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MIXED ARITHMETIC AND GEOMETRIC MEANS AND RELATED INEQUALITIES

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ABSTRACT. Mixed arithmetic and geometric means, with and without weights, are both considered. Related to mixed arithmetic and geometric means, the following three types of inequalities and their generalizations, from three variables to a general n variables, are studied. For arbitrary $x, y, z \geq 0$ we have

(A) $$\left[\frac{x+y+z}{3}(xyz)^{1/3}\right]^{1/2} \leq \left(\frac{x+y}{2}\cdot\frac{y+z}{2}\cdot\frac{z+x}{2}\right)^{1/3},$$

(B) $$\frac{1}{3}\left(\sqrt{xy}+\sqrt{yz}+\sqrt{zx}\right) \leq \frac{1}{2}\left[\frac{x+y+z}{3}+(xyz)^{1/3}\right],$$

(D) $$\left[\frac{1}{3}(xy+yz+zx)\right]^{1/2} \leq \left(\frac{x+y}{z}\cdot\frac{y+z}{2}\cdot\frac{z+x}{2}\right)^{1/3}.$$

The main results include generalizations of J.C. Burkill's inequalities (J.C. Burkill; The concavity of discrepancies in inequalities of means and of Hölder, *J. London Math. Soc.* (2), **7** (1974), 617–626), and a positive solution for the conjecture considered by B.C. Carlson, R.K. Meany and S.A. Nelson (B.C. Carlson, R.K. Meany, S.A. Nelson; Mixed arithmetic and geometric means, *Pacific J. of Math.*, **38** (1971), 343–347).

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1. INTRODUCTION

In this paper, our inequalities concern generally arbitrary numbers of variables, however, the simplest most meaningful case for us is the case of three variables. Thus our motivation in this paper can be illustrated with three variables. Let x, y, z be any three non-negative numbers. By taking the arithmetic mean of two each of x, y, z we have three numbers $\frac{x+y}{2}$, $\frac{y+z}{2}$ and $\frac{z+x}{2}$. Taking the geometric mean of these three numbers, we have $\left(\frac{x+y}{2}\cdot\frac{y+z}{2}\cdot\frac{z+x}{2}\right)^{\frac{1}{3}}$. If our process of taking the arithmetic means and geometric means is reversed, first we have $\sqrt{xy}$,

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$\sqrt{yz}$ and $\sqrt{zx}$, then we have $\frac{1}{3}\left(\sqrt{xy}+\sqrt{yz}+\sqrt{zx}\right)$. The two numbers $\left(\frac{x+y}{2}\cdot\frac{y+z}{2}\cdot\frac{z+x}{2}\right)^{\frac{1}{3}}$ and $\frac{1}{3}\left(\sqrt{xy}+\sqrt{yz}+\sqrt{zx}\right)$ are called the mixed arithmetic and geometric means, or simply the mixed means, of x, y, z. Mixed arithmetic and geometric means appear in many branches of mathematics. However in this paper our interest is stimulated by the following inequality (C), which was proved by B.C. Carlson, R.K. Meany and S.A. Nelson, and simply referred to as CMN, see [2] and [3],

$$\text{(C)}\qquad \frac{1}{3}\left(\sqrt{xy}+\sqrt{yz}+\sqrt{zx}\right)\le\left(\frac{x+y}{2}\cdot\frac{y+z}{2}\cdot\frac{z+x}{2}\right)^{\frac{1}{3}}.$$

Besides inequality (C), our main concern in this paper is to study the following three types of inequalities, which are all related to mixed arithmetic and geometric means:

$$\text{(A)}\qquad \left[\frac{x+y+z}{3}\cdot(xyz)^{\frac{1}{3}}\right]^{\frac{1}{2}}\le\left(\frac{x+y}{2}\cdot\frac{y+z}{2}\cdot\frac{z+x}{2}\right)^{\frac{1}{3}},$$

$$\text{(B)}\qquad \frac{1}{3}\left(\sqrt{xy}+\sqrt{yz}+\sqrt{zx}\right)\le\frac{1}{2}\left[\frac{x+y+z}{3}+(xyz)^{\frac{1}{3}}\right],$$

$$\text{(D)}\qquad \left[\frac{1}{3}\left(xy+yz+zx\right)\right]^{\frac{1}{2}}\le\left(\frac{x+y}{2}\cdot\frac{y+z}{2}\cdot\frac{z+x}{2}\right)^{\frac{1}{3}}.$$

Because of the convexity of the square function; x^2, we have

$$\frac{1}{3}\left(\sqrt{xy}+\sqrt{yz}+\sqrt{zx}\right)\le\left[\frac{1}{3}\left(xy+yz+zx\right)\right]^{\frac{1}{2}},$$

thus the inequality (D) is stronger than the inequality (C), that is, (D) implies (C).

Except for (C), among the three inequalities (A), (B) and (D) there is no such relationship that one is stronger than another, namely they are independent of each other. One special relationship between (A) and (D) should be mentioned here, (A) and (D) can be transformed into each other through a transformation; $(x,y,z)\to\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right)$, $x,\ y,\ z>0$. We add a few more remarks. The inequalities (A) and (B) are special cases of more general known inequalities, which were proved by J.C. Burkill [1]. Further generalizations of Burkill's inequalities will be discussed later. The inequality (C) above is also the simplest case of the more general inequality proved by CMN [3], which will be mentioned later.

2. Definitions and Notations

Our main results in this paper are generalizations of (A), (B) and (D) from three variables to n variables. The first step toward generalization must be the formulation of mixed arithmetic and geometric means for n variables in general. This formulation, for the case of no weights, was given already in CMN [3].

Let $x_1,\dots,x_n\ge 0,\ n\ge 3$ be arbitrary non-negative numbers and denote $X=\{x_1,\dots,x_n\}$. For any non empty subset Y of X, denote $|Y|$ as the cardinal number of Y, and denote $S(Y)$ and $P(Y)$ as the sum of all numbers of Y and the product of all numbers of Y respectively. Denote further by $A(Y)$ and $G(Y)$ the arithmetic mean of Y and geometric mean of Y respectively. Namely we have

$$A(Y)=\frac{1}{|Y|}S(Y)\quad\text{and}\quad G(Y)=P(Y)^{\frac{1}{|Y|}}.$$

For any k with $1 \leq k \leq n$, we define the *k-th* mixed arithmetic and geometric mean of $\{x_1, \dots, x_n\} = X$ as follows, and we will use the notations

$$(G \circ A)_k (x_1, \dots, x_n) = (G \circ A)_k (X)$$

and

$$(A \circ G)_k (x_1, \dots, x_n) = (A \circ G)_k (X)$$

throughout the paper, where

(*k-th* $G \circ A$ mean) $$(G \circ A)_k (x_1, \dots, x_n) = \left[\prod_{Y \subset X, |Y|=k} A(Y) \right]^{\frac{1}{\binom{n}{k}}}$$

and

(*k-th* $A \circ G$ mean) $$(A \circ G)_k (x_1, \dots, x_n) = \frac{1}{\binom{n}{k}} \sum_{Y \subset X, |Y|=k} G(Y)$$

In CMN [3], they prove the following inequality (C) ([3, Theorem 2]), which is identical to the previous (C) if $n = 3$ and $k = l = 2$,

(C) $$(A \circ G)_l (x_1, \dots, x_n) \leq (G \circ A)_k (x_1, \dots, x_n)$$

for any $x_1, \dots, x_n \geq 0$ and any k and l satisfying $1 \leq k,\ l \leq n$ and $n + 1 \leq k + l$.

Denote $P_k (x_1, \dots, x_n) = P_k (X)$ the *k-th* elementary symmetric function of $x_1, \dots, x_n$, namely

$$P_k (x_1, \dots, x_n) = \sum_{Y \subset X, |Y|=k} P(Y).$$

We define the *k-th* elementary symmetric mean of $\{x_1, \dots, x_n\} = X$, denoted by $q_k (x_1, \dots, x_n) = q_k (X)$, as

$$q_k (x_1, \dots, x_n) = \left[\frac{1}{\binom{n}{k}} P_k (x_1 \cdots x_n) \right]^{\frac{1}{k}}.$$

By employing these notations, our generalization of (A), (B) and (D) from 3 variables to $n \geq 3$ variables are as follows:

(A) $$A (x_1, \dots, x_n)^{\frac{k-1}{n-1}} \cdot G (x_1, \dots, x_n)^{\frac{n-k}{n-1}} \leq (G \circ A)_k (x_1, \dots, x_n),$$

(B) $$(A \circ G)_k (x_1, \dots, x_n) \leq \frac{n-k}{n-1} A (x_1, \dots, x_n) + \frac{k-1}{n-1} G (x_1, \dots, x_n),$$

(D) $$q_l (x_1, \dots, x_n) \leq (G \circ A)_k (x_1, \dots, x_n)$$

for any k and l satisfying $1 \leq k,\ l \leq n$ and $n + 1 \leq k + l$.

Because of the convexity of the function; x^l for $x \geq 0$, we have

$$(A \circ G)_l (x_1, \dots, x_n) \leq q_l (x_1, \dots, x_n).$$

Hence our inequality (D) above is stronger than the inequality (C). Actually in CMN [3] the inequality (D) is conjectured to be true.

The inequalities (A), (B) and (D) will be proved in separate sections. In Section 3, the mixed arithmetic and geometric means *with general weights* are considered. With respect to general weights, our final formulation of the inequalities (A) and (B) are given and they are proven in Theorems 3.1 and 3.2, which give generalizations of J.C. Burkill's inequalities. In Section 4, the inequality (D) is proven in Theorem 4.1, and entire section consists of proving (D) and checking the equality condition of (D). In Section 5, the inequality (C) with three variables and general weights is formulated and proved in Theorem 5.1.

