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## ON A F. QI INTEGRAL INEQUALITY

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Abstract

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## Abstract

Necessary and sufficient conditions under which the Qi integral inequality

$$\int_a^b f^t(x) dx \geq \left( \int_a^b f(x) dx \right)^{t-1}$$

or its reverse hold for all  $t \geq 1$  are given.

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*Key words:* Convexity, Qi type integral inequality.

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# 1. Introduction

In [5] Feng Qi formulated the following problem: Characterize a positive function  $f$  such that the inequality

$$(1.1) \quad \int_a^b f^t(x) dx \geq \left( \int_a^b f(x) dx \right)^{t-1}$$

holds for  $t > 1$ .

In [1, 2, 3, 4] and the references therein, several sufficient conditions and generalizations are given. In all the cited papers the authors look for the solution of the Qi inequality with restricted  $t$ . This paper is another contribution to this subject. We shall try to establish conditions under which the inequality holds for all  $t > 1$ .

Let  $(X, \mu)$  be a finite measure space and  $f$  be a positive measurable function. Define for  $t \in \mathbb{R}$

$$(1.2) \quad H(t) = H(t, f) = \ln \int f^t d\mu - (t-1) \ln \left( \int f d\mu \right).$$

It is clear that inequality (1.1) is equivalent to  $H(t) \geq 0$  for  $t > 1$ . We will say that for the function  $f$  the Qi Inequality (QI) holds if  $H(t, f)$  is nonnegative for all  $t \geq 1$ . We will also say that for the function  $f$  the Reverse Qi Inequality (RQI) holds if  $H(t, f)$  is non-positive for all  $t \geq 1$ .

By the Cauchy-Schwarz integral inequality, we have for  $p, q \in \mathbb{R}$

$$(1.3) \quad \left( \int f^{\frac{p+q}{2}} d\mu \right)^2 \leq \int f^p d\mu \int f^q d\mu$$



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which means that the function  $H(t)$  is convex, that is

$$(1.4) \quad H\left(\frac{t_1 + t_2}{2}\right) \leq \frac{H(t_1) + H(t_2)}{2}$$

holds for  $t_1, t_2 \in \mathbb{R}$ , so its derivative

$$(1.5) \quad H'(t) = \frac{\int f^t \ln f \, d\mu}{\int f^t \, d\mu} - \ln\left(\int f \, d\mu\right)$$

is increasing in  $t \in \mathbb{R}$ .

Let

$$M = \operatorname{ess\,sup}_{x \in X} f(x) \quad \text{and} \quad \mu_M = \mu(\{x : f(x) = M\}).$$

The following lemmas will be useful.

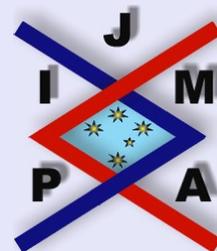
Note that from now on we will use the convention that  $\ln \infty = \infty$  and  $\ln 0 = -\infty$ .

**Lemma 1.1.** *The following formula holds:*

$$(1.6) \quad \lim_{t \rightarrow \infty} \frac{H(t)}{t} = \ln \frac{M}{\int f \, d\mu}.$$

*Proof.* For  $\varepsilon > 0$  let  $m_\varepsilon = \mu(x : f(x) > M - \varepsilon)$ . Then

$$(M - \varepsilon)^t m_\varepsilon \leq \int f^t \, d\mu \leq M^t \mu(X),$$




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so

$$(1.7) \quad t \ln(M - \varepsilon) + \ln m_\varepsilon \leq H(t) + (t - 1) \ln \left( \int f \, d\mu \right) \\ \leq t \ln M + \ln \mu(X).$$

Dividing by  $t$  on both sides of (1.7) yields

$$\ln \frac{M - \varepsilon}{\int f \, d\mu} \leq \liminf_{t \rightarrow \infty} \frac{H(t)}{t} \leq \limsup_{t \rightarrow \infty} \frac{H(t)}{t} \leq \ln \frac{M}{\int f \, d\mu}.$$

In case  $M = \infty$ ,  $M - \varepsilon$  stands for an arbitrary large number. This completes the proof.  $\square$

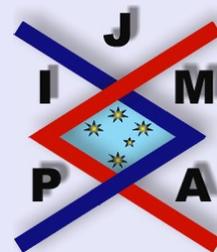
**Lemma 1.2.** *If  $M < \infty$  then*

$$(1.8) \quad \lim_{t \rightarrow \infty} \left( H(t) - t \ln \frac{M}{\int f \, d\mu} \right) = \ln \left( \mu_M \int f \, d\mu \right).$$

*Proof.* Direct computation yields

$$(1.9) \quad \lim_{t \rightarrow \infty} \left( H(t) - t \ln \frac{M}{\int f \, d\mu} \right) = \lim_{t \rightarrow \infty} \ln \int \left( \frac{f}{M} \right)^t \, d\mu + \ln \int f \, d\mu \\ = \ln \left( \mu_M \int f \, d\mu \right)$$

as  $(f/M)^t$  tends monotonically to the characteristic function of  $\{x : f(x) = M\}$ .  $\square$




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## 2. Feng Qi Integral Inequality

In this section we consider the problem: Characterize positive functions  $f$  that satisfy (QI).

