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ON A F. QI INTEGRAL INEQUALITY

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ABSTRACT. Necessary and sufficient conditions under which the Qi integral inequality

$$\int_a^b f^t(x)\,\mathrm{d}x \geq \left(\int_a^b f(x)\,\mathrm{d}x\right)^{t-1}$$

or its reverse hold for all $t \geq 1$ are given.

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1. INTRODUCTION

In [5] Feng Qi formulated the following problem: Characterize a positive function f such that the inequality

$$\int_a^b f^t(x)\,\mathrm{d}x \geq \left(\int_a^b f(x)\,\mathrm{d}x\right)^{t-1} \tag{1.1}$$

holds for $t > 1$.

In [1, 2, 3, 4] and the references therein, several sufficient conditions and generalizations are given. In all the cited papers the authors look for the solution of the Qi inequality with restricted t. This paper is another contribution to this subject. We shall try to establish conditions under which the inequality holds for all $t > 1$.

Let (X, μ) be a finite measure space and f be a positive measurable function. Define for $t \in \mathbb{R}$

$$H(t) = H(t, f) = \ln \int f^t\,\mathrm{d}\mu - (t-1)\ln\left(\int f\,\mathrm{d}\mu\right). \tag{1.2}$$

It is clear that inequality (1.1) is equivalent to $H(t) \geq 0$ for $t > 1$. We will say that for the function f the Qi Inequality (QI) holds if $H(t, f)$ is nonnegative for all $t \geq 1$. We will also say

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that for the function f the Reverse Qi Inequality (RQI) holds if $H(t,f)$ is non-positive for all $t \geq 1$.

By the Cauchy-Schwarz integral inequality, we have for $p,q \in \mathbb{R}$

$$\left(\int f^{\frac{p+q}{2}}\,\mathrm{d}\mu\right)^2 \leq \int f^p\,\mathrm{d}\mu \int f^q\,\mathrm{d}\mu \tag{1.3}$$

which means that the function $H(t)$ is convex, that is

$$H\left(\frac{t_1+t_2}{2}\right) \leq \frac{H(t_1)+H(t_2)}{2} \tag{1.4}$$

holds for $t_1, t_2 \in \mathbb{R}$, so its derivative

$$H'(t) = \frac{\int f^t \ln f\,\mathrm{d}\mu}{\int f^t\,\mathrm{d}\mu} - \ln\left(\int f\,\mathrm{d}\mu\right) \tag{1.5}$$

is increasing in $t \in \mathbb{R}$.

Let

$$M = \operatorname{ess\,sup}_{x\in X} f(x) \quad \text{and} \quad \mu_M = \mu(\{x : f(x) = M\}).$$

The following lemmas will be useful.

Note that from now on we will use the convention that $\ln\infty = \infty$ and $\ln 0 = -\infty$.

Lemma 1.1. *The following formula holds:*

$$\lim_{t\to\infty} \frac{H(t)}{t} = \ln \frac{M}{\int f\,\mathrm{d}\mu}. \tag{1.6}$$

Proof. For $\varepsilon > 0$ let $m_\varepsilon = \mu(x : f(x) > M - \varepsilon)$. Then

$$(M-\varepsilon)^t m_\varepsilon \leq \int f^t\,\mathrm{d}\mu \leq M^t \mu(X),$$

so

$$\begin{aligned} t\ln(M-\varepsilon) + \ln m_\varepsilon \\ &\leq H(t) + (t-1)\ln\left(\int f\,\mathrm{d}\mu\right) \\ &\leq t\ln M + \ln\mu(X). \end{aligned} \tag{1.7}$$

Dividing by t on both sides of (1.7) yields

$$\ln\frac{M-\varepsilon}{\int f\,\mathrm{d}\mu} \leq \liminf_{t\to\infty}\frac{H(t)}{t} \leq \limsup_{t\to\infty}\frac{H(t)}{t} \leq \ln\frac{M}{\int f\,\mathrm{d}\mu}.$$

In case $M = \infty$, $M - \varepsilon$ stands for an arbitrary large number. This completes the proof. □

Lemma 1.2. *If $M < \infty$ then*

$$\lim_{t\to\infty}\left(H(t) - t\ln\frac{M}{\int f\,\mathrm{d}\mu}\right) = \ln\left(\mu_M \int f\,\mathrm{d}\mu\right). \tag{1.8}$$

Proof. Direct computation yields

$$\begin{aligned} \lim_{t\to\infty}\left(H(t) - t\ln\frac{M}{\int f\,\mathrm{d}\mu}\right) \\ &= \lim_{t\to\infty}\ln\int\left(\frac{f}{M}\right)^t\mathrm{d}\mu + \ln\int f\,\mathrm{d}\mu \\ &= \ln\left(\mu_M\int f\,\mathrm{d}\mu\right) \end{aligned} \tag{1.9}$$

as $(f/M)^t$ tends monotonically to the characteristic function of $\{x : f(x) = M\}$. □

2. FENG QI INTEGRAL INEQUALITY

In this section we consider the problem: Characterize positive functions f that satisfy (QI).

Theorem 2.1. *A constant function M satisfies (QI) if and only if $\mu(X) \le 1$ and $M \ge 1/\mu(X)$.*

Proof. $H(t) \ge 0$ is equivalent to $M \ge \mu(X)^{t-2}$. This can be valid for all $t > 1$ only if the conditions of the theorem are fulfilled. □

From now on we assume that f is not constant, in which case the function H is strictly convex.

