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ANOTHER REFINEMENT OF BERNSTEIN'S INEQUALITY

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ABSTRACT. Given a polynomial $p(z) = \sum_{j=0}^{n} a_j z^j$, we denote by $\| \ \|$ the maximum norm on the unit circle $\{z\colon |z| = 1\}$. We obtain a characterization of the best possible constant $x_n \geq \frac{1}{2}$ such that the inequality $\|zp'(z) - xa_n z^n\| \leq (n - x)\|p\|$ holds for $0 \leq x \leq x_n$.

Key words and phrases: Bernstein's inequality, Unit circle, Convolution method.

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1. INTRODUCTION AND STATEMENTS OF THE RESULTS

We denote by $\mathcal{P}_n$ the class of all polynomials with complex coefficients, of degree $\leq n$:

$$p(z) = \sum_{j=0}^{n} a_j z^j. \tag{1.1}$$

Let $\|p\| := \max_{|z|=1} |p(z)|$. The classical inequality

$$\|p'\| \leq n\|p\| \tag{1.2}$$

is known as Bernstein's inequality. A great number of refinements and generalizations of (1.2) have been obtained. See [4, Part III] for an extensive study of that subject. An example of refinement is [2, p. 84]

$$\left\| zp'(z) - \frac{1}{2}a_n z^n + \frac{1}{4}a_0 \right\| + \gamma_n |a_0| \leq \left(n - \frac{1}{2} \right) \|p\|, \tag{1.3}$$

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where

$$\gamma_n = \begin{cases} \frac{1}{4}, & n \equiv 1(\operatorname{mod} 2),\ n \geq 1, \\ \frac{5}{12}, & n = 2, \\ \frac{11}{20}, & n = 4, \\ \frac{(n+3)}{4(n-1)}, & n \equiv 0(\operatorname{mod} 2),\ n \geq 6. \end{cases}$$

For each n, the constant γ_n is best possible in the following sense: given $\varepsilon > 0$, there exists a polynomial $p_\varepsilon \in \mathcal{P}_n$, $p_\varepsilon(z) = \sum_{j=0}^{n} a_j(\varepsilon) z^j$, such that

$$\left\| zp'_\varepsilon(z) - \frac{1}{2}a_n(\varepsilon)z^n + \frac{1}{4}a_0(\varepsilon) \right\| + (\gamma_n + \varepsilon)|a_0(\varepsilon)| > \left(n - \frac{1}{2}\right) \|p_\varepsilon\|.$$

The inequality (1.3) implies that

$$\left\| zp'(z) - \frac{1}{2}a_n z^n \right\| \leq \left(n - \frac{1}{2}\right) \|p\|. \tag{1.4}$$

In view of the inequality [4, p. 637] $|a_k| \leq \|p\|$, $0 \leq k \leq n$, and the triangle inequality, it follows from (1.4) that

$$\|zp'(z) - xa_n z^n\| \leq (n - x)\|p\| \tag{1.5}$$

for $0 \leq x \leq \frac{1}{2}$ (here x is a parameter independent of Re(z)). If $x > \frac{1}{2}$ then the same reasoning gives $(n+x-1)$ in the right-hand side of (1.5). But $(n+x-1) > (n-x)$ for $x > \frac{1}{2}$, so that the following natural question arises: what is the greatest constant $x_n \geq \frac{1}{2}$ such that the inequality (1.5) holds for $0 \leq x \leq x_n$?

The Chebyshev polynomials of the first and second kind are respectively

$$T_n(x) = \cos(n\theta)$$

and

$$U_n(x) = \frac{\sin((n+1)\theta)}{\sin(\theta)},$$

where $x = \cos(\theta)$. We prove the following result.