**Theorem 2.1.** *A constant function  $M$  satisfies (QI) if and only if  $\mu(X) \leq 1$  and  $M \geq 1/\mu(X)$ .*

*Proof.*  $H(t) \geq 0$  is equivalent to  $M \geq \mu(X)^{t-2}$ . This can be valid for all  $t > 1$  only if the conditions of the theorem are fulfilled.  $\square$

From now on we assume that  $f$  is not constant, in which case the function  $H$  is strictly convex.

It is clear that the necessary condition for (QI) is  $H(1) \geq 0$  or equivalently  $\int f \, d\mu \geq 1$ .

**Theorem 2.2.**

(a) *If  $H(1) = 0$  then (QI) holds if and only if  $H'(1) \geq 0$ .*

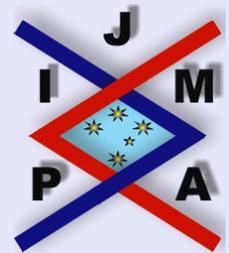
(b) *If  $H(1) > 0$  then*

(b1) *if  $H'(1) \geq 0$  then (QI) holds;*

(b2) *if  $H'(1) < 0$  and  $M < \int f \, d\mu$  then (QI) fails for large  $t$ ;*

(b3) *if  $H'(1) < 0$  and  $M = \int f \, d\mu$  then (QI) holds if and only if  $\mu_M M \geq 1$ ;*

(b4) *if  $H'(1) < 0$  and  $M > \int f \, d\mu$  then there exists a unique point  $t_0$  such that  $H'(t_0) = 0$  and (QI) holds if and only if  $H(t_0) \geq 0$ .*



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*Proof.* (a) and (b1) follow immediately from convexity of  $H$ .

From Lemma 1.1 we see that  $H$  becomes negative for large  $t$ , which proves (b2).

(b3) follows from Lemma 1.2 and from the fact that being convex the graph of  $H$  lies above its horizontal asymptote.

Finally (b4) follows from the fact that  $H'$  is strictly increasing and  $H'(t_0) = 0$  for some  $t_0$ , then  $H$  attains its minimum at  $t_0$ . Observe that in this case  $H$  may be infinite for some finite  $t_\infty$  and consequently for all  $t > t_\infty$ .  $\square$

From the above theorem we obtain the following, surprising

**Corollary 2.3.** *If  $\mu(X) < 1$  then (QI) holds if and only if  $H(1) \geq 0$ .*

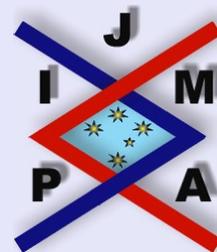
*Proof.* We will show that if  $\mu(X) < 1$  then  $H'(1) \geq 0$  for all  $f$ , so the condition (b1) is satisfied.

Applying the integral Jensen Inequality to the convex function  $x \ln x$  we obtain

$$\begin{aligned} \frac{1}{\mu(X)} \int f \ln f \, d\mu &\geq \left( \frac{1}{\mu(X)} \int f \, d\mu \right) \ln \left( \frac{1}{\mu(X)} \int f \, d\mu \right) \\ &\geq \frac{1}{\mu(X)} \left( \int f \, d\mu \right) \ln \left( \int f \, d\mu \right) \end{aligned}$$

which is equivalent to  $H'(1) \geq 0$ .  $\square$

In the case (b4), solving the equation  $H'(t) = 0$  may not be an easy task, but the following corollaries may be helpful:




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**Corollary 2.4.** *Let*

$$t_L = \frac{\int f \ln f \, d\mu - \int f \, d\mu \ln \int f \, d\mu}{\int f \ln f \, d\mu}.$$

*If  $H'(t_L) \geq 0$  then (QI) holds.*

*Proof.*  $t_L$  is the point where the supporting line drawn at  $t = 1$  meets the OX-axis. The graph of  $H$  lies above it. In particular  $H(t_L) \geq 0$ . As  $H'(t)$  is nonnegative for  $t \geq t_L$  the proof is completed.  $\square$