3. INEQUALITIES (A) AND (B) WITH WEIGHTS

All inequalities mentioned in our introduction are with equal weights, one can say without weights or no weights. For inequalities with weights, the order of given variables is very significant. Thus inequalities with weights do not have symmetry with respect to variables. Here we define one type of mixed arithmetic and geometric mean with weights, and we lose the symmetry between variables in our inequalities.

Let $t_1, \ldots, t_n$ be weights for n variables, that is, $t_1, \ldots, t_n$ are all positive numbers and $t_1 + \cdots + t_n = 1$. For any non negative n numbers $x_1, \ldots, x_n \geq 0$ we define the arithmetic mean and the geometric mean of $\{x_1, \ldots, x_n\} = X$ with weights $\{t_1, \ldots, t_n\}$ as usual, denoted by $A_t(x_1, \ldots, x_n) = A_t(X)$ and $G_t(x_1, \ldots, x_n) = G_t(X)$,

$$A_t(x_1, \ldots, x_n) = \sum_{i=1}^{n} t_i x_i,$$

$$G_t(x_1, \ldots, x_n) = \prod_{i=1}^{n} x_i^{t_i}.$$

With respect to the weights $\{t_1, \ldots, t_n\}$, similarly for any non-empty subset Y of $\{x_1, \ldots, x_n\} = X$, we define the arithmetic mean $A_t(Y)$ and the geometric mean $G_t(Y)$ as follows. Let Y be $\{x_{i_1}, \ldots, x_{i_k}\}$ for instance,

$$A_t(Y) = \frac{1}{t_{i_1} + \cdots + t_{i_k}} \left(t_{i_1} x_{i_1} + \cdots + t_{i_k} x_{i_k}\right),$$

$$G_t(Y) = \left(x_{i_1}^{t_{i_1}}, \ldots, x_{i_k}^{t_{i_k}}\right)^{\frac{1}{t_{i_1} + \cdots + t_{i_k}}}.$$

Next, the following number t_Y can be regarded as a weight for Y,

$$t_Y = \frac{1}{\binom{n-1}{k-1}} \left(t_{i_1} + \cdots + t_{i_k}\right),$$

because we have $t_Y > 0$ and $\sum_{Y \subset X, |Y|=k} t_Y = 1$.

Now we define the *k-th* mixed arithmetic and geometric means with weights $\{t_1, \ldots, t_n\}$ for any k of $1 \leq k \leq n$, denoted by

$$(G \circ A)_{k,t}(x_1, \ldots, x_n) = (G \circ A)_{k,t}(X)$$

and

$$(A \circ G)_{k,t}(x_1, \ldots, x_n) = (A \circ G)_{k,t}(X),$$

as follows:

(*k-th* $G \circ A$ mean) $$(G \circ A)_{k,t}(x_1, \ldots, x_n) = \prod_{Y \subset X, |Y|=k} A_t(Y)^{t_Y}$$

(*k-th* $A \circ G$ mean) $$(A \circ G)_{k,t}(x_1, \ldots, x_n) = \sum_{Y \subset X, |Y|=k} t_Y G_t(Y).$$

It is apparent that we have

$$(G \circ A)_{1,t}(X) = G_t(X) \quad \text{and} \quad (A \circ G)_{1,t}(X) = A_t(X) \quad \text{for} \quad k = 1$$

and

$$(G \circ A)_{n,t}(X) = A_t(X) \quad \text{and} \quad (G \circ A)_{n,t}(X) = G_t(X) \quad \text{for} \quad k = n.$$

And it can be seen that $(G \circ A)_{k,t}(X)$ is increasing with respect to k from $G_t(X)$ to $A_t(X)$. On the other hand, $(A \circ G)_{k,t}(X)$ is decreasing with respect to k from $A_t(X)$ to $G_t(X)$.

However, this property will not be used in the sequal, hence we omit the proof. The same property is proved for the case of no weights, see CMN [3].

Now we can formulate our inequalities (A) and (B) with weights and give our proof for them. We first prove (A).

Theorem 3.1. *Suppose k and n are positive integers and $1 \le k \le n$, and suppose $t_1, \ldots, t_n$ are weights. For any non-negative numbers $x_1, \ldots, x_n \ge 0$ we have*

$$A_t(x_1, \ldots, x_n)^{\frac{k-1}{n-1}} G_t(x_1, \ldots, x_n)^{\frac{n-k}{n-1}} \le (G \circ A)_{k,t}(x_1, \ldots, x_n). \tag{A}$$

For $k = 1$ or $k = n$, (A) is a trivial identity of either $G_t(x_1, \ldots, x_n) = G_t(x_1, \ldots, x_n)$ or $A_t(x_1, \ldots, x_n) = A_t(x_1, \ldots, x_n)$. For $2 \le k \le n-1$, the equality of (A) holds if and only if $x_1 = \cdots = x_n$ or the number of zeros among $x_1, \ldots, x_n$ is equal to k or larger than k.

Proof. There is nothing to prove if $k = 1$ or $k = n$. Thus we assume $2 \le k \le n-1$ and $3 \le n$. We assume also that our all variables $x_1, \ldots, x_n$ are positive until the last step of our proof, because we want to avoid unnecessary confusion.

Let $L(x_1, \ldots, x_n)$ be the ratio of the right side versus the left side of (A), namely

$$L(x_1, \ldots . x_n) = \frac{(G \circ A)_{k,t}(x_1, \ldots, x_n)}{A_t(x_1, \ldots, x_n)^{\frac{k-1}{n-1}} G_t(x_1, \ldots, x_n)^{\frac{n-k}{n-1}}}.$$

It suffices to prove $L(x_1, \ldots, x_n) \ge 1$ for all $x_1, \ldots, x_n > 0$. Our proof is divided into two steps of (i) and (ii), and step (i) is the main part of our proof.

(i) Choose arbitrary positive numbers $a_1, \ldots, a_n > 0$ which are not equal, and these $a_1, \ldots, a_n$ are fixed throughout step (i). By changing the order of (a_i, t_i), $1 \le i \le n$ if it is necessary, we can assume

$$a_1 = \min_{1 \le i \le n} a_i < a_2 = \max_{1 \le i \le n} a_i.$$

Set $\bar{a} = \frac{1}{t_1 + t_2}(t_1 a_1 + t_2 a_2)$, then clearly we have $a_1 < \bar{a} < a_2$.

Define $a_1(\lambda)$ and $a_2(\lambda)$ for all λ of $0 \le \lambda \le 1$ such that

$$a_1(\lambda) = (1-\lambda) a_1 + \lambda \bar{a} \quad \text{and} \quad a_2(\lambda) = (1-\lambda) a_2 + \lambda \bar{a},$$

then we have for all λ of $0 \le \lambda \le 1$:

(1) $a_1 \le a_1(\lambda) \le \bar{a} \le a_2(\lambda) \le a_2$,
(2) $t_1 a_1(\lambda) + t_2 a_2(\lambda) = t_1 a_1 + t_2 a_2$,
(3) $\frac{d}{d\lambda} a_1(\lambda) = \bar{a} - a_1$ and $\frac{d}{d\lambda} a_2(\lambda) = \bar{a} - a_2$.

If we regard $(a_1(\lambda), a_2(\lambda), a_3, \ldots, a_n)$ as a point in $\mathbb{R}^n$, we are considering here the line segment joining two points $(a_1 a_2, \ldots, a_n)$ and $(\bar{a}, \bar{a}, a_3, \ldots, a_n)$ in $\mathbb{R}^n$. Our main purpose of part (i) is to prove the following claim:

$$L(a_1(\lambda), a_2(\lambda), a_3, \ldots, a_n) \text{ is strictly decreasing with respect to } \lambda \text{ at a neighbour of } \lambda = 0. \tag{*}$$

Set $X_\lambda = \{a_1(\lambda), a_2(\lambda), a_3, \ldots, a_n\}$ for $0 \le \lambda \le 1$, hence $X_0 = \{a_1, a_2, \ldots, a_n\}$ for $\lambda = 0$. We have

$$L(a_1(\lambda), a_2(\lambda), a_3, \ldots, a_n) = \prod_{Y \subset X_\lambda,\ |Y| = k} \frac{A_t(Y)^{t_Y}}{A_t(X_\lambda)^{\frac{k-1}{n-1}} G_t(X_\lambda)^{\frac{n-k}{n-1}}}.$$

Note that $L(x_1, \ldots, x_n)$ decreases if and only if $\log L(x_1, \ldots, x_n)$ decreases.

Set

$$\phi(\lambda) = \log L(a_1(\lambda), a_2(\lambda), a_3, \dots, a_n) \quad \text{for} \quad 0 \le \lambda \le 1.$$

Then we have

$$\phi(\lambda) = \sum_{Y \subset X_\lambda,\ |Y|=k} t_Y \log A_t(Y) - \frac{k-1}{n-1} \log A_t(X_\lambda) - \frac{n-k}{n-1} \log G_t(X_\lambda).$$

Consider the derivative of $\phi(\lambda)$, note here $\frac{d}{d\lambda}[t_Y \log A_t(Y)] = 0$ if either of $a_1(\lambda)$ and $a_2(\lambda)$ belongs to Y or neither of $a_1(\lambda)$ and $a_2(\lambda)$ belongs to Y, and

$$\frac{d}{d\lambda}[t_Y \log A_t(Y)] = \frac{t_1(\bar{a} - a_1)}{\binom{n-1}{k-1} A_t(Y)} \quad \text{or} \quad \frac{t_2(\bar{a} - a_2)}{\binom{n-1}{k-1} A_t(Y)}$$

if $a_1(\lambda)$ belongs to Y but $a_2(\lambda)$ does not or $a_2(\lambda)$ belongs to Y but $a_1(\lambda)$ does not. Thus, denote Y by V if $a_1(\lambda) \in Y$ but $a_2(\lambda) \notin Y$, and by W if $a_1(\lambda) \notin Y$ but $a_2(\lambda) \in Y$. Then we have

$$\frac{d}{d\lambda}\phi(\lambda) = \sum_{V \subset X_\lambda} \frac{t_1(\bar{a} - a_1)}{\binom{n-1}{k-1} A_t(V)} + \sum_{W \subset X_\lambda} \frac{t_2(\bar{a} - a_2)}{\binom{n-1}{k-1} A_t(W)} - \frac{n-k}{n-1}\left[\frac{t_1(\bar{a} - a_1)}{a_1(\lambda)} + \frac{t_2(\bar{a} - a_2)}{a_2(\lambda)}\right],$$

since

$$\begin{aligned} &t_1(\bar{a} - a_1) + t_2(\bar{a} - a_2) = 0 \\ &= t_1(\bar{a} - a_1)\left[\sum_{V \subset X_\lambda} \frac{1}{\binom{n-1}{k-1} A_t(V)} - \sum_{W \subset X_\lambda} \frac{1}{\binom{n-1}{k-1} A_t(W)} - \frac{n-k}{n-1}\left(\frac{1}{a_1(\lambda)} - \frac{1}{a_2(\lambda)}\right)\right]. \end{aligned}$$

