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AN INSTABILITY THEOREM FOR A CERTAIN VECTOR DIFFERENTIAL EQUATION OF THE FOURTH ORDER

CEMIL TUNÇ

EDUCATION FACULTY
DEPARTMENT OF MATHEMATICS
YÜZÜNCÜ YIL UNIVERSITY,
65080 VAN, TURKEY.
cemtunc@yahoo.com

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ABSTRACT. In this paper sufficient conditions for the instability of the zero solution of the equation (1.1) are given.

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1. INTRODUCTION AND STATEMENT OF THE RESULT

This paper is concerned with the study of the instability of the trivial solution $X = 0$ of the vector differential equations of the form:

$$X^{(4)} + \Psi(\ddot{X})\dddot{X} + \Phi(\dot{X})\ddot{X} + H(\dot{X}) + F(X) = 0 \tag{1.1}$$

in the real Euclidean space R^n (with the usual norm, denoted in what follows by $\|.\|$) where Ψ and Φ are continuous $n \times n$ symmetric matrices depending, in each case, on the arguments shown, H and F are continuous n-vector functions and $H(0) = F(0) = 0$.

It will be convenient to consider, instead of the equation (1.1), the equivalent system

$$\begin{cases} \dot{X} = Y,\ \dot{Y} = Z,\ \dot{Z} = W, \\ \dot{W} = -\Psi(Z)W - \Phi(Y)Z - H(Y) - F(X) \end{cases} \tag{1.2}$$

obtained as usual by setting $\dot{X} = Y, \ddot{X} = Z, \dddot{X} = W$ in (1.1).

Let $J_F(X), J_H(Y), J_\Phi(Y)$ and $J_\Psi(Z)$ denote the Jacobian matrices corresponding to the functions $F(X), H(Y)$ and the matrices $\Phi(Y),\ \Psi(Z)$, respectively, that is, $J_F(X) = \left(\frac{\partial f_i}{\partial x_j}\right)$,

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$J_H(Y) = \left(\frac{\partial h_i}{\partial y_j}\right)$, $J_\Phi(Y) = \left(\frac{\partial \phi_i}{\partial y_j}\right)$ and $J_\Psi(Z) = \left(\frac{\partial \psi_i}{\partial z_j}\right)$ $(i, j = 1, 2, \ldots, n)$, where $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n)$, $(z_1, z_2, \ldots, z_n)$, $(f_1, f_2, \ldots, f_n)$, $(h_1, h_2, \ldots, h_n)$, $(\phi_1, \phi_2, \ldots, \phi_n)$ and $(\psi_1, \psi_2, \ldots, \psi_n)$ are the components of X, Y, Z, F, H, Φ and Ψ, respectively. Other than these, it is assumed that the Jacobian matrices $J_F(X)$, $J_H(Y)$, $J_\Phi(Y)$ and $J_\Psi(Z)$ exist and are continuous. The symbol $\langle X, Y\rangle$ corresponding to any pair X, Y in $\mathbb{R}^n$ stands for the usual scalar product $\sum_{i=1}^n x_i y_i$, and $\lambda_i(A)$ $(i = 1, 2, \ldots, n)$ are the eigenvalues of the $n \times n$ matrix A.

In the relevant literature, the instability properties for various third-, fourth-, fifth-, sixth- and eighth order nonlinear differential equations have been considered by many authors, see, for example, Berketoğlu [1], Ezeilo ([3] – [7]), Li and Yu [8], Li and Duan [9], Miller and Michel [10], Sadek [12], Skrapek ([13, 14]), Tiryaki ([15] – [17]) and the references therein. However, with respect to our observations in the relevant literature, in the case $n = 1$, the instability properties of solutions of nonlinear differential equations of the fourth order have been studied by Ezeilo ([3, 6]), Li and Yu [8], Skrapek [13] and Tiryaki [15]. Recently, the author in [12] also discussed the same subject for the vector differential equation as follows:

$$X^{(4)} + A\dddot{X} + H(X, \dot{X}, \ddot{X}, \dddot{X})\ddot{X} + G(X)\dot{X} + F(X) = 0.$$

Also, according to our observations in the relevant literature, we have not been able to locate results on the instability of solutions of certain nonlinear vector differential equations of the fourth order. The present investigation is a different attempt than the result in Sadek [12] to obtain sufficient conditions for the instability of the trivial of solutions of certain nonlinear vector differential equations of the fourth order. The motivation for the present study has come from the paper of Sadek [12] and the papers mentioned above. Our aim is to acquire a similar result for certain nonlinear vector differential equation of (1.1).