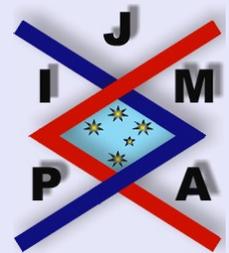
**Corollary 2.5.** *If  $0 < \mu_M, M < \infty$  let*

$$t_R = -\frac{\ln(\mu_M \int f \, d\mu)}{\ln(M / \int f \, d\mu)}.$$

*If  $H'(t_R) \leq 0$  or  $t_R \leq t_L$  then (QI) holds.*

*Proof.*  $t_R$  is the point where the supporting line drawn at  $\infty$  (it exists by Lemma 1.2) meets the OX-axis. If  $t_R \leq t_L$  the two supporting lines meet above the OX-axis.

If  $H'(t_R) \leq 0$  we use an argument similar to that in the proof of the previous corollary.  $\square$




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### 3. Reversed Feng Qi Inequality

In this section we give sufficient and necessary conditions for the reversed problem: Characterize positive functions  $f$  that satisfy (RQI).

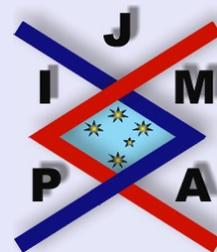
**Theorem 3.1.** *A constant function satisfies (RQI) if and only if  $\mu(X) \geq 1$  and  $M \leq 1/\mu(X)$ .*

The proof is similar to that of Theorem 2.1.

**Theorem 3.2.** *For a non constant function  $f$  (RQI) holds if and only if  $H(1) \leq 0$  and  $M \leq \int f d\mu$ .*

*Proof.* As  $\mu_M M < \int f d\mu = \exp(H(1)) \leq 1$  it follows from Lemma 1.1 and 1.2 that  $H$  is negative for large  $t$ . Being convex and non-positive at  $t = 1$ , it must be decreasing.

On the other hand if  $M > \int f d\mu$  then  $H$  is positive for large  $t$  by Lemma 1.1.  $\square$



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## 4. Final Remark

Finally we prove the following

**Theorem 4.1.** *For every positive function  $f$  there exists a constant  $c > 0$  such that  $cf$  satisfies (QI) or (RQI).*

*Proof.* One can easily see that

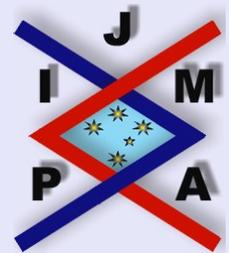
$$H(t, cf) = \ln c + H(t, f),$$

so  $cf$  satisfies (QI) for certain  $c$  if and only if  $H(t, f)$  is bounded from below. Similarly  $cf$  satisfies (RQI) only if  $H(t, f)$  is bounded from above.

It follows immediately from Lemma 1.1 and Lemma 1.2 that the function  $H(t)$  is bounded from below if and only if  $M > \int f \, d\mu$  or  $M = \int f \, d\mu$  and  $\mu_M > 0$  and is bounded from above if and only if  $M \leq \int f \, d\mu$ .

This completes the proof of our theorem.  $\square$

Note that in case  $M = \int f \, d\mu$  and  $\mu_M > 0$  we can find constants  $c_1$  and  $c_2$  such that (QI) holds for  $c_1 f$  and (RQI) holds for  $c_2 f$ .



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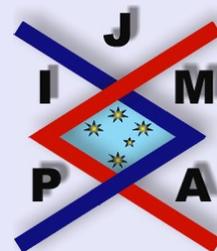
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## References

- [1] L. BOUGOFFA, Notes on Qi type integral inequalities, *J. Inequal. Pure Appl. Math.*, **4**(4) (2003), Art. 77. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=318>].
- [2] S. MAZOUZI AND F. QI, On an open problem regarding an integral inequality, *J. Inequal. Pure Appl. Math.*, **4**(2) (2003), Art. 31. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=269>].
- [3] J. PECARIĆ AND T. PEJKOVIĆ, Note on Feng Qi's integral inequality, *J. Inequal. Pure Appl. Math.*, **5**(3) (2004), Art. 51. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=418>].
- [4] T.K. POGANY, On an open problem of F. Qi, *J. Inequal. Pure Appl. Math.*, **3**(4) (2002), Art. 54. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=206>].
- [5] F. QI, Several integral inequalities, *J. Inequal. Pure Appl. Math.*, **1**(2) (2000), Art. 19. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=113>]. *RGMIA Res. Rep. Coll.*, **2**(7) (1999), Art. 9, 1039–1042. [ONLINE: <http://rgmia.vu.edu.au/v2n7.html>].



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