(z) = \sum_{j=0}^n a_j z^j$ , we denote by  $\| \|$  the maximum norm on the unit circle  $\{z: |z| = 1\}$ . We obtain a characterization of the best possible constant  $x_n \geq \frac{1}{2}$  such that the inequality  $\|zp'(z) - xa_n z^n\| \leq (n-x)\|p\|$  holds for  $0 \leq x \leq x_n$ .

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*Key words:* Bernstein's inequality, Unit circle, Convolution method.

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# 1. Introduction and Statements of the Results

We denote by  $\mathcal{P}_n$  the class of all polynomials with complex coefficients, of degree  $\leq n$ :

$$(1.1) \quad p(z) = \sum_{j=0}^n a_j z^j.$$

Let  $\|p\| := \max_{|z|=1} |p(z)|$ . The classical inequality

$$(1.2) \quad \|p'\| \leq n\|p\|$$

is known as Bernstein's inequality. A great number of refinements and generalizations of (1.2) have been obtained. See [4, Part III] for an extensive study of that subject. An example of refinement is [2, p. 84]

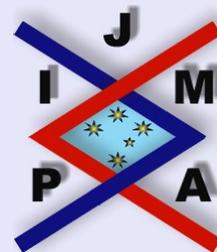
$$(1.3) \quad \left\| zp'(z) - \frac{1}{2}a_n z^n + \frac{1}{4}a_0 \right\| + \gamma_n |a_0| \leq \left( n - \frac{1}{2} \right) \|p\|,$$

where

$$\gamma_n = \begin{cases} \frac{1}{4}, & n \equiv 1 \pmod{2}, \quad n \geq 1, \\ \frac{5}{12}, & n = 2, \\ \frac{11}{20}, & n = 4, \\ \frac{(n+3)}{4(n-1)}, & n \equiv 0 \pmod{2}, \quad n \geq 6. \end{cases}$$

For each  $n$ , the constant  $\gamma_n$  is best possible in the following sense: given  $\varepsilon > 0$ , there exists a polynomial  $p_\varepsilon \in \mathcal{P}_n$ ,  $p_\varepsilon(z) = \sum_{j=0}^n a_j(\varepsilon) z^j$ , such that

$$\left\| zp'_\varepsilon(z) - \frac{1}{2}a_n(\varepsilon) z^n + \frac{1}{4}a_0(\varepsilon) \right\| + (\gamma_n + \varepsilon) |a_0(\varepsilon)| > \left( n - \frac{1}{2} \right) \|p_\varepsilon\|.$$



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The inequality (1.3) implies that

$$(1.4) \quad \left\| zp'(z) - \frac{1}{2}a_n z^n \right\| \leq \left( n - \frac{1}{2} \right) \|p\|.$$

In view of the inequality [4, p. 637]  $|a_k| \leq \|p\|$ ,  $0 \leq k \leq n$ , and the triangle inequality, it follows from (1.4) that

$$(1.5) \quad \|zp'(z) - xa_n z^n\| \leq (n-x)\|p\|$$

for  $0 \leq x \leq \frac{1}{2}$  (here  $x$  is a parameter independent of  $\text{Re}(z)$ ). If  $x > \frac{1}{2}$  then the same reasoning gives  $(n+x-1)$  in the right-hand side of (1.5). But  $(n+x-1) > (n-x)$  for  $x > \frac{1}{2}$ , so that the following natural question arises: what is the greatest constant  $x_n \geq \frac{1}{2}$  such that the inequality (1.5) holds for  $0 \leq x \leq x_n$ ?

The Chebyshev polynomials of the first and second kind are respectively

$$T_n(x) = \cos(n\theta)$$

and

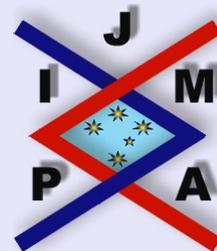
$$U_n(x) = \frac{\sin((n+1)\theta)}{\sin(\theta)},$$

where  $x = \cos(\theta)$ . We prove the following result.