Thus, we have

$$\left.\frac{d}{d\lambda}\phi(\lambda)\right|_{\lambda=0} = t_1(\bar{a} - a_1)\left[\sum_{V \subset X_0} \frac{1}{\binom{n-1}{k-1} A_t(V)} - \sum_{W \subset X_0} \frac{1}{\binom{n-1}{k-1} A_t(W)} - \frac{n-k}{n-1}\left(\frac{1}{a_1} - \frac{1}{a_2}\right)\right].$$

Because $a_1 = \min_{1 \le i \le n} a_i$ and $a_2 = \max_{1 \le i \le n} a_i$, we have $a_1 \le A_t(V)$ for all $V \subset X_0$ and $A_t(W) \le a_2$ for all $W \subset X_0$, hence

$$\sum_{V \subset X_0} \frac{1}{\binom{n-1}{k-1} A_t(V)} \le \frac{\binom{n-2}{k-1}}{\binom{n-1}{k-1} a_1} = \frac{n-k}{n-1} \cdot \frac{1}{a_1}$$

and

$$\sum_{W \subset X_0} \frac{1}{\binom{n-1}{k-1} A_t(W)} \ge \frac{\binom{n-2}{k-1}}{\binom{n-1}{k-1} a_2} = \frac{n-k}{n-1} \cdot \frac{1}{a_2}.$$

However, note that at least one of the above two has a strict inequality, because one can observe that $A_t(V) = a_1$ for all $V \subset X_0$ is equivalent to $a_3 = \cdots = a_n = a_1$ and $A_t(W) = a_2$ for all $W \subset X_0$ is equivalent to $a_3 = \cdots = a_n = a_2$.

Thus we have

$$\left.\frac{d}{d\lambda}\phi(\lambda)\right|_{\lambda=0} < t(\bar{a} - a_1)\left[\frac{n-k}{n-1}\left(\frac{1}{a_1} - \frac{1}{a_2}\right) - \frac{n-k}{n-1}\left(\frac{1}{a_1} - \frac{1}{a_2}\right)\right] = 0.$$

Hence $\phi(\lambda)$ is strictly decreasing at a neighbour of $\lambda = 0$. This completes the proof of the claim (*).

(ii) For any ε, $0 < \varepsilon < 1$, consider a bounded closed region $D_\varepsilon = \left[\varepsilon, \frac{1}{\varepsilon}\right]^n$ of $\mathbb{R}_+^n = (0, \infty)^n$. It is apparent that $\bigcup_{0<\varepsilon<1} D_\varepsilon = \mathbb{R}_+^n$. Regarding $L(x_1, \dots, x_n)$ as a continuous function on $\mathbb{R}_+^n$, $L(x_1, \dots, x_n)$ attains the minimum value over the region D_ε for every ε, $0 < \varepsilon < 1$. We claim the following (**) for this minimum value.

(**) The minimum value of $L(x_1, \dots, x_n)$ over D_ε is 1 for every $\varepsilon,\ 0 < \varepsilon < 1$ and the minimum value is attained only at identical points of $x_1 = x_2 = \cdots = x_n$.

Suppose $(a_1, a_2, \dots, a_n)$ is any point of D_ε which gives the minimum value of $L(x_1, \dots, x_n)$ over D_ε. Suppose $(a_1, \dots, a_n)$ is not an identical point. Now, we can use the result proved in part (i). Without loss of generality we assume $a_1 = \min_{1\le i\le n} a_i$ and $a_2 = \max_{1\le i\le n} a_i$. It is clear that the whole line segment $(a_1(\lambda), a_2(\lambda), a_3, \dots, a_n)$ for $0 \le \lambda \le 1$, which is constructed in part (i), belongs to the region D_ε. Hence we have

$$L(a_1, \dots, a_n) \le L(a_1(\lambda), a_2(\lambda), a_3, \dots, a_n) \quad \text{for all } \lambda,\ 0 \le \lambda \le 1.$$

On the other hand the claim (*) guarantees

$$L(a_1(\lambda), a_2(\lambda), a_3, \dots, a_n) < L(a_1, \dots, a_n)$$

for λ which is sufficiently close to 0. Thus we have a contradiction. Hence we can conclude that $a_1 = a_2 = \cdots = a_n$ and also the minimum value of $L(x_1, \dots, x_n)$ over D_ε must be 1, because $L(a_1, a_2, \dots, a_n) = 1$ if $a_1 = a_2 = \cdots = a_n$. Thus the claim (**) is proved.

We have proved so far that among positive variables $x_1, \dots, x_n > 0$ the inequality (A) holds and the equality of (A) holds if and only if $x_1 = x_2 = \cdots = x_n > 0$. By continuity, it is trivially clear that our inequality (A) holds for any non-negative variables $x_1, \dots, x_n \ge 0$. The only point remaining unproven is the equality condition of (A) for non-negative variables $x_1, \dots, x_n$ which include 0. Suppose we have 0 among $x_1, \dots, x_n \ge 0$, then we have clearly $G_t(x_1, \dots, x_n) = 0$, thus the left side of (A) is 0. On the other hand, it is easy to see that the right side of (A) is 0 if and only if we have k or more than k many zeros among $x_1, \dots, x_n \ge 0$. Finally we can conclude that the equality of (A) for $x_1, \dots, x_n \ge 0$ holds if and only if $x_1 = x_2 = \cdots = x_n \ge 0$ or we have k or more than k many zeros among $x_1, \dots, x_n \ge 0$. This completes the proof of Theorem 3.1. □

Theorem 3.2. *Suppose k and n are positive integers and $1 \le k \le n$ and suppose $t_1, \dots, t_n$ are weights. For any non-negative numbers $x_1, \dots, x_n \ge 0$ we have*

$$\text{(B)} \qquad (A \circ G)_{k,t}(x_1, \dots, x_n) \le \frac{n-k}{n-1} A_t(x_1, \dots, x_n) + \frac{k-1}{n-1} G_t(x_1, \dots, x_n).$$

For $k = 1$ or $k = n$, (B) is actually a trivial identity,

$$A_t(x_1, \dots, x_n) = A_t(x_1, \dots, x_n) \quad \textit{or} \quad G_t(x_1, \dots, x_n) = G_t(x_1, \dots, x_n).$$

For $2 \le k \le n-1$, the equality of (B) holds if and only if $x_1 = \cdots = x_n$ or one of $x_1, \dots, x_n$ is zero and the others are equal.

There is a certain similarity between our inequalities (A) and (B), although it may not be clear what the essence of this similarity is. Thus, it is not a surprise that our proof of (B) is similar to the proof of (A).

Proof. There is nothing to prove if $k = 1$ or $k = n$. Thus we assume $2 \le k \le n-1$ and $3 \le n$. We assume also that all variables $x_1, \dots, x_n$ are positive until indicated otherwise.

Let $L(x_1,\dots,x_n)$ be the difference of the right side and the left side of (B), namely

$$L(x_1,\dots,x_n)=\frac{n-k}{n-1}A_t(x_1,\dots,x_n)\\ +\frac{k-1}{n-1}G_t(x_1,\dots,x_n)-(A\circ G)_{k,t}(x_1,\dots,x_n).$$

It suffices to prove $L(x_1,\dots,x_n)\ge 0$ for all $x_1,\dots,x_n>0$. Our proof is divided into the three parts of (i), (ii) and (iii). The equality condition of (B) is discussed in (iii).

(i) Choose arbitrary positive numbers $a_1,\dots,a_n>0$ which are not equal, and these $a_1,\dots,a_n$ are fixed through part (i). By changing the order of (a_i,t_i), $1\le i\le n$ if it is necessary, we can assume $a_1=\min_{1\le i\le n}a_i<a_2=\max_{1\le i\le n}a_i$.

Set $\hat a=\left(a_1^{t_1}a_2^{t_2}\right)^{\frac{1}{t_1+t_2}}$, then we have clearly $a_1<\hat a<a_2$.

Define $a_1(\lambda)$ and $a_2(\lambda)$ for all λ, $0\le\lambda\le 1$ such that $a_1(\lambda)=a_1^{1-\lambda}\hat a^{\lambda}$ and $a_2(\lambda)=a_2^{1-\lambda}\hat a^{\lambda}$, then we have for all λ, $0\le\lambda\le 1$:

(1) $a_1\le a_1(\lambda)\le\hat a\le a_2(\lambda)\le a_2$,
(2) $a_1(\lambda)^{t_1}a_2(\lambda)^{t_2}=a_1^{t_1}a_2^{t_2}$,
(3) $\frac{d}{d\lambda}a_1(\lambda)=\log\left(\frac{\hat a}{a_1}\right)a_1(\lambda)$ and $\frac{d}{d\lambda}a_2(\lambda)=\log\left(\frac{\hat a}{a_2}\right)a_2(\lambda)$.