Theorem 1.1. *Let x_n be the smallest root of the equation*

$$\sqrt{1 - \frac{1}{2x}} = \frac{n}{2x} U_{2n+1}\left(\sqrt{1 - \frac{1}{2x}}\right) - T_{2n+1}\left(\sqrt{1 - \frac{1}{2x}}\right) \tag{1.6}$$

in the interval $(\frac{1}{2}, \infty)$. The inequality (1.5) *then holds for $0 \leq x \leq x_n$. The constant x_n is best possible.*

It will be clear that all the roots of the equation (1.6) are $> \frac{1}{2}$. Consider the polynomial, of degree $(n+1)$, defined by

$$D(n,x) := \frac{(-1)^{n+1}}{2} x^{n+1} + \frac{(-1)^n}{2} x^n \left(\frac{\frac{n}{2} U_{2n+1}\left(\sqrt{1 - \frac{1}{2x}}\right) - xT_{2n+1}\left(\sqrt{1 - \frac{1}{2x}}\right)}{\sqrt{1 - \frac{1}{2x}}} \right). \tag{1.7}$$

The solutions of the equation (1.6) are the roots of the polynomial $D(n,x)$. We also have the following asymptotic result.

Theorem 1.2. *For any complex number c, we have*

$$\lim_{n\to\infty} \frac{2^{n+1}}{n^2} D\left(n, \frac{1}{2} + \frac{c^2}{8n^2}\right) = \frac{\sin(c)}{c}, \tag{1.8}$$

where $D(n,x)$ is defined by (1.7)*. In particular, if x_n is the constant of Theorem* 1.1 *then*

$$x_n \sim \frac{1}{2} + \frac{\pi^2}{8n^2}, \quad n \to \infty. \tag{1.9}$$

2. PROOFS OF THE THEOREMS

Given two analytic functions

$$f(z) = \sum_{j=0}^{\infty} a_j z^j, \quad g(z) = \sum_{j=0}^{\infty} b_j z^j \quad (|z| \le K),$$

the function

$$(f \star g)(z) := \sum_{j=0}^{\infty} a_j b_j z^j \quad (|z| \le K)$$

is said to be their Hadamard product.

Let $\mathcal{B}_n$ be the class of polynomials Q in $\mathcal{P}_n$ such that

$$\|Q \star p\| \le \|p\| \quad \text{for every} \quad p \in \mathcal{P}_n.$$

To $p \in \mathcal{P}_n$ we associate the polynomial $\tilde{p}(z) := z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$. Observe that

$$Q \in \mathcal{B}_n \iff \tilde{Q} \in \mathcal{B}_n.$$

Let us denote by $\mathcal{B}_n^0$ the subclass of $\mathcal{B}_n$ consisting of polynomials R in $\mathcal{B}_n$ for which $R(0) = 1$.

Lemma 2.1. [4, p. 414] *The polynomial $R(z) = \sum_{j=0}^{n} b_j z^j$, where $b_0 = 1$, belongs to $\mathcal{B}_n^0$ if and only if the matrix*

$$M(b_0, b_1, \dots, b_n) := \begin{pmatrix} b_0 & b_1 & \cdots & b_{n-1} & b_n \\ \bar{b}_1 & b_0 & \cdots & b_{n-2} & b_{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ \bar{b}_{n-1} & \bar{b}_{n-2} & \cdots & b_0 & b_1 \\ \bar{b}_n & \bar{b}_{n-1} & \cdots & \bar{b}_1 & b_0 \end{pmatrix}$$

is positive semi-definite.

The following well-known result enables us to study the definiteness of the matrix $M(1, b_1, \dots, b_n)$ associated with the polynomial

$$R(z) = \tilde{Q}(z) = 1 + \sum_{j=1}^{n} b_j z^j.$$

Lemma 2.2. [3, p. 274] *The hermitian matrix*

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad a_{ij} = \bar{a}_{ji},$$

is positive semi-definite if and only if all its eigenvalues are non-negative.