**Theorem 1.1.** *Let  $x_n$  be the smallest root of the equation*

$$(1.6) \quad \sqrt{1 - \frac{1}{2x}} = \frac{n}{2x} U_{2n+1} \left( \sqrt{1 - \frac{1}{2x}} \right) - T_{2n+1} \left( \sqrt{1 - \frac{1}{2x}} \right)$$

*in the interval  $(\frac{1}{2}, \infty)$ . The inequality (1.5) then holds for  $0 \leq x \leq x_n$ . The constant  $x_n$  is best possible.*



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It will be clear that all the roots of the equation (1.6) are  $> \frac{1}{2}$ . Consider the polynomial, of degree  $(n + 1)$ , defined by

$$(1.7) \quad D(n, x) := \frac{(-1)^{n+1}}{2} x^{n+1} + \frac{(-1)^n}{2} x^n \left( \frac{\frac{n}{2} U_{2n+1} \left( \sqrt{1 - \frac{1}{2x}} \right) - x T_{2n+1} \left( \sqrt{1 - \frac{1}{2x}} \right)}{\sqrt{1 - \frac{1}{2x}}} \right).$$

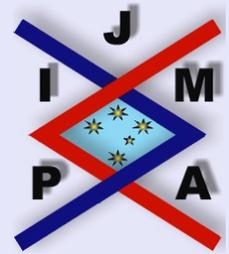
The solutions of the equation (1.6) are the roots of the polynomial  $D(n, x)$ . We also have the following asymptotic result.

**Theorem 1.2.** *For any complex number  $c$ , we have*

$$(1.8) \quad \lim_{n \rightarrow \infty} \frac{2^{n+1}}{n^2} D \left( n, \frac{1}{2} + \frac{c^2}{8n^2} \right) = \frac{\sin(c)}{c},$$

where  $D(n, x)$  is defined by (1.7). In particular, if  $x_n$  is the constant of Theorem 1.1 then

$$(1.9) \quad x_n \sim \frac{1}{2} + \frac{\pi^2}{8n^2}, \quad n \rightarrow \infty.$$



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## 2. Proofs of the Theorems

Given two analytic functions

$$f(z) = \sum_{j=0}^{\infty} a_j z^j, \quad g(z) = \sum_{j=0}^{\infty} b_j z^j \quad (|z| \leq K),$$

the function

$$(f \star g)(z) := \sum_{j=0}^{\infty} a_j b_j z^j \quad (|z| \leq K)$$

is said to be their Hadamard product.

Let  $\mathcal{B}_n$  be the class of polynomials  $Q$  in  $\mathcal{P}_n$  such that

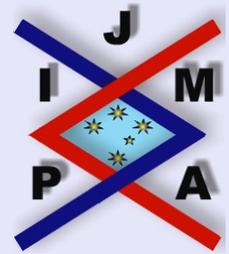
$$\|Q \star p\| \leq \|p\| \quad \text{for every } p \in \mathcal{P}_n.$$

To  $p \in \mathcal{P}_n$  we associate the polynomial  $\tilde{p}(z) := z^n \overline{p(\frac{1}{\bar{z}})}$ . Observe that

$$Q \in \mathcal{B}_n \iff \tilde{Q} \in \mathcal{B}_n.$$

Let us denote by  $\mathcal{B}_n^0$  the subclass of  $\mathcal{B}_n$  consisting of polynomials  $R$  in  $\mathcal{B}_n$  for which  $R(0) = 1$ .

**Lemma 2.1.** [4, p. 414] *The polynomial  $R(z) = \sum_{j=0}^n b_j z^j$ , where  $b_0 = 1$ , belongs to  $\mathcal{B}_n^0$  if and only if the matrix*



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$$M(b_0, b_1, \dots, b_n) := \begin{pmatrix} b_0 & b_1 & \cdots & b_{n-1} & b_n \\ \bar{b}_1 & b_0 & \cdots & b_{n-2} & b_{n-1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \bar{b}_{n-1} & \bar{b}_{n-2} & \cdots & b_0 & b_1 \\ \bar{b}_n & \bar{b}_{n-1} & \cdots & \bar{b}_1 & b_0 \end{pmatrix}$$

is positive semi-definite.

