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ON THE FEKETE-SZEGÖ PROBLEM FOR SOME SUBCLASSES OF ANALYTIC FUNCTIONS

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ABSTRACT. In this present investigation, the authors obtain Fekete-Szegö's inequality for certain normalized analytic functions $f(z)$ defined on the open unit disk for which $\frac{zf'(z)+\alpha z^2 f''(z)}{(1-\alpha)f(z)+\alpha z f'(z)}$ $(\alpha \geq 0)$ lies in a region starlike with respect to 1 and is symmetric with respect to the real axis. Also certain applications of the main result for a class of functions defined by convolution are given. As a special case of this result, Fekete-Szegö's inequality for a class of functions defined through fractional derivatives is obtained. The Motivation of this paper is to give a generalization of the Fekete-Szegö inequalities obtained by Srivastava and Mishra .

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1. INTRODUCTION

Let $\mathcal{A}$ denote the class of all *analytic* functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \Delta := \{z \in \mathbb{C} | \, |z| < 1\}) \tag{1.1}$$

and $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions. Let $\phi(z)$ be an analytic function with positive real part on Δ with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disk Δ onto a region

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starlike with respect to 1 which is symmetric with respect to the real axis. Let $S^*(\phi)$ be the class of functions in $f \in \mathcal{S}$ for which

$$\frac{zf'(z)}{f(z)} \prec \phi(z), \quad (z \in \Delta)$$

and $C(\phi)$ be the class of functions in $f \in \mathcal{S}$ for which

$$1 + \frac{zf''(z)}{f'(z)} \prec \phi(z), \quad (z \in \Delta),$$

where $\prec$ denotes the subordination between analytic functions. These classes were introduced and studied by Ma and Minda [10]. They have obtained the Fekete-Szegö inequality for the functions in the class $C(\phi)$. Since $f \in C(\phi)$ if and only if $zf'(z) \in S^*(\phi)$, we get the Fekete-Szegö inequality for functions in the class $S^*(\phi)$. For a brief history of the Fekete-Szegö problem for the class of starlike, convex and close-to-convex functions, see the recent paper by Srivastava *et al.* [7].

In the present paper, we obtain the Fekete-Szegö inequality for functions in a more general class $M_\alpha(\phi)$ of functions which we define below. Also we give applications of our results to certain functions defined through convolution (or the Hadamard product) and in particular we consider a class $M_\alpha^\lambda(\phi)$ of functions defined by fractional derivatives. The motivation of this paper is to give a generalization of the Fekete-Szegö inequalities of Srivastava and Mishra [6].

Definition 1.1. Let $\phi(z)$ be a univalent starlike function with respect to 1 which maps the unit disk Δ onto a region in the right half plane which is symmetric with respect to the real axis, $\phi(0) = 1$ and $\phi'(0) > 0$. A function $f \in \mathcal{A}$ is in the class $M_\alpha(\phi)$ if

$$\frac{zf'(z) + \alpha z^2 f''(z)}{(1-\alpha)f(z) + \alpha z f'(z)} \prec \phi(z) \quad (\alpha \geq 0).$$

For fixed $g \in \mathcal{A}$, we define the class $M_\alpha^g(\phi)$ to be the class of functions $f \in \mathcal{A}$ for which $(f * g) \in M_\alpha(\phi)$.

To prove our main result, we need the following:

Lemma 1.1. [10] *If $p_1(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is an analytic function with positive real part in Δ, then*

$$|c_2 - vc_1^2| \leq \begin{cases} -4v + 2 & \text{if } v \leq 0; \\ 2 & \text{if } 0 \leq v \leq 1; \\ 4v - 2 & \text{if } v \geq 1. \end{cases}$$

When $v < 0$ or $v > 1$, the equality holds if and only if $p_1(z)$ is $(1+z)/(1-z)$ or one of its rotations. If $0 < v < 1$, then the equality holds if and only if $p_1(z)$ is $(1+z^2)/(1-z^2)$ or one of its rotations. If $v = 0$, the equality holds if and only if

$$p_1(z) = \left(\frac{1}{2} + \frac{1}{2}\lambda\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\lambda\right)\frac{1-z}{1+z} \quad (0 \leq \lambda \leq 1)$$

or one of its rotations. If $v = 1$, the equality holds if and only if p_1 is the reciprocal of one of the functions such that the equality holds in the case of $v = 0$.