If we regard $(a_1(\lambda),a_2(\lambda),a_3,\dots,a_n)$ as a point in $\mathbb{R}^n$, we are considering a curve joining two points of $(a_1,a_2,\dots,a_n)$ and $(\hat a_1\hat a_2,a_3,\dots,a_n)$ in $\mathbb{R}^n$. The main purpose of part (i) is to prove the following claim.

(*) $L(a_1(\lambda),a_2(\lambda),a_3,\dots,a_n)$ is strictly decreasing with respect to λ at a neighbour of $\lambda=0$.

Set $X_\lambda=\{a_1(\lambda),a_2(\lambda),a_3,\dots,a_n\}$ for $0\le\lambda\le 1$, thus $X_0=\{a_1,\dots,a_n\}$ for $\lambda=0$. We have

$$L(a_1(\lambda),a_2(\lambda),a_3,\dots,a_n)=\frac{n-k}{n-1}A_t(X_\lambda)+\frac{k-1}{n-1}G_t(X_\lambda)-\sum_{Y\subset X_\lambda,|Y|=k}t_YG_t(Y).$$

Denote simply $L(a_1(\lambda),a_2(\lambda),a_3,\dots,a_n)$ by $\phi(\lambda)$ and consider the derivative of $\phi(\lambda)$. Note here

$$\frac{d}{d\lambda}A_t(X_\lambda)=t_1\log\left(\frac{\hat a}{a_1}\right)a_1(\lambda)+t_2\log\left(\frac{\hat a}{a_2}\right)a_2(\lambda),$$
$$\frac{d}{d\lambda}G_t(X_\lambda)=0\quad\text{and}\quad\frac{d}{d\lambda}t_YG_t(Y)=0$$

if either of $a_1(\lambda)$ and $a_2(\lambda)$ belongs to Y or neither of them belongs to Y;

$$\frac{d}{d\lambda}t_YG_t(Y)=\frac{1}{\binom{n-1}{k-1}}t_1\log\left(\frac{\hat a}{a_1}\right)G_t(Y)\quad\text{or}\quad\frac{1}{\binom{n-1}{k-1}}t_2\log\left(\frac{\hat a}{a_2}\right)G_t(Y)$$

if $a_1(\lambda)$ belongs to Y but $a_2(\lambda)$ does not or $a_2(\lambda)$ belongs to Y but $a_1(\lambda)$ does not.

Thus, denote Y by V if $a_1(\lambda) \in Y$ but $a_2(\lambda) \notin Y$ and by W if $a_1(\lambda) \notin Y$ but $a_2(\lambda) \in Y$. Then we have

$$\frac{d}{d\lambda}\phi(\lambda) = \frac{n-k}{n-1}\left[t_1 \log\left(\frac{\hat{a}}{a_1}\right) a_1(\lambda) + t_2 \log\left(\frac{\hat{a}}{a_2}\right) a_2(\lambda)\right] - \frac{1}{\binom{n-1}{k-1}}\left[\sum_{V\subset X_\lambda} t_1 \log\left(\frac{\hat{a}}{a_1}\right) G_t(V) + \sum_{W\subset X_\lambda} t_2 \log\left(\frac{\hat{a}}{a_2}\right) G_t(W)\right].$$

Thus, we have

$$\left.\frac{d}{d\lambda}\phi(\lambda)\right|_{\lambda=0} = \frac{n-k}{n-1}\left[t_1 \log\left(\frac{\hat{a}}{a_1}\right) a_1 + t_2 \log\left(\frac{\hat{a}}{a_2}\right) a_2\right] - \frac{1}{\binom{n-1}{k-1}}\left[\sum_{V\subset X_0} t_1 \log\left(\frac{\hat{a}}{a_1}\right) G_t(V) + \sum_{W\subset X_0} t_2 \log\left(\frac{\hat{a}}{a_2}\right) G_t(W)\right],$$

and since $t_1 \log\left(\frac{\hat{a}}{a_1}\right) + t_2 \log\left(\frac{\hat{a}}{a_2}\right) = 0$,

$$\left.\frac{d}{d\lambda}\phi(\lambda)\right|_{\lambda=0} = t_1 \log\left(\frac{\hat{a}}{a_1}\right)\left\{\frac{n-k}{n-1}(a_1 - t_2) - \frac{1}{\binom{n-1}{k-1}}\left[\sum_{V\subset X_0} G_t(V) - \sum_{W\subset X_0} G_t(W)\right]\right\}.$$

Since $a_1 = \min_{1\le i\le n} a_i$ and $a_2 = \max_{1\le i\le n} a_i$, we have $a_1 \le G_t(V)$ for all $V \subset X_0$ and $a_2 \ge G_t(W)$ for all $W \subset X_0$, hence

$$\frac{1}{\binom{n-1}{k-1}}\sum_{V\subset X_0} G_t(V) \ge \frac{\binom{n-2}{k-1}}{\binom{n-1}{k-1}} a_1 = \frac{n-k}{n-1} a_1,$$

$$\frac{1}{\binom{n-1}{k-1}}\sum_{W\subset X_0} G_t(W) \le \frac{\binom{n-2}{k-1}}{\binom{n-1}{k-1}} a_2 = \frac{n-k}{n-1} a_2.$$

However, note that at least one of the above two has a strict inequality, because one can observe $G_t(V) = a_1$ for all $V \subset X_0$ is equivalent to $a_3 = \cdots = a_n = a_1$ and $G_t(W) = a_2$ for all $W \subset X_0$ is equivalent to $a_3 = \cdots = a_n = a_2$. Thus we have

$$\left.\frac{d}{d\lambda}\phi(\lambda)\right|_{\lambda=0} < t_1 \log\left(\frac{\hat{a}}{a_1}\right)\left[\frac{n-k}{n-1}(a_1 - a_2) - \frac{n-k}{n-1} a_1 + \frac{n-k}{n-1} a_2\right] = 0.$$

Hence $\phi(\lambda)$ is strictly decreasing at a neighbour of $\lambda = 0$. This completes the proof of the claim (*).

(ii) Based upon the claim (*) and exactly by the same arguments employed in part (ii) of our proof of Theorem 3.1, one can see that the following (**) is true. We omit its details.

(**) The minimum value of $L(x_1, \dots, x_n)$ over $\mathbb{R}_t^n = (0, \infty)^n$ is 0 and the minimum value is attained only at identical points of $x_1 = x_2 = \cdots = x_n > 0$.

Now we have proved that among positive variables $x_1, \dots, x_n > 0$ the inequality (B) holds and the equality of (B) holds if and only if $x_1 = \cdots = x_n > 0$. By continuity, it is trivially obvious that the inequality (B) holds for any non-negative variables $x_1, \dots, x_n \ge 0$. The only point left unproven is when the equality of (B) happens for non-negative variables which include 0. This is checked in the next step.

(iii) Suppose $x_1, \ldots, x_n \geq 0$ are given and at least one of them is 0, and suppose the number of positive x_i is l. Then we have $1 \leq l \leq n-1$. Without loss of generality we can assume $x_1, \ldots, x_l > 0$ and $x_{l+1} = \cdots = x_n = 0$.

Then the right side of (B)

$$= \frac{n-k}{n-1} A_t (x_1, \ldots, x_n) = \frac{n-k}{n-1} (t_1 x_1 + \cdots + t_l x_l) > 0.$$

On the other hand, if $l < k$, then we have the left side of $(B) = 0$, thus we have a strict inequality of (B) for this case. If $l \geq k$, let Y_0 be $\{x_1, \ldots, x_l\}$, then the left side of (B)

$$\begin{aligned}
&= \sum_{Y \subset Y_0, |Y|=k} t_Y G_t (Y) \leq \sum_{Y \subset Y_0, |Y|=k} t_Y A_t (Y) = \sum_{Y \subset Y_0, |Y|=k} \frac{1}{\binom{n-1}{k-1}} S_t (Y) \\
&= \frac{\binom{l-1}{k-1}}{\binom{n-1}{k-1}} (t_1 x_1 + \cdots + t_l x_l) \leq \frac{\binom{n-2}{k-1}}{\binom{n-1}{k-1}} (t_1 x_1 + \cdots + t_l x_l) \\
&= \frac{n-k}{n-1} (t_1 x_1 + \cdots + t_l x_l).
\end{aligned}$$

In the above, $S_t (Y)$ means the sum of all numbers of Y with respect to weights $\{t_1, \ldots, t_n\}$, for $Y = \{x_{i_1}, \ldots, x_{i_k}\} \subset Y_0 = \{x_1, \ldots, x_l\}$, for instance, we have $S_t (Y) = t_{i_1} x_{i_1} + \cdots + t_{i_k} x_{i_k}$.

Thus, from the above, the left side of (B) = the right side of (B) if and only if $G_t (Y) = A_t (Y)$ for all $Y \subset Y_0$ with $|Y| = k$ and $\binom{n-2}{k-1} = \binom{l-1}{k-1}$, and this is equivalent to $x_1 = \cdots = x_l$ and $l = n-1$. Now we have proved that the equality of (B) for $x_1, \ldots, x_n \geq 0$ including 0 happens if and only if only one of x_i is 0 and the others are equal. This completes the proof of Theorem 3.2. □

Inequalities (A) and (B) with weights can be considered as natural generalizations of J.C. Burkill's inequalities [1], namely (A) and (B) for $n = 3$ and $k = 2$ are identical to Burkill's inequalities.

By employing the same notations as in [1], we state Burkill's inequalities as a corollary of (A) and (B).