Now, we consider, in the case $n = 1$, the linear constant coefficient scalar differential equation of the form:

$$x^{(4)} + a_1\dddot{x} + a_2\ddot{x} + a_3\dot{x} + a_4 x = 0. \tag{1.3}$$

It should be pointed out that if $a_4 > \frac{1}{4}a_2^2$, then the trivial solution $x = 0$ of the equation (1.3) is unstable.

Our aim is to prove the following.

Theorem 1.1. *Suppose that the functions Ψ, Φ, H and F that appeared in (1.1) are continuously differentiable and there are positive constants a_1, a_2, a_3 and $a_4(\neq 0)$ with $a_4 > \frac{1}{4}a_2^2$ such that $\lambda_i(\Psi(Z)) \geq a_1$ for all $Z \in \mathbb{R}^n$, $\lambda_i(\Phi(Y)) \geq a_2$ and $\lambda_i(J_H(Y)) \geq a_3$ for all $Y \in \mathbb{R}^n$ and $\lambda_i(J_F(X)) \geq a_4$ for all $X(\neq 0) \in \mathbb{R}^n$ $(i = 1, 2, \ldots, n)$.*

Then the zero solution $X = 0$ of the system (1.2) is unstable.

In the subsequent discussion we require the following lemma.

Lemma 1.2. *Let A be a real symmetric $n \times n$ matrix and*

$$a' \geq \lambda_i(A) \geq a > 0 \ \ (i = 1, 2, \ldots, n), \text{where } a', a \text{ are constants.}$$

Then

$$a' \langle X, X\rangle \geq \langle AX, X\rangle \geq a \langle X, X\rangle$$

and

$$a'^2 \langle X, X\rangle \geq \langle AX, AX\rangle \geq a^2 \langle X, X\rangle .$$

Proof. See [2]. □

2. PROOF OF THE THEOREM

The proof is based on the use of Ceatev's instability criterion in [10]. For the proof of the theorem our main tool is the Lyapunov function $V = V(X, Y, Z, W)$ defined by:

$$V = \langle W, Z\rangle + \langle Y, F(X)\rangle + \int_0^1 \langle \sigma\Psi(\sigma Z)Z, Z\rangle \, d\sigma + \int_0^1 \langle \Phi(\sigma Y)Z, Y\rangle \, d\sigma + \int_0^1 \langle H(\sigma Y), Y\rangle \, d\sigma. \tag{2.1}$$

It is clear that $V(0, 0, 0, 0) = 0$.

Since $\frac{\partial}{\partial\sigma} \langle H(\sigma Y), Y\rangle = \langle J_H(\sigma Y)Y, Y\rangle$ and $H(0) = 0$, then

$$\langle H(Y), Y\rangle = \int_0^1 \langle J_H(\sigma Y)Y, Y\rangle \, d\sigma \geq \int_0^1 \langle a_3 Y, Y\rangle \, d\sigma = a_3 \langle Y, Y\rangle .$$

Therefore

$$\int_0^1 \langle H(\sigma Y), Y\rangle \, d\sigma \geq a_3 \int_0^1 \langle \sigma Y, Y\rangle \, d\sigma = \frac{1}{2} a_3 \|Y\|^2 . \tag{2.2}$$

By using the assumptions of the theorem, the above lemma and (2.2) it can be easily obtained that:

$$V(X, Y, Z, W) \geq \frac{1}{2} a_1 \|Z\|^2 + \frac{1}{2} a_3 \|Y\|^2 + \langle W, Z\rangle + \langle Y, F(X)\rangle + \int_0^1 \langle \Phi(\sigma Y)Z, Y\rangle \, d\sigma.$$

and hence

$$\begin{aligned} V(0, \varepsilon, \varepsilon, 0) &\geq \frac{1}{2} a_1 \|\varepsilon\|^2 + \frac{1}{2} a_3 \|\varepsilon\|^2 + \int_0^1 \langle \Phi(\sigma\varepsilon)\varepsilon, \varepsilon\rangle \, d\sigma \\ &\geq \frac{1}{2}(a_1 + a_2 + a_3) \|\varepsilon\|^2 > 0 \end{aligned}$$

for all arbitrary $\varepsilon \in \mathbb{R}^n$. So, in every neighborhood of $(0, 0, 0, 0)$ there exists a point (ξ, η, ζ, μ) such that $V(\xi, \eta, \zeta, \mu) > 0$ for all ξ, η, ζ and μ in R^n. Let $(X, Y, Z, W) = (X(t), Y(t), Z(t), W(t))$ be an arbitrary solution of (1.2). We obtain from (2.1) and (1.2) that