Also the above upper bound is sharp, and it can be improved as follows when $0 < v < 1$:

$$|c_2 - vc_1^2| + v|c_1|^2 \leq 2 \quad \left(0 < v \leq \frac{1}{2}\right)$$

and

$$|c_2 - vc_1^2| + (1-v)|c_1|^2 \leq 2 \quad \left(\frac{1}{2} < v \leq 1\right).$$

2. FEKETE-SZEGÖ PROBLEM

Our main result is the following:

Theorem 2.1. *Let* $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. *If* $f(z)$ *given by (1.1) belongs to* $M_\alpha(\phi)$, *then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{B_2}{2(1+2\alpha)} - \frac{\mu}{(1+\alpha)^2}B_1^2 + \frac{1}{2(1+2\alpha)(1+\alpha)^2}B_1^2 & \textit{if} \quad \mu \leq \sigma_1; \\ \frac{B_1}{2(1+2\alpha)} & \textit{if} \quad \sigma_1 \leq \mu \leq \sigma_2; \\ -\frac{B_2}{2(1+2\alpha)} + \frac{\mu}{(1+\alpha)^2}B_1^2 - \frac{1}{2(1+2\alpha)(1+\alpha)^2}B_1^2 & \textit{if} \quad \mu \geq \sigma_2, \end{cases}$$

where

$$\sigma_1 := \frac{(1+\alpha)^2(B_2 - B_1) + (1+\alpha^2)B_1^2}{2(1+2\alpha)B_1^2},$$
$$\sigma_2 := \frac{(1+\alpha)^2(B_2 + B_1) + (1+\alpha^2)B_1^2}{2(1+2\alpha)B_1^2}.$$

The result is sharp.

Proof. For $f(z) \in M_\alpha(\phi)$, let

$$p(z) := \frac{zf'(z) + \alpha z^2 f''(z)}{(1-\alpha)f(z) + \alpha z f'(z)} = 1 + b_1 z + b_2 z^2 + \cdots. \tag{2.1}$$

From (2.1), we obtain

$$(1+\alpha)a_2 = b_1 \quad \text{and} \quad (2+4\alpha)a_3 = b_2 + (1+\alpha^2)a_2^2.$$

Since $\phi(z)$ is univalent and $p \prec \phi$, the function

$$p_1(z) = \frac{1 + \phi^{-1}(p(z))}{1 - \phi^{-1}(p(z))} = 1 + c_1 z + c_2 z^2 + \cdots$$

is analytic and has a positive real part in Δ. Also we have

$$p(z) = \phi\left(\frac{p_1(z) - 1}{p_1(z) + 1}\right) \tag{2.2}$$

and from this equation (2.2), we obtain

$$b_1 = \frac{1}{2}B_1 c_1$$

and

$$b_2 = \frac{1}{2}B_1(c_2 - \frac{1}{2}c_1^2) + \frac{1}{4}B_2 c_1^2.$$

Therefore we have

$$a_3 - \mu a_2^2 = \frac{B_1}{4(1+2\alpha)}\left\{c_2 - vc_1^2\right\}, \tag{2.3}$$

where

$$v := \frac{1}{2}\left[1 - \frac{B_2}{B_1} + \frac{(2\mu - 1) + \alpha(4\mu - \alpha)}{(1+\alpha)^2}B_1\right].$$

Our result now follows by an application of Lemma 1.1. To show that the bounds are sharp, we define the functions $K_\alpha^{\phi_n}$ $(n = 2, 3, \ldots)$ by

$$\frac{z[K_\alpha^{\phi_n}]'(z) + \alpha z^2[K_\alpha^{\phi_n}]''(z)}{(1-\alpha)[K_\alpha^{\phi_n}](z) + \alpha z[K_\alpha^{\phi_n}]'(z)} = \phi(z^{n-1}), \quad K_\alpha^{\phi_n}(0) = 0 = [K_\alpha^{\phi_n}]'(0) - 1$$

and the function F_α^λ and G_α^λ $(0 \le \lambda \le 1)$ by

$$\frac{z[F_\alpha^\lambda]'(z) + \alpha z^2[F_\alpha^\lambda]''(z)}{(1-\alpha)[F_\alpha^\lambda](z) + \alpha z[F_\alpha^\lambda]'(z)} = \phi\left(\frac{z(z+\lambda)}{1+\lambda z}\right), \quad F^\lambda(0) = 0 = (F^\lambda)'(0) - 1$$