Corollary 3.3 (Burkill)**.** *Let $a, b, c > 0$ and $a + b + c = 1$. For any non-negative three numbers $x, y, z \geq 0$ we have:*

(A) $$(ax + by + cz)\, x^a y^b z^c \leq \left(\frac{ax + by}{a + b}\right)^{a+b} \cdot \left(\frac{by + cz}{b + c}\right)^{b+c} \cdot \left(\frac{cz + ax}{c + a}\right)^{c+a},$$

(B) $$(a + b) \left(x^a y^b\right)^{\frac{1}{a+b}} + (b + c) \left(y^b z^c\right)^{\frac{1}{b+c}} + (c + a) \left(z^c x^a\right)^{\frac{1}{c+a}} \leq ax + by + cz + x^a y^b z^c.$$

The equality of (A) holds if and only if $x = y = z$ or two of x, y, z are 0. The equality of (B) holds if and only if $x = y = z$ or one of x, y, z is 0 and the other two are equal.

4. INEQUALITIES (D) AND (C)

Before we start our proof of (D), our method of proof may be explained in a few lines. Elementary symmetric means $q_l (x_1, \ldots, x_n)$ are decreasing with respect to l for $1 \leq l \leq n$;

$$q_{l-1} (x_1, \ldots, x_n) \geq q_l (x_1, \ldots, x_n), \quad 2 \leq l \leq n.$$

This inequality is due to C. Maclaurin. Hardy, Littlewood and Pólya [4] give two kinds of proof for the Maclaurin inequality. The second proof, which is given on page 53 of [4], suggests that the inequality can be proven by examining the minimum value of $q_{l-1} (x_1, \ldots, x_n)$ over certain regions on which $q_l (x_1, \ldots, x_n)$ stays constant. We employ this method here. In our case,

$q_{l-1}(x_1,\ldots,x_n)$ is replaced by $(G\circ A)_k(x_1,\ldots,x_n)$ and we examine the minimum value of $(G\circ A)_k(x_1,\ldots,x_n)$ over certain regions on which $q_l(x_1,\ldots,x_n)$ stays unchanged. Another small remark should be added here. Since the Maclaurin inequality is available, it is sufficient for us to prove the inequality (D) for the case of $k+l=n+1$ only. However our proof will be done without the help of the Maclaurin inequality.

Theorem 4.1. *Suppose k, l and n are positive integers such that $1\le k,l\le n$ and $n+1\le k+l$. For any non-negative numbers $x_1,\ldots,x_n\ge 0$ we have*

$$q_l(x_1,\ldots,x_n)\le (G\circ A)_k(x_1,\ldots,x_n). \tag{D}$$

For $(k,l)=(n,1)$ or $(1,n)$, (D) is a trivial identity,

$$A(x_1,\ldots,x_n)=A(x_1,\ldots,x_n)\quad \text{or}\quad G(x_1,\ldots,x_n)=G(x_1,\ldots,x_n).$$

For $(k,l)\ne(n,1)$ and $(1,n)$, the equality condition of (D) is as follows,

(1) *$q_l(x_1,\ldots,x_n)=(G\circ A)_k(x_1,\ldots,x_n)>0$ if and only if $x_1=\cdots=x_n>0$,*
(2) *$q_l(x_1,\ldots,x_n)=(G\circ A)_k(x_1,\ldots,x_n)=0$ if and only if k or more than k many x_i are zero.*

Proof. Our proof is divided into three parts. A preliminary lemma is given in part (i), part (ii) contains the main arguments of our proof, and the equality condition of (D) is examined in part (iii).

(i) The assumption of $n+1\le k+l$ in our inequality (D) is very crucial, namely (D) does not hold without this assumption. The condition of $n+1\le k+l$ is needed only in the following situation. Suppose X is a set of cardinality n, then for any subsets U and V of X, whose cardinality are k and l respectively, we have a non empty intersection $U\cap V\ne\phi$ if $k+l\ge n+1$. Throughout our proof of (D), the following preliminary lemma is the only place where the condition of $n+1\le k+l$ is used.

Suppose $x_1,\ldots,x_n$ are positive numbers and set $X=\{x_1,\ldots,x_n\}$. As defined in the introduction, $P_l(X)$ stands for the *l-th* elementary symmetric function of $x_1,\ldots,x_n$, $S(V)$ stands for the sum of all numbers belonging to $V\subset X$ and $P_{l-1}(X)=P_0(X)$ for $l=1$ is defined as the constant 1.

Lemma 4.2. *Suppose $1\le k,l\le n$ and $n+1\le k+l$. For any subset V of X with $|V|=k$, we have $S(V)P_{l-1}(X)\ge P_l(X)$. The equality holds if and only if $k=n$ and $l=1$.*

Proof of Lemma 4.2. Suppose $l=1$, then we have $k=n$ because of our assumption $k+l\ge n+1$. Thus we have $P_1(X)=S(X)$, $V=X$ and $P_0(X)=1$, hence $S(V)P_{l-1}(X)=S(X)$. We have the equality of $S(V)P_{l-1}(X)=P_l(X)$. Suppose $l\ge 2$ and $V\subset X$ with $|V|=k$ is given. One can assume $V=\{x_1,\ldots,x_k\}$ without loss of generality. Then we have

$$S(V)P_{l-1}(X)=\sum_{i=1}^{k}\sum_{W\subset X,|W|=l-1}x_iP(W) \tag{4.1}$$

and

$$P_l(X)=\sum_{V\subset X,|V|=l}P(V) \tag{4.2}$$

Since $U\cap V\ne\phi$ for all $U\subset X$ with $|U|=l$, let x_{i_u} be the member of $U\cap V=U\cap\{x_1,\ldots,x_k\}$ which has the smallest suffix and let W_u be the subset $U\setminus\{x_{i_u}\}$. Then it is obvious that the correspondence: $U\to(x_{i_u},W_u)$ is one to one and we have $P(U)=x_{i_u}P(W_u)$ for all $U\subset X$ with $|U|=l$. Compare the two summations of (4.1) and (4.2) above, and cancel off equal terms which correspond to each other. Every term $P(U)$ of (4.2) can be cancelled by the corresponding term $x_{i_u} P(W_u)$ of (4.1) and every term $x_i P(W)$ satisfying $x_i \in W$ of (4.1) is not cancelled and left as it is. Hence we can conclude that $S(V) P_{l-1}(X) > P_l(X)$. This completes the proof of Lemma 4.2. $\square$

(ii) There is nothing to prove if $(k, l) = (1, n)$ or $(n, 1)$. Because of our assumption $n+1 \leq k+l$, if $k = 1$ then $l = n$, thus we have

$$q_l(X) = q_n(X) = G(X) \quad \text{and} \quad (G \circ A)_k(X) = (G \circ A)_1(X) = G(X),$$

hence our inequality (D) turns into an identity of $G(X) = G(X)$. Similarly (D) turns into $A(X) = A(X)$ if $l = 1$. If $n = 2$ and $k = l = 2$, then (D) turns into the inequality $G(X) \leq A(X)$, which holds. Thus we consider only the case of $2 \leq k, l \leq n,\ 3 \leq n$ and $n + 1 \leq k + l$.

We suppose also that all variables $x_1, \ldots, x_n$ are positive throughout part (ii).

Choose fixed arbitrary variables $a_1, \ldots, a_n > 0$ in what follows. If $a_1, \ldots, a_n$ are equal, $a_1 = \cdots = a_n = a$, then our inequality (D) holds trivially as $q_l(a_1, \ldots, a_n) = a = (G \circ A)_k (a_1, \ldots, a_n)$. Thus we assume $a_1, \ldots, a_n$ are not identical. The following (*) is what we have to prove.

(*) $$q_l(a_1, \ldots, a_n) < (G \circ A)_k (a_1, \ldots, a_n).$$

Depending on $(a_1, \ldots, a_n)$, consider a bounded closed region D_a of $\mathbb{R}^n_+ = (0, \infty)^n$ as follows,

$$D_a = \left\{ (x_1, \ldots, x_n) \ \mid q_l(x_1, \ldots, x_n) = q_l(a_1, \ldots, a_n), \right.$$
$$\left. \min_{1 \leq i \leq n} a_i \leq x_i \leq \max_{1 \leq i \leq n} a_i \quad \text{for all } 1 \leq i \leq n \right\}.$$

Clearly the point $(a_1, \ldots, a_n)$ belongs to D_a.

Our second claim is as follows,

(**) The minimum value of $(G \circ A)_k (x_1, \ldots, x_n)$ over the region D_a is equal to $q_l(a_1, \ldots, a_n)$ and the minimum value is attained only at an identical point of D_a.

Since an identical point which belongs to D_a is only one point of $(x_1, \ldots, x_n)$ with $x_i = q_l(a_1, \ldots, a_n)$ for all $1 \leq i \leq n$, the second half of (**) implies the first half of (**). It is also clear that the claim (*) follows from the claim (**). Thus we can concentrate on proving the second half of (**). Now we employ the method of contradiction: reductio ad absurdum. Suppose the minimum value of $(G \circ A)_k (x_1, \ldots, x_n)$ over the region D_a is attained at a non-identical point $(b_1, \ldots, b_n)$ of D_a. We assume, without loss of generality, $\min_{1 \leq i \leq n} b_i = b_1 < b_2 = \max_{1 \leq i \leq n} b_i$.

Next, we are going to choose a suitable continuous curve $(x, \varphi(x), b_3, \ldots, b_n)$ with $b_1 \leq x \leq b_2$ within our region D_a. For this purpose the recurrence formulas on elementary symmetric functions are useful.

The following recurrence formula is easily seen.

$$\begin{aligned} P_l &= (x_1, \ldots, x_n) \\ &= P_l^{(n-2)}(x_3, \ldots, x_n) + (x_1 + x_2) P_{l-1}^{(n-2)}(x_3, \ldots, x_n) + x_1 x_2 P_{l-2}^{(n-2)}(x_3, \ldots, x_n), \end{aligned}$$

where $P_l^{(n-2)}(x_3, \ldots, x_n)$ denotes the *l-th* elementary symmetric function of $(n-2)$ variables $x_3 \cdots x_n$. More precisely, if $l = n$ then the first and second terms of the right side of the formula disappear, and if $l = n - 1$ then the first term disappears. Thus, in the following arguments we have to change our expressions a little bit for the case of $l = n$ or $l = n - 1$. However, since we are not losing generality, we will keep the recurrence formula above and omit details for the case of $l = n$ or $n - 1$.

