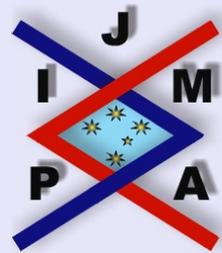


INEQUALITIES RELATED TO THE UNITARY ANALOGUE OF LEHMER PROBLEM

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Abstract

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Abstract

Observing that $\phi(n)$ divides $n - 1$ if n is a prime, where $\phi(n)$ is the well known Euler function, Lehmer has asked whether there is any composite number n with this property. For this unsolved problem, partial answers were given by several researchers. Considering the unitary analogue $\phi^*(n)$ of $\phi(n)$, Subbarao noted that $\phi^*(n)$ divides $n - 1$, if n is the power of a prime; and sought for integers n other than prime powers which satisfy this condition. In this paper we improve two inequalities, established by Subbarao and Siva Rama Prasad [5], to be satisfied by n for $\phi^*(n)$ which divides $n - 1$.

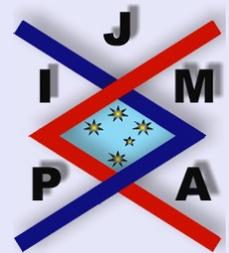
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Key words: Lehmer Problem, Unitary analogue of Lehmer problem.

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1. Introduction

Let $\phi(n)$ denote, as usual the number of positive integers not exceeding n that are relatively prime to n . Noting that $\phi(n) \mid n - 1$ if n is a prime, Lehmer [2] asked, in 1932, whether there is a composite number n for which $\phi(n) \mid n - 1$.

Equivalently, if

$$(1.1) \quad S_M = \{n : M\phi(n) = n - 1\} \quad \text{for } M = 1, 2, 3, \dots,$$

then the Lehmer problem seeks composite numbers in $S = \bigcup_{M>1} S_M$. For this problem, which has not been settled so far, several partial answers were provided, the details of which can be found in [5]. Lehmer [2] has shown that

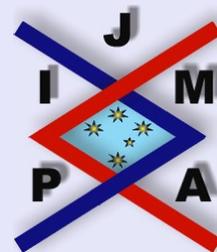
$$(1.2) \quad \text{If } n \in S, \text{ then } n \text{ is square free.}$$

It is well known that a divisor $d > 0$ of a positive integer n for which $(d, n/d) = 1$ is called a *unitary divisor* of n . For positive integers a and b , the greatest divisor of a which is a unitary divisor of b is denoted by $(a, b)^*$.

E. Cohen [1] has defined $\phi^*(n)$, the unitary analogue of the Euler totient function, as the number of integers a with $1 \leq a \leq n$ and $(a, n)^* = 1$. It can be seen that $\phi^*(1) = 1$ and if $n > 1$ with $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_r^{\alpha_r}$, then

$$(1.3) \quad \phi^*(n) = (p_1^{\alpha_1} - 1)(p_2^{\alpha_2} - 1) \cdots (p_r^{\alpha_r} - 1)$$

Noting that $\phi^*(n) \mid n - 1$ whenever n is a prime power, Subbarao [3] has asked whether non-prime powers n exist with this property and this is the unitary



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analogue of the Lehmer problem. If

$$(1.4) \quad S_M^* = \{n : M\phi^*(n) = n - 1\} \quad \text{for } M = 1, 2, 3, \dots,$$

the problem seeks non-prime powers in $S_M^* = \bigcup_{M>1} S_M^*$.

For excellent information on the Lehmer problem, its generalizations and extensions, we refer readers to the book of J. Sandor and B. Crstici ([3, p. 212-215]).

Let Q denote the set of all square free numbers. Since $\phi^*(n) = \phi(n)$ for $n \in Q$, it follows that $S_M^* \cap Q = S_M$ for each $M > 1$ and therefore $S^* \cap Q = S$, showing $S \subset S^*$ and hence a separate study of S^* is meaningful.

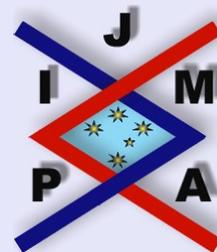
In a study of certain analogues of the Lehmer problem, Subbarao and Siva Rama Prasad [5] have proved, among other things, that if $\omega(n) = r$ is the number of distinct prime factors of $n \in S^*$ then

$$(1.5) \quad \omega(n) \geq 11$$

and that

$$(1.6) \quad n < (r - 1)^{2^r - 1}$$

The purpose of this paper is to prove Theorems **A** and **B** (see Section 3) which improve (1.5) and (1.6) respectively.



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2. Preliminaries

We state below the results proved in [4] which are needed for our purpose.

(2.1) If $n \in S^*$, then n is odd and is not a powerful number.

A number is said to be powerful if each prime dividing it is of multiplicity at least 2.

(2.2) If $n \in S^*$ and p, q are primes such that p divides n and $q^\beta \equiv 1 \pmod{p}$, then q^β cannot be a unitary divisor of n .

(2.3) If $n \in S^*$ and $3|n$ then $\omega(n) \geq 1850$.