The following well-known result enables us to study the definiteness of the matrix  $M(1, b_1, \dots, b_n)$  associated with the polynomial

$$R(z) = \tilde{Q}(z) = 1 + \sum_{j=1}^n b_j z^j.$$

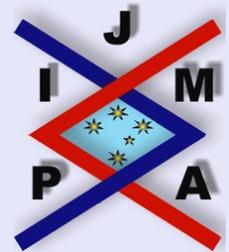
**Lemma 2.2.** [3, p. 274] *The hermitian matrix*

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad a_{ij} = \bar{a}_{ji},$$

is positive semi-definite if and only if all its eigenvalues are non-negative.

*Proof of Theorem 1.1.* The preceding lemmas are applied to a polynomial of the form

$$(2.1) \quad R(z) = \tilde{Q}(z) = 1 + \sum_{j=1}^n \binom{n-j}{n-x} z^j.$$



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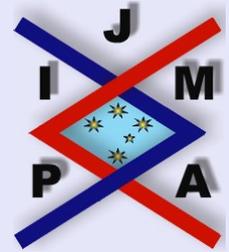
We study the definiteness of the matrix  $M(n - x, n - 1, \dots, 2, 1, 0)$ . Let

$$(2.2) \quad F(n, x) := \begin{vmatrix} n-x & n-1 & n-2 & \cdots & 2 & 1 & 0 \\ n-1 & n-x & n-1 & \cdots & 3 & 2 & 1 \\ n-2 & n-1 & n-x & \cdots & 4 & 3 & 2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 1 & 2 & 3 & \cdots & n-1 & n-x & n-1 \\ 0 & 1 & 2 & \cdots & n-2 & n-1 & n-x \end{vmatrix}.$$

We will prove that  $F(n, x) \equiv D(n, x)$ , where  $D(n, x)$  is defined by (1.7). Let  $x_n$  be the smallest positive root of the equation  $F(n, x) = 0$ . The smallest eigenvalue  $\lambda$  of  $M(n - x, n - 1, \dots, 2, 1, 0)$  is the one for which  $\lambda + x = x_n$ ; we thus have  $\lambda \geq 0$  whenever  $0 \leq x \leq x_n$ . For  $n > 1$ , it will be clear that  $F(n, x^*) < 0$  for some  $x^* > x_n$ ; the constant  $x_n$  is thus the greatest one for which an inequality of the form (1.5) holds.

In order to evaluate explicitly the determinant (2.2) we perform on it a sequence of operations. We denote by  $L_i$  the  $i$ -th row of the determinant in consideration. After each operation we continue to denote by  $L_i$  the new  $i$ -th row.

1.  $L_i - L_{i+1}$ ,  $1 \leq i \leq n$ , i.e., we subtract its  $(i + 1)$ -st row from its  $i$ -th row for  $i = 1, 2, \dots, n$ .
2.  $L_{i+1} - L_i$ ,  $1 \leq i < n$ , i.e., we subtract the new  $i$ -th row from its new  $(i + 1)$ -st row for  $i = 1, 2, \dots, (n - 1)$ .



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After these two steps, we obtain

$$(2.3) \quad F(n, x)$$

$$= \begin{vmatrix} 1-x & x-1 & -1 & -1 & \cdots & -1 & -1 & -1 & -1 \\ x & 2-2x & x & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & x & 2-2x & x & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & x & 2-2x & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & x & 2-2x & x & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & x & 2-2x & x \\ 0 & 1 & 2 & 3 & \cdots & n-3 & n-2 & n-1 & n-x \end{vmatrix}.$$