For any x and y we have

$$\begin{aligned} P_l &= (x, y, b_3, \dots, b_n) \\ &= P_l^{(n-2)} (b_3, \dots, b_n) + (x + y) P_{l-1}^{(n-2)} (b_3, \dots, b_n) + xy P_{l-2}^{(n-2)} (b_3, \dots, b_n) . \end{aligned}$$

We simplify our notations by setting Q_l, Q_{l-1} and Q_{l-2} as

$$Q_l = P_l^{(n-2)} (b_3, \dots, b_n) , \quad Q_{l-1} = P_{l-1}^{(n-2)} (b_3, \dots, b_n)$$
$$\text{and} \quad Q_{l-2} = P_{l-2}^{(n-2)} (b_3, \dots, b_n) .$$

Then we have

$$P_l (b_1, b_2, \dots, b_n) = Q_l + (b_1 + b_2) Q_{l-1} + b_1 b_2 Q_{l-2}, \tag{4.3}$$

$$P_l (x, y, b_3, \dots, b_n) = Q_l + (x + y) Q_{l-1} + xy Q_{l-2}. \tag{4.4}$$

Now we can solve the equation

$$P_l (b_1, b_2, \dots, b_n) = P_l (x, y, b_3, \dots, b_n) ,$$

by solving (4.3) and (4.4) above simultaneously. For any given $x > 0$ there is a y uniquely denoted by $\varphi (x)$, such that

$$y = \varphi (x) = \frac{(b_1 + b_2 - x) \; Q_{l-1} + b_1 b_2 Q_{l-2}}{Q_{l-1} + x \, Q_{l-2}}, \tag{4.5}$$

$$P_l (b_1, b_2, \dots, b_n) = P_l (x, \varphi (x) , b_3, \dots, b_n) . \tag{4.6}$$

From expression (4.5), it follows that $\varphi (b_1) = b_2$, $\varphi (b_2) = b_1$ and $\varphi (x)$ decreases from b_2 to b_1 if x increases from b_1 to b_2. Thus, for all x with $b_1 \le x \le b_2$ we have

$$\min_{1 \le i \le n} \; a_i \le b_1 \le x, \quad \varphi (x) \le b_2 \le \max_{1 \le i \le n} \; a_i.$$

From (4.6), we have also

$$\begin{aligned} q_l (a_1, a_2, \dots, a_n) &= q_l (b_1, \dots, b_n) \\ &= \left[\frac{1}{\binom{n}{l}} P_l (b_1, b_2, \dots, b_n) \right]^{\frac{1}{l}} \\ &= \left[\frac{1}{\binom{n}{l}} P_l (x, \varphi (x) , b_3, \dots, b_n) \right]^{\frac{1}{l}} \\ &= q_l (x, \varphi (x) , b_3, \dots, b_n) . \end{aligned}$$

Hence, our continuous curve $(x, \varphi (x) , b_3, \dots, b_n)$ for $b_1 \le x \le b_2$ is located within our region D_a. Since the minimum value of $(G \circ A)_k (x_1, \dots, x_n)$ over D_a is attained at $(b_1, b_2, \dots, b_n)$, we have for all x of $b_1 \le x \le b_2$:

$$(G \circ A)_k (x, \varphi (x) , b_3, \dots, b_n) \ge (G \circ A)_k (b_1, b_2, \dots, b_n) \tag{4.7}$$

Next, we will see that $(G \circ A)_k (x, \varphi (x) , b_3, \dots, b_n)$ is strictly decreasing at a neighbour of $x = b_1$.

Denote

$$\phi (x) = \log \; [(G \circ A)_k (x, \varphi (x) , b_3, \dots, b_n)]$$

and calculate the derivative $\frac{d}{dx}\phi(x)=\phi'(x)$. Setting $B=\{b_3,\dots,b_n\}$,

$$\begin{aligned}\phi'(x)&=\frac{d}{dx}\left[\frac{1}{\binom{n}{k}}\sum_{Y\subset\{x,\varphi(x),b_3,\dots,b_n\},|Y|=k}\log A(Y)\right]\\&=\frac{1}{\binom{n}{k}}\sum_{V\subset B,|V|=k-1}\left[\frac{1}{S(V)+x}+\frac{\varphi'(x)}{S(V)+\varphi(x)}\right]\\&\quad+\frac{1}{\binom{n}{k}}\sum_{W\subset B,|W|=k-2}\frac{1+\varphi'(x)}{S(W)+x+\varphi(x)}.\end{aligned}$$

Hence we have

$$\begin{aligned}\phi'(b_1)=\frac{1}{\binom{n}{k}}&\sum_{V\subset B,|V|=k-1}\left[\frac{1}{S(V)+b_1}+\frac{\varphi'(b_1)}{S(V)+b_2}\right]\\&+\frac{1}{\binom{n}{k}}\sum_{W\subset B,|W|=k-2}\frac{1+\varphi'(b_1)}{S(W)+b_1+b_2}.\end{aligned}$$

Let L be the first summation and let M be the second summation in the above, namely,

$$L=\frac{1}{\binom{n}{k}}\sum_{V\subset B,|V|=k-1}\left[\frac{1}{S(V)+b_1}+\frac{\varphi'(b_1)}{S(V)+b_2}\right],$$

$$M=\frac{1}{\binom{n}{k}}\sum_{W\subset B,|W|=k-2}\frac{1+\varphi'(b_1)}{S(W)+b_1+b_2}.$$

Using the expression (4.5) of $\varphi(x)$, we get

$$\varphi'(b_1)=-\frac{Q_{l-1}+b_2Q_{l-2}}{Q_{l-1}+b_1Q_{l-2}}<-1. \tag{4.8}$$

Thus, we have

$$\begin{aligned}\frac{1}{S(V)+b_1}+\frac{\varphi'(b_1)}{S(V)+b_2}&=\frac{1}{S(V)+b_1}-\frac{Q_{l-1}+b_2Q_{l-2}}{[S(V)+b_2][Q_{l-1}+b_1Q_{l-2}]}\\&=-\frac{(b_2-b_1)[S(V)Q_{l-2}-Q_{l-1}]}{[S(V)+b_1][S(V)+b_2][Q_{l-1}+b_1Q_{l-2}]},\end{aligned}$$

hence

$$L=\frac{1}{\binom{n}{k}}\sum_{V\subset B,|V|=k-1}\frac{-(b_2-b_1)[S(V)Q_{l-2}-Q_{l-1}]}{[S(V)+b_1][S(V)+b_2][Q_{l-1}+b_1Q_{l-2}]}.$$

We apply our lemma to $S(V)Q_{l-2}-Q_{l-1}$,

$$S(V)Q_{l-2}-Q_{l-1}=S(V)P_{l-2}^{(n-2)}(b_3,\dots,b_n)-P_{l-1}^{(n-2)}(b_3,\dots,b_n),$$

$V\subset B=\{b_3,\dots,b_n\}$, $|V|=k-1$.

Since $(k-1)+(l-1)\geq(n-2)+1$, by Lemma 4.2 in part (i), we can conclude that $S(V)Q_{l-2}-Q_{l-1}\geq 0$ for all $V\subset B$ with $|V|=k-1$, hence $L\leq 0$. On the other hand, from (4.8) we have $1+\varphi'(b_1)<0$, hence $M<0$. Finally, we have $\varphi'(b_1)=L+M<0$. This means that $\log[(G\circ A)_k(x,\varphi(x),b_3,\dots,b_n)]$ is strictly decreasing at a neighbour of b_1. Now we have for all $x>b_1$, sufficiently close to b_1 :

$$(G\circ A)_k(x,\varphi(x),b_3,\dots,b_n)<(G\circ A)_k(b_1,b_2,\dots,b_n). \tag{4.9}$$

Clearly (4.9) contradicts (4.7). Thus we complete the proof of our claim (**).

What we have proved so far is the following: With respect to positive variables, inequality (D) holds for every $x_1, \dots, x_n > 0$ and the equality of (D) holds if and only if $x_1 = \cdots = x_n > 0$.

By continuity, it is obvious that the inequality (D) holds for all non-negative variables $x_1, \dots, x_n \geq 0$. The only remaining unproven point is when the equality of (D) happens for non-negative variables which include 0.

(iii) As mentioned at the beginning of part (ii), (D) is actually an identity if $(k, l) = (1, n)$ or $(n, 1)$. Hence the equality condition of (D) should be examined for the case of $2 \leq k, l \leq n$ and $n + 1 \leq k + l$.

Suppose $x_1, \dots, x_n \geq 0$, which include 0, are given and suppose the number of positive x_i is m, then we have $1 \leq m \leq n - 1$. Without loss of generality we can assume $x_1, \dots, x_m > 0$ and $x_{m+1} = \cdots = x_n = 0$.

Set $X = \{x_1, \dots, x_n\}$ and $X_+ = \{x_1, \dots, x_m\}$. First, the following is easy to observe.

$$\text{If } n - m \geq k, \text{ then } q_l(X) = 0 = (G \circ A)_k(X), \quad \text{hence we have an equality for } (D). \tag{4.10}$$

As there is a subset $Y \subset X$ with $|Y| = k$ such that Y consists of k many 0, hence $A(Y) = 0$ and we have $(G \circ A)_k(X) = 0$.

On the other hand, since $l \geq n + 1 - k \geq m + 1$, every subset $Z \subset X$with $|Z| = l$ contains 0, hence $P(Z) = 0$, thus we have $q_l(X) = 0$.