Proof of Theorem 1.1. The preceding lemmas are applied to a polynomial of the form

$$R(z) = \tilde{Q}(z) = 1 + \sum_{j=1}^{n} \binom{n-j}{n-x} z^j. \tag{2.1}$$

We study the definiteness of the matrix $M(n-x, n-1, \dots 2, 1, 0)$. Let

$$F(n,x) := \begin{vmatrix} n-x & n-1 & n-2 & \cdots & 2 & 1 & 0 \\ n-1 & n-x & n-1 & \cdots & 3 & 2 & 1 \\ n-2 & n-1 & n-x & \cdots & 4 & 3 & 2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 1 & 2 & 3 & \cdots & n-1 & n-x & n-1 \\ 0 & 1 & 2 & \cdots & n-2 & n-1 & n-x \end{vmatrix}. \tag{2.2}$$

We will prove that $F(n,x) \equiv D(n,x)$, where $D(n,x)$ is defined by (1.7). Let x_n be the smallest positive root of the equation $F(n,x) = 0$. The smallest eigenvalue λ of $M(n-x, n-1, \dots, 2, 1, 0)$ is the one for which $\lambda + x = x_n$; we thus have $\lambda \geq 0$ whenever $0 \leq x \leq x_n$. For $n > 1$, it will be clear that $F(n, x^*) < 0$ for some $x^* > x_n$; the constant x_n is thus the greatest one for which an inequality of the form (1.5) holds.

In order to evaluate explicitly the determinant (2.2) we perform on it a sequence of operations. We denote by L_i the i-th row of the determinant in consideration. After each operation we continue to denote by L_i the new i-th row.

(1) $L_i - L_{i+1}$, $1 \leq i \leq n$, i.e., we subtract its $(i+1)$-st row from its i-th row for $i = 1, 2, \dots, n$.
(2) $L_{i+1} - L_i$, $1 \leq i < n$, i.e., we subtract the new i-th row from its new $(i+1)$-st row for $i = 1, 2, \dots, (n-1)$.

After these two steps, we obtain

$$F(n,x) = \begin{vmatrix} 1-x & x-1 & -1 & -1 & \cdots & -1 & -1 & -1 & -1 \\ x & 2-2x & x & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & x & 2-2x & x & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & x & 2-2x & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & x & 2-2x & x & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & x & 2-2x & x \\ 0 & 1 & 2 & 3 & \cdots & n-3 & n-2 & n-1 & n-x \end{vmatrix}. \tag{2.3}$$

Consider now the recurrence relations

$$y_k = z_{k-1} - \frac{(2-2x)}{x} y_{k-1} \tag{2.4}$$

for $1 \leq k < n$, and

$$z_k = (k+1) - y_{k-1} \tag{2.5}$$

for $1 \leq k < n-1$, with the initial values $y_0 = 0$, $z_0 = 1$. On the determinant (2.3), we perform the operations

(3) $L_{n+1} - \frac{y_{i-2}}{x} L_i$, $i = 3, 4, \dots, n$.
(4) $L_1 + L_2$.
(5) $L_2 - xL_1$.

We obtain

$$F(n,x)=\begin{vmatrix}1 & 1-x & x-1 & -1 & \cdots & -1 & -1 & -1 & -1\\ 0 & \alpha_1 & \beta_1 & x & \cdots & x & x & x & x\\ 0 & x & 2-2x & x & \cdots & 0 & 0 & 0 & 0\\ 0 & 0 & x & 2-2x & \cdots & 0 & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 0 & x & 2-2x & x\\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & y_{n-1} & z^*_{n-1}\end{vmatrix} \tag{2.6}$$

where $\alpha_1=(x-1)(x-2)$, $\beta_1=x(2-x)$ and

$$z^*_{n-1}=(n-x)-y_{n-2} \tag{2.7}$$

for $n=2,3,\ldots$.

We continue with the following operations on the determinant (2.6).

(6) $L_{i+2}-\frac{x}{\alpha_i}L_{i+1}$, $i=1,2,\ldots,(n-2)$, and $L_{n+1}-\frac{y_{n-1}}{\alpha_{n-1}}L_n$, where