Consider now the recurrence relations

$$(2.4) \quad y_k = z_{k-1} - \frac{(2-2x)}{x} y_{k-1}$$

for  $1 \leq k < n$ , and

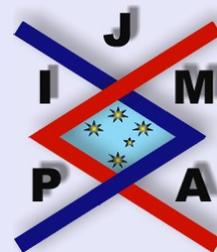
$$(2.5) \quad z_k = (k+1) - y_{k-1}$$

for  $1 \leq k < n-1$ , with the initial values  $y_0 = 0$ ,  $z_0 = 1$ . On the determinant (2.3), we perform the operations

$$(3) \quad L_{n+1} - \frac{y_{i-2}}{x} L_i, \quad i = 3, 4, \dots, n.$$

$$(4) \quad L_1 + L_2.$$

$$(5) \quad L_2 - xL_1.$$



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We obtain

(2.6)  $F(n, x)$

$$= \begin{vmatrix} 1 & 1-x & x-1 & -1 & \cdots & -1 & -1 & -1 & -1 \\ 0 & \alpha_1 & \beta_1 & x & \cdots & x & x & x & x \\ 0 & x & 2-2x & x & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & x & 2-2x & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & x & 2-2x & x \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & y_{n-1} & z_{n-1}^* \end{vmatrix}$$

where  $\alpha_1 = (x-1)(x-2)$ ,  $\beta_1 = x(2-x)$  and

(2.7) 
$$z_{n-1}^* = (n-x) - y_{n-2}$$

for  $n = 2, 3, \dots$

We continue with the following operations on the determinant (2.6).

(6)  $L_{i+2} - \frac{x}{\alpha_i} L_{i+1}$ ,  $i = 1, 2, \dots, (n-2)$ , and  $L_{n+1} - \frac{y_{n-1}}{\alpha_{n-1}} L_n$ , where

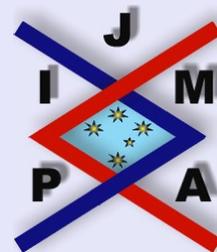
(2.8) 
$$\alpha_k = (2-2x) - x \frac{\beta_{k-1}}{\alpha_{k-1}}$$

for  $1 < k < n$ ,

(2.9) 
$$\beta_k = x + A_{k-1}$$

for  $1 < k < n$ , and

(2.10) 
$$A_k := \frac{(-1)^k x^{k+1}}{\alpha_1 \cdots \alpha_k}.$$



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We obtain

$$(2.11) \quad F(n, x) = \begin{vmatrix} 1 & 1-x & x-1 & -1 & -1 & \cdots & -1 & -1 & -1 & -1 \\ 0 & \alpha_1 & \beta_1 & x & x & \cdots & x & x & x & x \\ 0 & 0 & \alpha_2 & \beta_2 & A_1 & \cdots & A_1 & A_1 & A_1 & A_1 \\ 0 & 0 & 0 & \alpha_3 & \beta_3 & \cdots & A_2 & A_2 & A_2 & A_2 \\ 0 & 0 & 0 & 0 & \alpha_4 & \cdots & A_3 & A_3 & A_3 & A_3 \\ \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \alpha_{n-3} & \beta_{n-3} & A_{n-4} & A_{n-4} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \alpha_{n-2} & \beta_{n-2} & A_{n-3} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \alpha_{n-1} & \beta_{n-1} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & z_{n-1}^{**} \end{vmatrix}$$

where

$$(2.12) \quad z_{n-1}^{**} = z_{n-1}^* - \frac{\beta_{n-1}}{\alpha_{n-1}} y_{n-1}.$$

It follows from (2.11) that

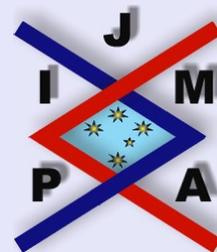
$$(2.13) \quad F(n, x) = \alpha_1 \alpha_2 \cdots \alpha_{n-1} z_{n-1}^{**}$$

for  $n = 2, 3, \dots$ . Let

$$(2.14) \quad \gamma_k := \alpha_1 \alpha_2 \cdots \alpha_k.$$

It is readily seen that

$$(2.15) \quad F(n, x) = (n - x - y_{n-2})\gamma_{n-1} - xy_{n-1}\gamma_{n-2} + (-1)^{n-1}x^{n-1}y_{n-1}.$$