and

$$\frac{z[G_\alpha^\lambda]'(z) + \alpha z^2[G_\alpha^\lambda]''(z)}{(1-\alpha)[G_\alpha^\lambda](z) + \alpha z[G_\alpha^\lambda]'(z)} = \phi\left(-\frac{z(z+\lambda)}{1+\lambda z}\right), \quad G^\lambda(0) = 0 = (G^\lambda)'(0).$$

Clearly the functions $K_\alpha^{\phi n}, F_\alpha^\lambda, G_\alpha^\lambda \in M_\alpha(\phi)$. Also we write $K_\alpha^\phi := K_\alpha^{\phi_2}$.

If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds if and only if f is K_α^ϕ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, the equality holds if and only if f is $K_\alpha^{\phi_3}$ or one of its rotations. If $\mu = \sigma_1$ then the equality holds if and only if f is F_α^λ or one of its rotations. If $\mu = \sigma_2$ then the equality holds if and only if f is G_α^λ or one of its rotations. □

Remark 2.2. If $\sigma_1 \le \mu \le \sigma_2$, then, in view of Lemma 1.1, Theorem 2.1 can be improved. Let σ_3 be given by

$$\sigma_3 := \frac{(1+\alpha)^2 B_2 + (1+\alpha^2)B_1^2}{2(1+2\alpha)B_1^2}.$$

If $\sigma_1 \le \mu \le \sigma_3$, then

$$|a_3 - \mu a_2^2| + \frac{(1+\alpha)^2}{2(1+2\alpha)B_1^2}\left[B_1 - B_2 + \frac{(2\mu-1) + \alpha(4\mu-\alpha)}{(1+2\alpha)}B_1^2\right]|a_2|^2 \le \frac{B_1}{2(1+2\alpha)}.$$

If $\sigma_3 \le \mu \le \sigma_2$, then

$$|a_3 - \mu a_2^2| + \frac{(1+\alpha)^2}{2(1+2\alpha)B_1^2}\left[B_1 + B_2 - \frac{(2\mu-1) + \alpha(4\mu-\alpha)}{(1+2\alpha)^2}B_1^2\right]|a_2|^2 \le \frac{B_1}{2(1+2\alpha)}.$$

3. Applications to Functions Defined by Fractional Derivatives

In order to introduce the class $M_\alpha^\lambda(\phi)$, we need the following:

Definition 3.1 (see [3, 4]; see also [8, 9])**.** Let $f(z)$ be analytic in a simply connected region of the z-plane containing the origin. The *fractional derivative* of f of order λ is defined by

$$D_z^\lambda f(z) := \frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z \frac{f(\zeta)}{(z-\zeta)^\lambda}d\zeta \quad (0 \le \lambda < 1),$$

where the multiplicity of $(z-\zeta)^\lambda$ is removed by requiring that $\log(z-\zeta)$ is real for $z - \zeta > 0$.

Using the above Definition 3.1 and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [3] introduced the operator $\Omega^\lambda : \mathcal{A} \to \mathcal{A}$ defined by

$$(\Omega^\lambda f)(z) = \Gamma(2-\lambda)z^\lambda D_z^\lambda f(z), \quad (\lambda \ne 2, 3, 4, \ldots).$$

The class $M_\alpha^\lambda(\phi)$ consists of functions $f \in \mathcal{A}$ for which $\Omega^\lambda f \in M_\alpha(\phi)$. Note that $M_0^0(\phi) \equiv S^*(\phi)$ and $M_\alpha^\lambda(\phi)$ is the special case of the class $M_\alpha^g(\phi)$ when

$$g(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} z^n. \tag{3.1}$$

Let

$$g(z) = z + \sum_{n=2}^{\infty} g_n z^n \qquad (g_n > 0).$$

Since

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in M_\alpha^g(\phi)$$

if and only if

$$(f * g) = z + \sum_{n=2}^{\infty} g_n a_n z^n \in M_\alpha(\phi),$$

we obtain the coefficient estimate for functions in the class $M_\alpha^g(\phi)$, from the corresponding estimate for functions in the class $M_\alpha(\phi)$. Applying Theorem 2.1 for the function $(f * g)(z) = z + g_2 a_2 z^2 + g_3 a_3 z^3 + \cdots$, we get the following Theorem 3.1 after an obvious change of the parameter μ:

Theorem 3.1. *Let the function $\phi(z)$ be given by $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. If $f(z)$ given by (1.1) belongs to $M_\alpha^g(\phi)$, then*

$$|a_3 - \mu a_2^2| \le \begin{cases} \frac{1}{g_3}\left[\frac{B_2}{2(1+2\alpha)} - \frac{\mu g_3}{(1+\alpha)^2 g_2^2} B_1^2 + \frac{1}{2(1+2\alpha)(1+\alpha)^2} B_1^2\right] & \text{if} \quad \mu \le \sigma_1; \\ \frac{1}{g_3}\left[\frac{B_1}{2(1+2\alpha)}\right] & \text{if} \quad \sigma_1 \le \mu \le \sigma_2; \\ \frac{1}{g_3}\left[-\frac{B_2}{2(1+2\alpha)} + \frac{\mu g_3}{(1+\alpha)^2 g_2^2} B_1^2 - \frac{1}{2(1+2\alpha)(1+\alpha)^2} B_1^2\right] & \text{if} \quad \mu \ge \sigma_2, \end{cases}$$

where

$$\sigma_1 := \frac{g_2^2}{g_3} \frac{(1+\alpha)^2(B_2 - B_1) + (1+\alpha^2)B_1^2}{2(1+2\alpha)B_1^2}$$

$$\sigma_2 := \frac{g_2^2}{g_3} \frac{(1+\alpha)^2(B_2 + B_1) + (1+\alpha^2)B_1^2}{2(1+2\alpha)B_1^2}.$$

The result is sharp.

Since

$$(\Omega^\lambda f)(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n z^n,$$

we have

$$g_2 := \frac{\Gamma(3)\Gamma(2-\lambda)}{\Gamma(3-\lambda)} = \frac{2}{2-\lambda} \tag{3.2}$$

and

$$g_3 := \frac{\Gamma(4)\Gamma(2-\lambda)}{\Gamma(4-\lambda)} = \frac{6}{(2-\lambda)(3-\lambda)}. \tag{3.3}$$

For g_2 and g_3 given by (3.2) and (3.3), Theorem 3.1 reduces to the following:

Theorem 3.2. *Let the function $\phi(z)$ be given by $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. If $f(z)$ given by (1.1) belongs to $M_\alpha^\lambda(\phi)$, then*

$$|a_3 - \mu a_2^2| \le \begin{cases} \frac{(2-\lambda)(3-\lambda)}{6}\gamma & \text{if} \quad \mu \le \sigma_1; \\ \frac{(2-\lambda)(3-\lambda)}{6} \cdot \frac{B_1}{2(1+2\alpha)} & \text{if} \quad \sigma_1 \le \mu \le \sigma_2; \\ \frac{(2-\lambda)(3-\lambda)}{6}\gamma & \text{if} \quad \mu \ge \sigma_2, \end{cases}$$

where

$$\gamma := \frac{B_2}{2(1+2\alpha)} - \frac{3(2-\lambda)}{2(3-\lambda)}\frac{\mu}{(1+\alpha)^2}B_1^2 + \frac{1}{2(1+2\alpha)(1+\alpha)^2}B_1^2,$$

$$\sigma_1 := \frac{2(3-\lambda)}{3(2-\lambda)}.\frac{(1+\alpha)^2(B_2-B_1)+(1+\alpha^2)B_1^2}{2(1+2\alpha)B_1^2}$$

$$\sigma_2 := \frac{2(3-\lambda)}{3(2-\lambda)}.\frac{(1+\alpha)^2(B_2+B_1)+(1+\alpha^2)B_1^2}{2(1+2\alpha)B_1^2}.$$

The result is sharp.

Remark 3.3. When $\alpha = 0$, $B_1 = 8/\pi^2$ and $B_2 = 16/(3\pi^2)$, the above Theorem 3.1 reduces to a recent result of Srivastava and Mishra[6, Theorem 8, p. 64] for a class of functions for which $\Omega^\lambda f(z)$ is a parabolic starlike function [2, 5].

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