$$\alpha_k=(2-2x)-x\frac{\beta_{k-1}}{\alpha_{k-1}} \tag{2.8}$$

for $1<k<n$,

$$\beta_k=x+A_{k-1} \tag{2.9}$$

for $1<k<n$, and

$$A_k:=\frac{(-1)^k x^{k+1}}{\alpha_1\cdots\alpha_k}. \tag{2.10}$$

We obtain

$$F(n,x)=\begin{vmatrix}1 & 1-x & x-1 & -1 & -1 & \cdots & -1 & -1 & -1 & -1\\ 0 & \alpha_1 & \beta_1 & x & x & \cdots & x & x & x & x\\ 0 & 0 & \alpha_2 & \beta_2 & A_1 & \cdots & A_1 & A_1 & A_1 & A_1\\ 0 & 0 & 0 & \alpha_3 & \beta_3 & \cdots & A_2 & A_2 & A_2 & A_2\\ 0 & 0 & 0 & 0 & \alpha_4 & \cdots & A_3 & A_3 & A_3 & A_3\\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & 0 & \cdots & \alpha_{n-3} & \beta_{n-3} & A_{n-4} & A_{n-4}\\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \alpha_{n-2} & \beta_{n-2} & A_{n-3}\\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \alpha_{n-1} & \beta_{n-1}\\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & z^{**}_{n-1}\end{vmatrix} \tag{2.11}$$

where

$$z^{**}_{n-1}=z^*_{n-1}-\frac{\beta_{n-1}}{\alpha_{n-1}}y_{n-1}. \tag{2.12}$$

It follows from (2.11) that

$$F(n,x)=\alpha_1\alpha_2\cdots\alpha_{n-1}z^{**}_{n-1} \tag{2.13}$$

for $n=2,3,\ldots$. Let

$$\gamma_k:=\alpha_1\alpha_2\cdots\alpha_k. \tag{2.14}$$

It is readily seen that

$$F(n,x)=(n-x-y_{n-2})\gamma_{n-1}-xy_{n-1}\gamma_{n-2}+(-1)^{n-1}x^{n-1}y_{n-1}. \tag{2.15}$$

The sequences y_k and γ_k satisfy the recurrence relations

$$xy_k + (2-2x)y_{k-1} + xy_{k-2} = kx \tag{2.16}$$

for $k \geq 2$, with $y_0 = 0$, $y_1 = 1$, and

$$\gamma_k - (2-2x)\gamma_{k-1} + x^2\gamma_{k-2} = (-1)^{k+1}x^k \tag{2.17}$$

for $k \geq 2$, with $\gamma_0 := 1-x$, $\gamma_1 = (x-1)(x-2)$. These recurrence relations can be solved by elementary means (a mathematical software may help). We find that

$$y_k = y_k(x) = \frac{((x-1)-\sqrt{1-2x})^{k+1} - ((x-1)+\sqrt{1-2x})^{k+1}}{4x^{k-1}\sqrt{1-2x}} + \frac{(k+1)x}{2} \tag{2.18}$$

and

$$\begin{aligned}\gamma_k = \gamma_k(x) = \frac{(-x)^{k+1}}{2} &+ \frac{((2-3x)+(2-x)\sqrt{1-2x})((1-x)+\sqrt{1-2x})^k}{4\sqrt{1-2x}} \\ &+ \frac{((3x-2)+(2-x)\sqrt{1-2x})((1-x)-\sqrt{1-2x})^k}{4\sqrt{1-2x}}.\end{aligned} \tag{2.19}$$

Substituting in the right-hand member of (2.15), we finally obtain an explicit representation for $F(n,x)$:

$$\begin{aligned}F(n,x) = \frac{1}{4\sqrt{1-2x}}\Big(&\big((1-x)-\sqrt{1-2x}\big)^n\big((n-x)\sqrt{1-2x}+(n+1)x-n\big) \\ &+ \big((1-x)+\sqrt{1-2x}\big)^n\big((n-x)\sqrt{1-2x}-(n+1)x+n\big) + 2(-1)^{n+1}x^{n+1}\sqrt{1-2x}\Big).\end{aligned} \tag{2.20}$$

It follows from (2.20) that

$$\begin{aligned}F(n,x) = \frac{(n-x)}{2}\sum_{j=0}^{[\frac{n}{2}]}\binom{n}{2j}(1-2x)^j(1-x)^{n-2j} \\ - \frac{1}{2}\big((n+1)x-n\big)\sum_{j=0}^{[\frac{n-1}{2}]}\binom{n}{2j+1}(1-2x)^j(1-x)^{n-2j-1} + \frac{(-1)^{n+1}}{2}x^{n+1}.\end{aligned} \tag{2.21}$$