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The sequences  $y_k$  and  $\gamma_k$  satisfy the recurrence relations

$$(2.16) \quad xy_k + (2 - 2x)y_{k-1} + xy_{k-2} = kx$$

for  $k \geq 2$ , with  $y_0 = 0$ ,  $y_1 = 1$ , and

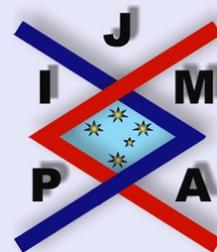
$$(2.17) \quad \gamma_k - (2 - 2x)\gamma_{k-1} + x^2\gamma_{k-2} = (-1)^{k+1}x^k$$

for  $k \geq 2$ , with  $\gamma_0 := 1 - x$ ,  $\gamma_1 = (x - 1)(x - 2)$ . These recurrence relations can be solved by elementary means (a mathematical software may help). We find that

$$(2.18) \quad \begin{aligned} y_k &= y_k(x) \\ &= \frac{((x - 1) - \sqrt{1 - 2x})^{k+1} - ((x - 1) + \sqrt{1 - 2x})^{k+1}}{4x^{k-1}\sqrt{1 - 2x}} \\ &\quad + \frac{(k + 1)x}{2} \end{aligned}$$

and

$$(2.19) \quad \begin{aligned} \gamma_k &= \gamma_k(x) \\ &= \frac{(-x)^{k+1}}{2} \\ &\quad + \frac{((2 - 3x) + (2 - x)\sqrt{1 - 2x})((1 - x) + \sqrt{1 - 2x})^k}{4\sqrt{1 - 2x}} \\ &\quad + \frac{((3x - 2) + (2 - x)\sqrt{1 - 2x})((1 - x) - \sqrt{1 - 2x})^k}{4\sqrt{1 - 2x}}. \end{aligned}$$



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Substituting in the right-hand member of (2.15), we finally obtain an explicit representation for  $F(n, x)$ :

$$(2.20) \quad F(n, x) = \frac{1}{4\sqrt{1-2x}} \left( ((1-x) - \sqrt{1-2x})^n ((n-x)\sqrt{1-2x} + (n+1)x - n) + ((1-x) + \sqrt{1-2x})^n ((n-x)\sqrt{1-2x} - (n+1)x + n) + 2(-1)^{n+1}x^{n+1}\sqrt{1-2x} \right).$$

It follows from (2.20) that

$$(2.21) \quad F(n, x) = \frac{(n-x)}{2} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j} (1-2x)^j (1-x)^{n-2j} - \frac{1}{2} ((n+1)x - n) \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2j+1} (1-2x)^j (1-x)^{n-2j-1} + \frac{(-1)^{n+1}}{2} x^{n+1}.$$

The identity

$$(2.22) \quad F(n, x) = D(n, x),$$

where  $D(n, x)$  is defined by (1.7), also follows from (2.20). It is a direct verification noticing that the well-known representation

$$T_m(x) = \frac{(x + \sqrt{x^2 - 1})^m + (x - \sqrt{x^2 - 1})^m}{2}$$



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and

$$U_m(x) = \frac{(x + \sqrt{x^2 - 1})^{m+1} - (x - \sqrt{x^2 - 1})^{m+1}}{2\sqrt{x^2 - 1}}$$

readily give

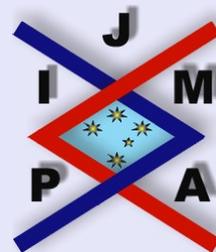
$$\begin{aligned} T_{2n+1} \left( \sqrt{1 - \frac{1}{2x}} \right) &= \frac{i(-1)^n}{2\sqrt{2x} x^n} \left( ((1-x) + \sqrt{1-2x})^n (1 + \sqrt{1-2x}) \right. \\ &\quad \left. - ((1-x) - \sqrt{1-2x})^n (1 - \sqrt{1-2x}) \right) \end{aligned}$$

and

$$\begin{aligned} U_{2n+1} \left( \sqrt{1 - \frac{1}{2x}} \right) &= \frac{i(-1)^n}{\sqrt{2x} x^n} \left( ((1-x) + \sqrt{1-2x})^{n+1} - ((1-x) - \sqrt{1-2x})^{n+1} \right). \end{aligned}$$