$$\begin{aligned} \dot{V} &= \frac{d}{dt} V(X, Y, Z, W) \\ &= \langle W, W\rangle - \langle \Psi(Z)W, Z\rangle - \langle \Phi(Y)Z, Z\rangle - \langle H(Y), Z\rangle + \langle Y, J_F(X)Y\rangle \\ &\quad + \frac{d}{dt}\int_0^1 \langle \sigma\Psi(\sigma Z)Z, Z\rangle \, d\sigma + \frac{d}{dt}\int_0^1 \langle \Phi(\sigma Y)Z, Y\rangle \, d\sigma + \frac{d}{dt}\int_0^1 \langle H(\sigma Y), Y\rangle \, d\sigma. \end{aligned} \tag{2.3}$$

But

$$\begin{aligned} \frac{d}{dt}\int_0^1 \langle H(\sigma Y), Y\rangle \, d\sigma &= \int_0^1 \sigma \langle J_H(\sigma Y)Z, Y\rangle \, d\sigma + \int_0^1 \langle H(\sigma Y), Z\rangle \, d\sigma \\ &= \int_0^1 \sigma \frac{\partial}{\partial\sigma} \langle H(\sigma Y), Z\rangle \, d\sigma + \int_0^1 \langle H(\sigma Y), Z\rangle \, d\sigma \\ &= \sigma \langle H(\sigma Y), Z\rangle \Big|_0^1 = \langle H(Y), Z\rangle , \end{aligned} \tag{2.4}$$

(2.5)
$$\begin{aligned}\frac{d}{dt}\int_0^1 \langle \Phi(\sigma Y)Z, Y\rangle\, d\sigma &\\ &= \int_0^1 \langle \Phi(\sigma Y)Z, Z\rangle\, d\sigma + \int_0^1 \sigma \langle J_\Phi(\sigma Y)ZY, Z\rangle\, d\sigma + \int_0^1 \langle \Phi(\sigma Y)W, Y\rangle\, d\sigma \\ &= \int_0^1 \langle \Phi(\sigma Y)Z, Z\rangle\, d\sigma + \int_0^1 \sigma \frac{\partial}{\partial\sigma} \langle \Phi(\sigma Y)Z, Z\rangle\, d\sigma + \int_0^1 \langle \Phi(\sigma Y)Y, W\rangle\, d\sigma \\ &= \langle \Phi(Y)Z, Z\rangle + \int_0^1 \langle \Phi(\sigma Y)Y, W\rangle\, d\sigma\end{aligned}$$

and

(2.6)
$$\begin{aligned}\frac{d}{dt}\int_0^1 \langle \sigma\Psi(\sigma Z)Z, Z\rangle\, d\sigma &\\ &= \int_0^1 \langle \sigma\Psi(\sigma Z)Z, W\rangle\, d\sigma + \int_0^1 \sigma^2 \langle J_\Psi(\sigma Z)ZW, Z\rangle\, d\sigma + \int_0^1 \langle \sigma\Psi(\sigma Z)W, Z\rangle\, d\sigma \\ &= \int_0^1 \langle \sigma\Psi(\sigma Z)W, Z\rangle\, d\sigma + \int_0^1 \sigma \frac{\partial}{\partial\sigma} \langle \sigma\Psi(\sigma Z)W, Z\rangle\, d\sigma \\ &= \sigma \langle \Psi(\sigma Z)W, Z\rangle \Big|_0^1 = \langle \Psi(Z)W, Z\rangle\,.\end{aligned}$$

On gathering the estimates (2.4) – (2.6) into (2.3) we obtain

$$\dot{V} = \langle W, W\rangle + \int_0^1 \langle \Phi(\sigma Y)Y, W\rangle\, d\sigma + \langle Y, J_F(X)Y\rangle\,. \tag{2.7}$$

Let

$$\Phi_1(Y) = \int_0^1 \Phi(\sigma Y)Y\, d\sigma.$$

Then

$$\int_0^1 \langle \Phi(\sigma Y)Y, W\rangle\, d\sigma = \Phi_1(Y)W.$$

Hence, by using the assumptions of the theorem and the lemma, we have

$$\begin{aligned}\dot{V} &= \left\| W + \frac{1}{2}\Phi_1(Y)\right\|^2 + \langle Y, J_F(X)Y\rangle - \frac{1}{4}\langle \Phi_1(Y), \Phi_1(Y)\rangle \\ &\geq \langle Y, J_F(X)Y\rangle - \frac{1}{4}\langle \Phi_1(Y), \Phi_1(Y)\rangle \\ &\geq \left(a_4 - \frac{1}{4}a_2^2\right)\|Y\|^2 > 0.\end{aligned}$$