The identity

$$F(n,x) = D(n,x), \tag{2.22}$$

where $D(n,x)$ is defined by (1.7), also follows from (2.20). It is a direct verification noticing that the well-known representation

$$T_m(x) = \frac{(x+\sqrt{x^2-1})^m + (x-\sqrt{x^2-1})^m}{2}$$

and

$$U_m(x) = \frac{(x+\sqrt{x^2-1})^{m+1} - (x-\sqrt{x^2-1})^{m+1}}{2\sqrt{x^2-1}}$$

readily give

$$\begin{aligned}T_{2n+1}\left(\sqrt{1-\frac{1}{2x}}\right) = \frac{i(-1)^n}{2\sqrt{2x}\,x^n}\Big(&\big((1-x)+\sqrt{1-2x}\big)^n(1+\sqrt{1-2x}) \\ &- \big((1-x)-\sqrt{1-2x}\big)^n(1-\sqrt{1-2x})\Big)\end{aligned}$$

and

$$U_{2n+1}\left(\sqrt{1-\frac{1}{2x}}\right)=\frac{i(-1)^n}{\sqrt{2x}\,x^n}\Big(\big((1-x)+\sqrt{1-2x}\big)^{n+1}-\big((1-x)-\sqrt{1-2x}\big)^{n+1}\Big).$$

Since $M(n-x,n-1,\dots,2,1,0)$ is a symmetric matrix we know from the general theory that all its eigenvalues are real. It is evident from (2.21) that $F(n,x)>0$ for $x\le 0$. The proof of Theorem 1.1 will be complete if we can show that $F(n,x)\neq 0$ for $0\le x\le \frac{1}{2}$. In fact, the polynomials $F(n,x)$ are decreasing in $\left[0,\frac{1}{2}\right]$, with

$$F(n,0)=n2^{n-1}\quad\text{and}\quad F\left(n,\frac{1}{2}\right)=\frac{1}{2^{n+1}}\left(n^2+\frac{((-1)^n+1)}{2}\right).$$

If n is even then the foregoing affirmation is evident since all the fundamental terms are decreasing in (2.21). If n is odd then all the fundamental terms are decreasing except $(-1)^{n+1}x^{n+1}=x^{n+1}$. In that case we note that $(n-x)(1-x)^n=(n-1)(1-x)^n+(1-x)^{n+1}$; it is then sufficient to observe that the function $(1-x)^{n+1}+x^{n+1}=:\varphi(x)$ is decreasing (we have $\varphi'(x)=(n+1)\big(x^n-(1-x)^n\big)\le 0$ for $0\le x\le\frac{1}{2}$). □

Proof of Theorem 1.2. The representation (2.21) gives

$$\text{(2.23)}\quad \frac{2^{n+1}}{n^2}F\left(n,\frac{1}{2}+\frac{c^2}{8n^2}\right)$$
$$=\left(1-\frac{1}{2n}+\frac{c^2}{8n^3}\right)\sum_{j=0}^{[\frac{n}{2}]}\frac{n!}{n^{2j+1}(n-2j)!}\left(1-\frac{c^2}{4n^2}\right)^{n-2j}\frac{(-c^2)^j}{(2j)!}$$
$$+\left(\frac{(n-1)}{n}-\frac{(n+1)c^2}{4n^3}\right)\sum_{j=0}^{[\frac{n-1}{2}]}\frac{n!}{n^{2j+1}(n-2j-1)!}\left(1-\frac{c^2}{4n^2}\right)^{n-2j-1}\frac{(-c^2)^j}{(2j+1)!}$$
$$+\frac{(-1)^{n+1}}{2n^2}\left(1+\frac{c^2}{4n^2}\right)^{n+1}.$$

For any fixed integer m we have $\frac{n!}{(n-m)!}\sim n^m$, as $n\to\infty$. It follows from (2.23) and the dominated convergence theorem that

$$\text{(2.24)}\qquad \lim_{n\to\infty}\frac{2^{n+1}}{n^2}F\left(n,\frac{1}{2}+\frac{c^2}{8n^2}\right)=\sum_{j=0}^{\infty}\frac{(-1)^jc^{2j}}{(2j+1)!}=\frac{\sin(c)}{c},$$

which is the relation (1.8).