Since  $M(n-x, n-1, \dots, 2, 1, 0)$  is a symmetric matrix we know from the general theory that all its eigenvalues are real. It is evident from (2.21) that  $F(n, x) > 0$  for  $x \leq 0$ . The proof of Theorem 1.1 will be complete if we can show that  $F(n, x) \neq 0$  for  $0 \leq x \leq \frac{1}{2}$ . In fact, the polynomials  $F(n, x)$  are decreasing in  $[0, \frac{1}{2}]$ , with

$$F(n, 0) = n2^{n-1} \quad \text{and} \quad F\left(n, \frac{1}{2}\right) = \frac{1}{2^{n+1}} \left( n^2 + \frac{((-1)^n + 1)}{2} \right).$$



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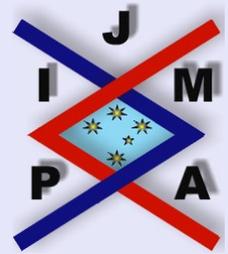


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If  $n$  is even then the foregoing affirmation is evident since all the fundamental terms are decreasing in (2.21). If  $n$  is odd then all the fundamental terms are decreasing except  $(-1)^{n+1}x^{n+1} = x^{n+1}$ . In that case we note that  $(n-x)(1-x)^n = (n-1)(1-x)^n + (1-x)^{n+1}$ ; it is then sufficient to observe that the function  $(1-x)^{n+1} + x^{n+1} =: \varphi(x)$  is decreasing (we have  $\varphi'(x) = (n+1)(x^n - (1-x)^n) \leq 0$  for  $0 \leq x \leq \frac{1}{2}$ ).  $\square$

*Proof of Theorem 1.2.* The representation (2.21) gives

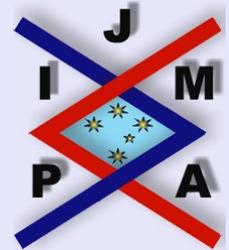
$$\begin{aligned}
 (2.23) \quad & \frac{2^{n+1}}{n^2} F\left(n, \frac{1}{2} + \frac{c^2}{8n^2}\right) \\
 &= \left(1 - \frac{1}{2n} + \frac{c^2}{8n^3}\right) \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{n^{2j+1}(n-2j)!} \left(1 - \frac{c^2}{4n^2}\right)^{n-2j} \frac{(-c^2)^j}{(2j)!} \\
 &+ \left(\frac{(n-1)}{n} - \frac{(n+1)c^2}{4n^3}\right) \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{n!}{n^{2j+1}(n-2j-1)!} \left(1 - \frac{c^2}{4n^2}\right)^{n-2j-1} \frac{(-c^2)^j}{(2j+1)!} \\
 &+ \frac{(-1)^{n+1}}{2n^2} \left(1 + \frac{c^2}{4n^2}\right)^{n+1}.
 \end{aligned}$$

For any fixed integer  $m$  we have  $\frac{n!}{(n-m)!} \sim n^m$ , as  $n \rightarrow \infty$ . It follows from (2.23) and the dominated convergence theorem that

$$(2.24) \quad \lim_{n \rightarrow \infty} \frac{2^{n+1}}{n^2} F\left(n, \frac{1}{2} + \frac{c^2}{8n^2}\right) = \sum_{j=0}^{\infty} \frac{(-1)^j c^{2j}}{(2j+1)!} = \frac{\sin(c)}{c},$$

which is the relation (1.8).