In our remaining arguments, we will show that (4.10) above is the only case for which the equality of (D) holds. Namely we claim the following:

$$\text{If } n - m < k, \text{ then we have } q_l(X) < (G \circ A)_k(X). \tag{***}$$

Our proof of (***) is completed as follows. Firstly, if $n - m < k$ and $m < l$, then we have

$$q_l(X) = 0 < (G \circ A)_k(X).$$

Since $n - m < k$, every subset $Y \subset X$ with $|Y| = k$ contains a positive number, hence $A(Y) > 0$, thus we have $(G \circ A)_k(X) > 0$. On the other hand, because of $m < l$, every subset $Z \subset X$ with $|Z| = l$ contains 0, hence $P(Z) = 0$, and we have $q_l(X) = 0$.

In order to prove (***), we limit ourselves to $n - m < k$ and $n - k + 1 \leq l \leq m$.

First we have

$$q_l(X) = \left[\frac{\binom{m}{l}}{\binom{n}{l}}\right]^{\frac{1}{l}} q_l(X_+), \tag{4.11}$$

since,

$$\begin{aligned} q_l(X) &= \left[\frac{1}{\binom{n}{l}} \sum_{Z \subset X, |Z| = l} P(Z)\right]^{\frac{1}{l}} \\ &= \left[\frac{1}{\binom{n}{l}} \sum_{Z \subset X_+, |Z| = l} P(Z)\right]^{\frac{1}{l}} \\ &= \left[\frac{\binom{m}{l}}{\binom{n}{l}}\right]^{\frac{1}{l}} \left[\frac{1}{\binom{m}{l}} \sum_{Z \subset X_+, |Z| = l} P(Z)\right]^{\frac{1}{l}} = \left[\frac{\binom{m}{l}}{\binom{n}{l}}\right]^{\frac{1}{l}} q_l(X_+). \end{aligned}$$

Next we examine a relationship between $(G\circ A)_k(X)$ and $(G\circ A)_k(X_+)$. For any subset $Y\subset X$ with $|Y|=k$, the cardinality of $Y\cap X_+$ is possibly between $k-(n-m)$ and $\min\{k,m\}$, namely

$$k-(n-m)\le |Y\cap X_+|\le \min\{k,m\}.$$

Denote $k_0=k-(n-m)$ and $k_1=\min\{k,m\}$, then $k_0<k_1\le k$.

Now we have

$$(G\circ A)_k(X)=\prod_{k_0\le p\le k_1}\left(\frac{p}{k}\right)^{\binom{m}{p}\binom{n-m}{k-p}/\binom{n}{k}}\cdot\prod_{k_0\le p\le k_1}(G\circ A)_p(X_+)^{\binom{m}{p}\binom{n-m}{k-p}/\binom{n}{k}}, \tag{4.12}$$

since,

$$\begin{aligned}(G\circ A)_k(X)^{\binom{n}{k}}&=\prod_{Y\subset X,|Y|=k}A(Y)\\&=\prod_{k_0\le p\le k_1}\left[\left(\frac{p}{k}\right)^{\binom{m}{p}\binom{n-m}{k-p}}\prod_{W\subset X_+,|W|=p}A(W)^{\binom{n-m}{k-p}}\right]\\&=\prod_{k_0\le p\le k_1}\left[\left(\frac{p}{k}\right)^{\binom{m}{p}\binom{n-m}{k-p}}\cdot(G\circ A)_p(X_+)^{\binom{m}{p}\binom{n-m}{k-p}}\right]\\&=\prod_{k_0\le p\le k_1}\left(\frac{p}{k}\right)^{\binom{m}{p}\binom{n-m}{k-p}}\cdot\prod_{k_0\le p\le k_1}(G\circ A)_p(X_+)^{\binom{m}{p}\binom{n-m}{k-p}}.\end{aligned}$$

In (4.11) and (4.12), by setting $x_1=\cdots=x_m=1$ we have

$$q_l(1,\dots,1,\ 0,\dots,0)=\left[\frac{\binom{m}{l}}{\binom{n}{l}}\right]^{\frac{1}{l}}\quad\text{and} \tag{4.13}$$

$$(G\circ A)_k(1,\dots,1,\ 0,\dots,0)=\prod_{k_0\le p\le k_1}\left(\frac{p}{k}\right)^{\binom{m}{p}\binom{n-m}{k-p}/\binom{n}{k}}.$$

Thus, (4.11) and (4.12) can be expressed as

$$q_l(X)=q_l(1,\dots,1,\ 0,\dots,0)\,q_l(X_+), \tag{4.14}$$

$$(G\circ A)_k(X)=(G\circ A)_k(1,\dots,1,\ 0,\dots,0)\cdot\prod_{k_0\le p\le k_1}(G\circ A)_p(X_+)^{\binom{m}{p}\binom{n-m}{k-p}/\binom{n}{k}}. \tag{4.15}$$

For any p of $k_0\le p\le k_1$, we have

$$p+l\ge k_0+l=k-(n-m)+l=k+l-n+m\ge 1+m,$$

namely $p+l\ge m+1$. Thus inequality (D) for X_+, which was already proven in (ii), yields

$$(G\circ A)_p(X_+)\ge q_l(X_+)\quad\text{for all } p,\ k_0\le p\le k_1.$$

And, since

$$\sum_{k_0\le p\le k_1}\frac{\binom{m}{p}\binom{n-m}{k-p}}{\binom{n}{k}}=1,$$

we have from (4.15)

$$(G\circ A)_k(X)\ge (G\circ A)_k(1,\dots,1,\ 0,\dots,0)\,q_l(X_+).$$

From (4.14) we have

$$(G\circ A)_k(X) \geq \frac{(G\circ A)_k(1,\dots,1,\ 0,\dots,0)}{q_l(1,\dots,1,\ 0,\dots,0)} q_l(X). \tag{4.16}$$

From (4.16), it is obvious that

$$(G\circ A)_k(1,\dots,1,\ 0,\dots,0) > q_l(1,\dots,1,\ 0,\dots,0)$$

yields

$$(G\circ A)_k(X) > q_l(X).$$

Thus our claim (***) is reduced to proving

$$(G\circ A)_k(1,\dots,1,\ 0,\dots,0) > q_l(1,\dots,1,\ 0,\dots,0).$$

Note that this is a very special case of (***).

$$(G\circ A)_k(1,\dots,1,\ 0,\dots,0) > q_l(1,\dots,1,\ 0,\dots,0). \tag{4.17}$$

Proof of (4.17). In (4.17), we replace 0 by non-negative variable $x \geq 0$, denote

$$L(x) = (G\circ A)_k(1,\dots,1,\ x,\dots,x) \quad \text{and} \quad M(x) = q_l(1,\dots,1,\ x,\dots,x).$$

Then, from the inequality (D) we have $L(x)/M(x) \geq 1$ for all $x \geq 0$. Thus, if we know that $L(x)/M(x)$ is strictly decreasing at $x = 0$, we can conclude $L(0)/M(0) > 1$, which is the same as (4.17). Set $f(x) = \log[L(x)/M(x)]$ and calculate the derivative of $f(x)$ at $x = 0$. According to the definitions of $(G\circ A)_k(X)$ and $q_l(X)$, we have

$$\begin{aligned} L(x) &= (G\circ A)_k(1,\dots,1,\ x,...,x) \\ &= \prod_{k_0\leq p\leq k_1}\left[\frac{p+(k-p)x}{k}\right]^{\binom{m}{p}\binom{n-m}{k-p}/\binom{n}{k}} \end{aligned}$$

and

$$M(x) = q_l(1,\dots,1,\ x,...,x) = \left[\sum_{0\leq r\leq l}\frac{\binom{n-m}{r}\binom{m}{l-r}}{\binom{n}{r}}x^r\right]^{\frac{1}{l}}.$$

Now, we can calculate $\frac{d}{dx}\left[\log L(x) - \log M(x)\right]\big|_{x=0} = f'(0)$,

$$\begin{aligned} f'(0) &= \sum_{k_0\leq p\leq k_1}\frac{\binom{m}{p}\binom{n-m}{k-p}}{\binom{n}{k}}\frac{k-p}{p} - \frac{1}{l}\frac{\binom{n-m}{1}\binom{m}{l-1}/\binom{n}{l}}{\binom{m}{l}/\binom{n}{l}} \\ &= \sum_{k_0\leq p\leq k_1}\frac{\binom{m}{p}\binom{n-m}{k-p}}{\binom{n}{k}}\frac{k-p}{p} - \frac{n-m}{m-l+1}. \end{aligned}$$

Note $\sum_{k_0\leq p\leq k_1}\binom{m}{p}\binom{n-m}{k-p}\Big/\binom{n}{k} = 1$ and $\frac{k-p}{p} < \frac{k-k_0}{k_0} = \frac{n-m}{k_0}$ for all p of $k_0 < p \leq k_1$.

The above expression of $f'(0)$ yields

$$f'(0) < \frac{n-m}{k_0} - \frac{n-m}{m-l+1} = (n-m)\frac{n+1-(k+l)}{k_0(m-l+1)} \leq 0.$$

This completes the proof of (4.17). □

Together with what we proved in (ii), we have completed the proof of the equality condition of (D) stated in Theorem 4.1, namely $q_l(x_1,\dots,x_n) = (G\circ A)_k(x_1,\dots,x_n) > 0$ if and only if $x_1 = \cdots = x_n > 0$ and $q_l(x_1,\dots,x_n) = (G\circ A)_k(x_1,\dots,x_n) = 0$ if and only if k or more than k many x_i are zero. This completes the proof of Theorem 4.1. □

As mentioned in the introduction, the inequality (C) can be regarded as corollary of Theorem 4.1. And it is easy to see that the equality condition of (C) is the same as the equality condition of (D).

Corollary 4.3 (Carlson, Meany and Nelson)**.** *Suppose k, l and n are positive integers such that $1 \leq k, l \leq n$ and $n+1 \leq k+l$. For any non-negative numbers $x_1, \dots, x_n \geq 0$ we have*

$$(A \circ G)_l (x_1, \dots, x_n) \leq (G \circ A)_k (x_1, \dots, x_n) . \tag{C}$$

The equality condition of (C) is the same as the equality condition of (D).