For large n, we deduce from (2.24) that $F\big(n,\frac{1}{2}+\frac{c^2}{8n^2}\big)>0$ if $0<c<\pi$ and that $F\big(n,\frac{1}{2}+\frac{c^2}{8n^2}\big)<0$ if $\pi<c<2\pi$. We obtain (1.9) by continuity. □

3. CONCLUDING REMARKS AND OPEN PROBLEMS

There exists inequalities similar to (1.4) that cannot be proved with the method of convolution. An example is

$$\text{(3.1)}\qquad \|zp'(z)-2a_nz^n\|\le(n-1)\|p\|$$

for $n>1$. The inequality (3.1) is a consequence of the particular case $\gamma=\pi$, $m=1$ of [1, Lemma 2]. If we wish to apply the method described at the beginning of Section 2 then the relevant polynomial should be $R(z)=\frac{(n-2)}{(n-1)}+\sum_{j=1}^{n}\frac{(n-j)}{(n-1)}z^j$. But $R(0)=\frac{(n-2)}{(n-1)}\neq 1$, so that Lemma 2.1 is not applicable.

The constant x_n of Theorem 1.1 can be computed explicitly for some values of n. We have $x_1 = 1$, $x_2 = 2-\sqrt{2}$, $x_3 = 2-\sqrt{2}$, $x_5 = 2(2-\sqrt{3})$, $x_7 = 4+2\sqrt{2}-\sqrt{2(10+7\sqrt{2})}$, $x_9 = 6+2\sqrt{5}-\sqrt{2(25+11\sqrt{5})}$ and $x_{11} = 8-3\sqrt{6}-\sqrt{2(49-20\sqrt{6})}$. The values x_4 and x_6 are more complicated. For other values of n, it seems difficult to express the roots of $D(n,x)$ by means of radicals. It is numerically evident that $x_{n+1} < x_n$.

The substitution $\sqrt{1-\frac{1}{2x}} \mapsto x$ permits us to write the equation (1.6) as

$$n(1-x^2)U_{2n+1}(x) - T_{2n+1}(x) = x. \tag{3.2}$$

We thus have

$$x_n = \frac{1}{2(1-y_n^2)}, \tag{3.3}$$

where y_n is the smallest positive root of the equation (3.2). The identities $(1-x^2)U_m(x) = xT_{m+1}(x) - T_{m+2}(x)$ and $T_{\ell+m}(x) + T_{\ell-m}(x) = 2T_\ell(x)T_m(x)$ lead us to the factorization

$$n(1-x^2)U_{2n+1}(x) - T_{2n+1}(x) - x = T_{n+1}(x)\big((n-2)T_n(x) - nT_{n+2}(x)\big). \tag{3.4}$$

It follows that the value y_n defined by (3.3) is the least positive root of the polynomial $T_{n+1}(x)$ or the least positive root of the equation $(n-2)T_n(x) = nT_{n+2}(x)$.

Conjecture 3.1. *If n is odd then $y_n = \sin\left(\frac{\pi}{2(n+1)}\right)$ (so that $x_n = \frac{1}{2\cos^2\left(\frac{\pi}{2(n+1)}\right)}$). If n is even then y_n is the smallest positive root of the equation $(n-2)T_n(x) = nT_{n+2}(x)$.*

We finally mention the following (not proved) representation of $D(n,x)$:

$$D(n,x) = \sum_{k=0}^{n} (-1)^k \frac{(2n-k+1)!(n^2-(k-1)n+k)}{(2n-2k+2)!k!} 2^{n-k}x^k + (-1)^{n+1}x^{n+1}. \tag{3.5}$$

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