It is clear that the necessary condition for (QI) is $H(1) \ge 0$ or equivalently $\int f \, \mathrm{d}\mu \ge 1$.

Theorem 2.2. (a) *If $H(1) = 0$ then (QI) holds if and only if $H'(1) \ge 0$.*
(b) *If $H(1) > 0$ then*
(b1) *if $H'(1) \ge 0$ then (QI) holds;*
(b2) *if $H'(1) < 0$ and $M < \int f \, \mathrm{d}\mu$ then (QI) fails for large t;*
(b3) *if $H'(1) < 0$ and $M = \int f \, \mathrm{d}\mu$ then (QI) holds if and only if $\mu_M M \ge 1$;*
(b4) *if $H'(1) < 0$ and $M > \int f \, \mathrm{d}\mu$ then there exists an unique point t_0 such that $H'(t_0) = 0$ and (QI) holds if and only if $H(t_0) \ge 0$.*

Proof. (a) and (b1) follow immediately from convexity of H.

From Lemma 1.1 we see that H becomes negative for large t, which proves (b2).

(b3) follows from Lemma 1.2 and from the fact that being convex the graph of H lies above its horizontal asymptote.

Finally (b4) follows from the fact that H' is strictly increasing and $H'(t_0) = 0$ for some t_0, then H attains its minimum at t_0. Observe that in this case H may be infinite for some finite t_∞ and consequently for all $t > t_\infty$. □

From the above theorem we obtain the following, surprising

Corollary 2.3. *If $\mu(X) < 1$ then (QI) holds if and only if $H(1) \ge 0$.*

Proof. We will show that if $\mu(X) < 1$ then $H'(1) \ge 0$ for all f, so the condition (b1) is satisfied. Applying the integral Jensen Inequality to the convex function $x \ln x$ we obtain

$$\begin{aligned}\frac{1}{\mu(X)} \int f \ln f \, \mathrm{d}\mu &\ge \left(\frac{1}{\mu(X)} \int f \, \mathrm{d}\mu\right) \ln \left(\frac{1}{\mu(X)} \int f \, \mathrm{d}\mu\right) \\ &\ge \frac{1}{\mu(X)} \left(\int f \, \mathrm{d}\mu\right) \ln \left(\int f \, \mathrm{d}\mu\right)\end{aligned}$$

which is equivalent to $H'(1) \ge 0$. □

In the case (b4), solving the equation $H'(t) = 0$ may not be an easy task, but the following corollaries may be helpful:

Corollary 2.4. *Let*

$$t_L = \frac{\int f \ln f \, \mathrm{d}\mu - \int f \, \mathrm{d}\mu \ln \int f \, \mathrm{d}\mu}{\int f \ln f \, \mathrm{d}\mu}.$$

If $H'(t_L) \ge 0$ then (QI) holds.

Proof. t_L is the point where the supporting line drawn at $t = 1$ meets the OX-axis. The graph of H lies above it. In particular $H(t_L) \ge 0$. As $H'(t)$ is nonnegative for $t \ge t_L$ the proof is completed. □

Corollary 2.5. *If* $0 < \mu_M, M < \infty$ *let*

$$t_R = -\frac{\ln(\mu_M \int f \,\mathrm{d}\mu)}{\ln(M/\int f \,\mathrm{d}\mu)}.$$

If $H'(t_R) \le 0$ *or* $t_R \le t_L$ *then (QI) holds.*

Proof. t_R is the point where the supporting line drawn at ∞ (it exists by Lemma 1.2) meets the OX-axis. If $t_R \le t_L$ the two supporting lines meet above the OX-axis.
If $H'(t_R) \le 0$ we use an argument similar to that in the proof of the previous corollary. □

3. Reversed Feng Qi Inequality

In this section we give sufficient and necessary conditions for the reversed problem: Characterize positive functions f that satisfy (RQI).

Theorem 3.1. *A constant function satisfies (RQI) if and only if* $\mu(X) \ge 1$ *and* $M \le 1/\mu(X)$.

The proof is similar to that of Theorem 2.1.

Theorem 3.2. *For a non constant function* f *(RQI) holds if and only if* $H(1) \le 0$ *and* $M \le \int f \,\mathrm{d}\mu$.

Proof. As $\mu_M M < \int f \,\mathrm{d}\mu = \exp(H(1)) \le 1$ it follows from Lemma 1.1 and 1.2 that H is negative for large t. Being convex and non-positive at $t = 1$, it must be decreasing.
On the other hand if $M > \int f \,\mathrm{d}\mu$ then H is positive for large t by Lemma 1.1. □

4. Final remark

Finally we prove the following

Theorem 4.1. *For every positive function* f *there exists a constant* $c > 0$ *such that* cf *satisfies (QI) or (RQI).*

Proof. One can easily see that

$$H(t, cf) = \ln c + H(t, f),$$

so cf satisfies (QI) for certain c if and only if $H(t, f)$ is bounded from below. Similarly cf satisfies (RQI) only if $H(t, f)$ is bounded from above.
It follows immediately from Lemma 1.1 and Lemma 1.2 that the function $H(t)$ is bounded from below if and only if $M > \int f \,\mathrm{d}\mu$ or $M = \int f \,\mathrm{d}\mu$ and $\mu_M > 0$ and is bounded from above if and only if $M \le \int f \,\mathrm{d}\mu$.
This completes the proof of our theorem. □

Note that in case $M = \int f \,\mathrm{d}\mu$ and $\mu_M > 0$ we can find constants c_1 and c_2 such that (QI) holds for $c_1 f$ and (RQI) holds for $c_2 f$.

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