Thus, the assumption $a_4 > \frac{1}{4}a_2^2$ shows that $\dot{V}_0$ is positive semi-definite. Also $\dot{V}_0 = 0 \ \ (t \geq 0)$ necessarily implies that $Y = 0$ for all $t \geq 0$, and therefore also that $X = \xi$ (a constant vector), $Z = \dot{Y} = 0$, $W = \ddot{Y} = 0$,$\dddot{Y} = \dot{W} = 0$, for $t \geq 0$. Substituting the estimates

$$X = \xi, Y = Z = W = 0$$

in (1.2) it follows that $F(\xi) = 0$ which necessarily implies that $\xi = 0$ because of $F(0) = 0$. So

$$X = Y = Z = W = 0 \ \text{ for all } t \geq 0.$$

Therefore, the function V has all the requisite Ceatev criterion proved in [10] subject to the conditions in the theorem, which now follows. The basic properties of $V(X, Y, Z, W)$, which are proved above justify that the zero solution of (1.2) is unstable. (See Theorem 1.15 in Reissig [11] and Miller and Michel [10].) The system of equations (1.2) is equivalent to the differential equation (1.1). Consequently, it follows the original statement of the theorem.

REFERENCES

[1] H. BEREKETOĞLU, On the instability of trivial solutions of a class of eighth-order differential equations, *Indian J. Pure. Appl. Math.*, **22**(3) (1991), 199–202.

[2] R. BELLMAN, *Introduction to Matrix Analysis*, Reprint of the second (1970) edition. Classics in Applied Mathematics, 19. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1997.

[3] J.O.C. EZEILO, An instability theorem for a certain fourth order differential equation, *Bull. London Math. Soc.* **10**(2) (1978), 184–185.

[4] J.O.C. EZEILO, Instability theorems for certain fifth-order differential equations, *Math. Proc. Cambridge. Philos. Soc.*, **84**(2) (1978), 343–350.

[5] J.O.C. EZEILO, A further instability theorem for a certain fifth-order differential equation. *Math. Proc. Cambridge Philos. Soc.*, **86**(3) (1979), 491–493.

[6] J.O.C. EZEILO, Extension of certain instability theorems for some fourth and fifth order differential equations, *Atti. Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Natur.*, (8) **66**(4) (1979), 239–242.

[7] J.O.C. EZEILO, An instability theorem for a certain sixth order differential equation, *J. Austral. Math. Soc. Ser. A*, **32**(1) (1982), 129–133.

[8] W. J. LI AND Y.H. YU, Instability theorems for some fourth-order and fifth-order differential equations, (Chinese) *J. Xinjiang Univ. Natur. Sci.*, **7**(2) (1990), 7–10.

[9] W. J. LI AND K.C. DUAN, Instability theorems for some nonlinear differential systems of fifth order, *J. Xinjiang Univ. Natur. Sci.*, **17**(3) (2000), 1–5.

[10] R.K. MILLER AND A. N. MICHEL, *Ordinary Differential Equations*, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1982.

[11] R. REISSIG, G. SANSONE AND R. CONTI, *Non-linear Differential Equations of Higher Order*, translated from the German, Noordhoff International Publishing, Leyden, (1974)

[12] A.I. SADEK, Instability results for certain systems of fourth and fifth order differential equations, *Applied Mathematics and Computation*, **145**(2-3) (2004), 541–549.

[13] W.A. SKRAPEK, Instability results for fourth-order differential equations, *Proc. Roy. Soc. Edinburgh Sect. A*, **85**(3-4) (1980), 247–250.

[14] W.A. SKRAPEK, Some instability theorems for third order ordinary differential equations, *Math. Nachr.*, **96** (1980), 113-117.

[15] A. TIRYAKI, Extension of an instability theorem for a certain fourth order differential equation, *Bull. Inst. Math. Acad. Sinica.*, **16**(2) (1988), 163–165.

[16] A. TIRYAKI, An instability theorem for a certain sixth order differential equation, *Indian J. Pure. Appl. Math.*, **21**(4) (1990), 330–333.

[17] A. TIRYAKI, Extension of an instability theorem for a certain fifth order differential equation, National Mathematics Symposium (Trabzon, 1987). *J. Karadeniz Tech. Univ. Fac. Arts Sci. Ser. Math. Phys.*, **11** (1988), 225–227.

[18] C. TUNÇ, A global stability result for a certain system of fifth order nonlinear differential equations, *Acta Comment. Univ. Tartu. Math.*, No.2 (1998), 3–13.