For large  $n$ , we deduce from (2.24) that  $F\left(n, \frac{1}{2} + \frac{c^2}{8n^2}\right) > 0$  if  $0 < c < \pi$  and that  $F\left(n, \frac{1}{2} + \frac{c^2}{8n^2}\right) < 0$  if  $\pi < c < 2\pi$ . We obtain (1.9) by continuity.  $\square$



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### 3. Concluding Remarks and Open Problems

There exists inequalities similar to (1.4) that cannot be proved with the method of convolution. An example is

$$(3.1) \quad \|zp'(z) - 2a_n z^n\| \leq (n-1)\|p\|$$

for  $n > 1$ . The inequality (3.1) is a consequence of the particular case  $\gamma = \pi$ ,  $m = 1$  of [1, Lemma 2]. If we wish to apply the method described at the beginning of Section 2 then the relevant polynomial should be  $R(z) = \frac{(n-2)}{(n-1)} + \sum_{j=1}^n \frac{(n-j)}{(n-1)} z^j$ . But  $R(0) = \frac{(n-2)}{(n-1)} \neq 1$ , so that Lemma 2.1 is not applicable.

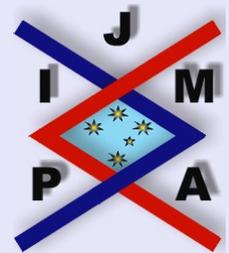
The constant  $x_n$  of Theorem 1.1 can be computed explicitly for some values of  $n$ . We have  $x_1 = 1$ ,  $x_2 = 2 - \sqrt{2}$ ,  $x_3 = 2 - \sqrt{2}$ ,  $x_5 = 2(2 - \sqrt{3})$ ,  $x_7 = 4 + 2\sqrt{2} - \sqrt{2(10 + 7\sqrt{2})}$ ,  $x_9 = 6 + 2\sqrt{5} - \sqrt{2(25 + 11\sqrt{5})}$  and  $x_{11} = 8 - 3\sqrt{6} - \sqrt{2(49 - 20\sqrt{6})}$ . The values  $x_4$  and  $x_6$  are more complicated. For other values of  $n$ , it seems difficult to express the roots of  $D(n, x)$  by means of radicals. It is numerically evident that  $x_{n+1} < x_n$ .

The substitution  $\sqrt{1 - \frac{1}{2x}} \mapsto x$  permits us to write the equation (1.6) as

$$(3.2) \quad n(1 - x^2)U_{2n+1}(x) - T_{2n+1}(x) = x.$$

We thus have

$$(3.3) \quad x_n = \frac{1}{2(1 - y_n^2)},$$



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where  $y_n$  is the smallest positive root of the equation (3.2). The identities  $(1 - x^2)U_m(x) = xT_{m+1}(x) - T_{m+2}(x)$  and  $T_{\ell+m}(x) + T_{\ell-m}(x) = 2T_\ell(x)T_m(x)$  lead us to the factorization

$$(3.4) \quad n(1-x^2)U_{2n+1}(x) - T_{2n+1}(x) - x = T_{n+1}(x)((n-2)T_n(x) - nT_{n+2}(x)).$$

It follows that the value  $y_n$  defined by (3.3) is the least positive root of the polynomial  $T_{n+1}(x)$  or the least positive root of the equation  $(n-2)T_n(x) = nT_{n+2}(x)$ .

**Conjecture 3.1.** *If  $n$  is odd then  $y_n = \sin\left(\frac{\pi}{2(n+1)}\right)$  (so that  $x_n = \frac{1}{2\cos^2\left(\frac{\pi}{2(n+1)}\right)}$ ). If  $n$  is even then  $y_n$  is the smallest positive root of the equation  $(n-2)T_n(x) = nT_{n+2}(x)$ .*

We finally mention the following (not proved) representation of  $D(n, x)$ :

$$(3.5) \quad D(n, x) = \sum_{k=0}^n (-1)^k \frac{(2n-k+1)!(n^2-(k-1)n+k)}{(2n-2k+2)!k!} 2^{n-k} x^k + (-1)^{n+1} x^{n+1}.$$



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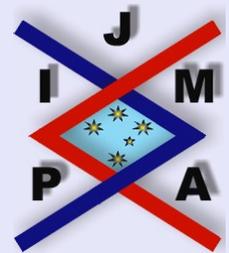
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