5. INEQUALITY (C) WITH WEIGHTS (THREE VARIABLES)

The process of making the mixed means involves two stages. We can consider weights differently at each stage, that is, our weights at the second stage may be unrelated to the weights at the first stage. Suppose $t_1, \dots, t_n > 0$ are weights for variables $x_1, \dots, x_n \geq 0$ and $1 \leq k \leq n$. For any subset $Y \subset X = \{x_1, \dots, x_n\}$ with $|Y| = k$, $A_t(Y)$ and $G_t(Y)$ are defined as before. At the second stage, let s_Y be weights for Y with $|Y| = k$, namely we have $s_Y > 0$ for all Y with $|Y| = k$ and $\sum_{Y \subset X, |Y| = k} s_Y = 1$. Here this second weight $\{s_Y\}$ can be chosen independently of the first weight $\{t_i\}$. Now, consider

$$\sum_{Y \subset X, |Y| = k} s_Y G_t(Y) \quad \text{and} \quad \prod_{Y \subset X, |Y| = k} A_t(Y)^{s_Y} .$$

These two numbers, denoted by $(A \circ G)_{k,t,s}(X)$ and $(G \circ A)_{k,t,s}(X)$, can be regarded as the *k-th* mixed arithmetic and geometric means with weights in the most general sense.

In relation to (C), one can ask the following question. Suppose the first weight $\{t_i\}$ and $2 \leq k, l \leq n$ with $n+1 \leq k+l$ are given. Do there exist second weights $\{s_Y\}$ $(Y \subset X, |Y| = k)$ and $\{s_Z\}$ $(Z \subset X, Z = l)$ such that

$$(A \circ G)_{l,t,s}(x_1, \dots, x_n) \leq (G \circ A)_{k,t,s}(x_1, \dots, x_n)$$

holds for all $x_1, \dots, x_n \geq 0$? If the answer is yes, it means that we can have the inequality (C) with weights. The author does not have the answer in general. However, there is one positive answer for the simplest case of $n = 3$ (three variables) and $k = l = 2$.

Our notation goes back to the three variables case. Suppose $x.y, z \geq 0$ are non-negative three variables and $a, b, c > 0$ with $a + b + c = 1$ are the first weights. If we choose the second weights s_Y as

$$\frac{ab}{ab+bc+ca}, \quad \frac{bc}{ab+bc+ca} \quad \text{and} \quad \frac{ca}{ab+bc+ca}$$

for $Y = \{x, y\}$, $\{y, z\}$ and $\{z, x\}$ respectively, then we can generalize our inequality (C) of three variables from without weights to with weights.

Theorem 5.1. *Suppose a, b, c are positive numbers with $a + b + c = 1$. For any non-negative numbers $x, y, z \geq 0$, we have, denoted by $\Delta = ab + bc + ca$,*

$$\frac{ab}{\Delta}\left(x^a y^b\right)^{\frac{1}{a+b}} + \frac{bc}{\Delta}\left(y^b z^c\right)^{\frac{1}{b+c}} + \frac{ca}{\Delta}\left(z^c x^a\right)^{\frac{1}{c+a}} \leq \left(\frac{ax+by}{a+b}\right)^{\frac{ab}{\Delta}} \cdot \left(\frac{by+cz}{b+c}\right)^{\frac{bc}{\Delta}} \cdot \left(\frac{cz+ax}{c+a}\right)^{\frac{ca}{\Delta}} . \tag{5.1}$$

The equality holds if and only if $x = y = z \geq 0$ or two of x, y, z are 0.

Proof. Our proof can be done using an idea which is almost the same, but slightly general, as the idea used in CMN [3]. We assume all variables x, y, z are positive until the equality condition of (5.1) is discussed. With respect to our first weights $a, b, c > 0$, the arithmetic mean and the geometric mean of any subset $Y \subset X = \{x, y, z\}$, $Y \neq \phi$, are simply denoted by $A(Y)$ and $G(Y)$, for instance, $A(Y) = \frac{ax+by}{a+b}$ and $G(Y) = \left(x^a y^b\right)^{\frac{1}{a+b}}$ for $Y = \{x, y\}$. To make sure, our second weights s_Y are $\frac{ab}{\Delta}, \frac{bc}{\Delta}$ and $\frac{ca}{\Delta}$ for $Y = \{x, y\}$, $\{y, z\}$ and $\{z, x\}$ respectively. Using these notations, our inequality (5.1) above is expressed equivalently as

(C')
$$\sum_{Y \subset X, |Y|=2} s_Y G(Y) \leq \sum_{Y \subset X, |Y|=2} A(Y)^{s_Y}.$$

We will use this expression of (5.1).

First, as in CMN [3], a variational form of Hölder's inequality will be used. Suppose $a_{i,j}$, $1 \leq i \leq n$ and $1 \leq j \leq m$, are $n \times m$ many positive numbers and suppose t_i, $1 \leq i \leq n$ and s_j, $1 \leq j \leq m$ are $n + m$ many positve numbers satisfying $t_1 + \cdots + t_n = 1$.

Then we have

(H)
$$\sum_{j=1}^{m} s_j \left(\prod_{i=1}^{n} a_{i,j}^{t_i}\right) \leq \prod_{i=1}^{n} \left(\sum_{j=1}^{m} s_j a_{i,j}\right)^{t_i}$$

The equality holds if and only if $a_{i,j} = b_i c_j$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$ for some $b_i > 0$, $1 \leq i \leq n$ and $c_j > 0$, $1 \leq j \leq m$. (We will not need this equality condition here).

A proof of (H) can be given simply by applying the inequality between the arithmetic means and the geometric means to the ratio of the left side divided by the right side of (H).

Next, we have the following identities 1) and 2).

For all $1 \leq j \leq m$ with $|Y| = 2$ we have

1) $A(Y) = \sum_{Z \subset X, |Z|=2} s_Z A(Y \cap Z)$,

2) $G(Y) = \prod_{Z \subset X, |Z|=2} G(Y \cap Z)^{s_Z}$.

Proof of 1). Suppose $Y = \{x, y\}$,, then

$$\begin{aligned}\sum_{Z \subset X, |Z|=2} s_Y A(Y \cap Z) &= s_{\{x,y\}} A(\{x, y\}) + s_{\{y,z\}} A(\{y\}) + s_{\{x,z\}} A(\{x\}) \\ &= \frac{ab}{\Delta} \cdot \frac{ax + by}{a + b} + \frac{bc}{\Delta} y + \frac{ca}{\Delta} x \\ &= \frac{ab}{\Delta} \frac{ax + by}{a + b} + \frac{c}{\Delta}(by + ax) \\ &= \frac{ax + by}{\Delta}\left(\frac{ab}{a + b} + c\right) = \frac{ax + by}{a + b} = A(Y).\end{aligned}$$

□

Proof of 2). This can be done exactly the same as 1) above. □

Now, our proof of (C') goes as follows, note that our notations $\overset{1)}{=}$ and $\overset{(H)}{\leq}$ mean the equality = follows from 1) and the inequality $\leq$ follows from (H).

$$\begin{aligned}\sum_{Y\subset X,|Y|=2} s_Y G(Y) &\overset{2)}{=} \sum_{Y\subset X,|Y|=2} s_Y \prod_{Z\subset X,|Z|=2} G(Y\cap Z)^{s_Z}\\ &\overset{(H)}{\leq} \prod_{Z\subset X,|Z|=2}\left[\sum_{Y\subset X,|Y|=2} s_Y G(Y\cap Z)\right]^{s_Z}\\ &\leq \prod_{Z\subset X,|Z|=2}\left[\sum_{Y\subset X,|Y|=2} s_Y A(Y\cap Z)\right]^{s_Z}\\ &\overset{1)}{=} \prod_{Z\subset X,|Z|=2} A(Z)^{s_Z}.\end{aligned}$$

Thus, we have proved (C') for positive $x, y, z > 0$. And one can see that the equality of (C') holds if and only if we have the equality of (H) and $G(Y\cap Z) = A(Y\cap Z)$ for all Y and Z with $|Y| = 2$ and $|Z| = 2$. However, the latter is equivalent to $x = y = z$ and the latter yields the equality of (H). Hence we can conclude that the equality of (C') for $x, y, z > 0$ holds if and only if $x = y = z > 0$.

By continuity again, it is obvious that the inequality (C') holds for any non-negative $x, y, z \geq 0$. The only point we have to check is the equality condition for $x, y, z \geq 0$ which include 0. If two of $x, y, z \geq 0$ are 0, it is clear that both sides of (C') are 0. Suppose only one of $x, y, z \geq 0$ is 0, then for *all* Z with $|Z| = 2$ we have

$$0 < \sum_{Y\subset X,|Y|=2} s_Y G(Y\cap Z) \leq \sum_{Y\subset X,|Y|=2} s_Y A(Y\cap Z)$$

and for *some* Z with $|Z| = 2$ we have

$$\sum_{Y\subset X,|Y|=2} s_Y G(Y\cap Z) < \sum_{Y\subset X,|Y|=2} s_Y A(Y\cap Z).$$

Thus, we have

$$\begin{aligned}0 &< \prod_{Z\subset X,|Z|=2}\left[\sum_{Y\subset X,|Y|=2} s_Y G(Y\cap Z)\right]^{s_Z}\\ &< \prod_{Z\subset X,|Z|=2}\left[\sum_{Y\subset X,|Y|=2} s_Y A(Y\cap Z)\right]^{s_Z}.\end{aligned}$$

This means that the last inequality of the previous arguments proving (C') is strict. Hence we have a strict inequality of (C'). This completes the proof of Theorem 5.1. □

Finally a question is left open. It is verified easily that the inequalities (A) and (D) of three variables are transformed into each other by the transformation: $(x, y, z) \to \left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$. It seems natural to ask whether there is a reasonable relationship between (A) and (D) of general n variables, which extends the relationship for three variables.

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