(2.4) If $n \in S^*$, $3 \nmid n$ and $5 | n$ then $\omega(n) \geq 11$.

(2.5) If $n \in S^*$, $3 \nmid n$ and $5 \nmid n$ then $\omega(n) \geq 17$.

(2.6) If $n \in S^*$, with $2 < \omega(n) \leq 16$ then $n \in S_2^*$, $3 \nmid n$, $5 | n$ and $7 | n$.



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3. Main Results

Theorem A. *If $n \in S^*$ and 455 is not a unitary divisor of n then $\omega(n) \geq 17$.*

Proof. (2.3) and (2.5) respectively prove the theorem in the cases $3|n$ and $15 \nmid n$.

Therefore we assume that $3 \nmid n$ and $5 | n$.

Let n be of the form (2.8) with $\omega(n) \leq 16$ then by (2.6), $n \in S_2^*$, $5|n$ and $7|n$. That is $p_1 = 5, p_2 = 7$ and so $n = 5^{\alpha_1} 7^{\alpha_2} p_3^{\alpha_3} \cdots p_r^{\alpha_r}$, where $p_i \not\equiv 1 \pmod{5}$ and $p_i \not\equiv 1 \pmod{7}$ for $i \geq 3$, in view of (2.2).

Suppose A is a set of primes (in increasing order) containing 5 and 7; and those primes p with $p \not\equiv 1 \pmod{5}$ and $p \not\equiv 1 \pmod{7}$. Denote the i^{th} element of A by a_i so that $a_1 = 5, a_2 = 7, a_3 = 13, a_4 = 17, a_5 = 19, a_6 = 23, a_7 = 37, \dots$

Now since

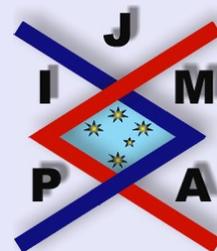
$$\frac{n}{\phi^*(n)} = \prod_{i=1}^r \frac{p_i^{\alpha_i}}{p_i^{\alpha_i} - 1}$$

increases with r and $r \leq 16$, we consider the case $r = 16$ and prove that the product on the right is < 2 in this case, which contradicts (2.7).

Therefore $r \leq 16$ cannot hold, proving the theorem.

If $r = 16$ and $p_3 \neq a_3$, then $p_i \geq a_{i+1}$ for $i \geq 3$ so that, in view of the fact that $x/(x-1)$ is decreasing, we get

$$\frac{n}{\phi^*(n)} = \frac{5^{\alpha_1}}{5^{\alpha_1} - 1} \cdot \frac{7^{\alpha_2}}{7^{\alpha_2} - 1} \cdot \prod_{i=3}^{16} \frac{p_i^{\alpha_i}}{p_i^{\alpha_i} - 1} < \frac{5}{4} \cdot \frac{7}{6} \prod_{i=3}^{16} \frac{a_{i+1}}{a_{i+1} - 1} < 2$$



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Hence $p_3 = a_3$. Now since $13^2 \equiv 1 \pmod{7}$ we get, by (2.2), $2 \nmid \alpha_3$ and so $n = 5^{\alpha_1} 7^{\alpha_2} 13^{\alpha_3} \cdots p_{16}^{\alpha_{16}}$, where α_3 is odd. Further since 455 is not a unitary divisor of n , we must have $\alpha_1 \alpha_2 \alpha_3 > 1$.

If $\alpha_1 \alpha_2 = 1$ or $\alpha_1 \alpha_2 > 1$, we get contradiction to (2.7). In fact in case $\alpha_1 \alpha_2 = 1$, we must have $\alpha_3 \geq 3$ so that

$$\frac{p_3^{\alpha_3}}{p_3^{\alpha_3} - 1} \leq \frac{13^3}{13^3 - 1} = \frac{2197}{2196}$$

and therefore

$$\frac{n}{\phi^*(n)} < \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{2197}{2196} \prod_{i=4}^{16} \frac{a_i}{a_i - 1} < 2$$

and in case $\alpha_1 \alpha_2 > 1$, it is enough to consider the case $\alpha_3 = 1$, so that in this case

$$\frac{n}{\phi^*(n)} < \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{13}{12} \prod_{i=4}^{16} \frac{a_i}{a_i - 1} < 2$$

Finally the case $\alpha_1 > 1$, $\alpha_2 > 1$, and $\alpha_3 > 1$ can be handled similarly. \square

Theorem B. *If $n \in S^*$ with $\omega(n) = r$ and 455 does not divide n unitarily then $n < \left(r - \frac{23}{10}\right)^{2^r - 1}$.*

Proof. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_r^{\alpha_r}$, where $p_1^{\alpha_1} < p_2^{\alpha_2} < \cdots < p_r^{\alpha_r}$. By (2.10) and Theorem A, we have

$$(3.1) \quad p_1^{\alpha_1} < 2 + 2 \left(\frac{r}{3}\right) < r - \frac{18}{5}, \quad \text{for } r \geq 17.$$



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Now by (2.9) and (3.1), we successively have

$$p_1^{\alpha_1} < r - \frac{18}{5} < \left(r - \frac{23}{10}\right)$$

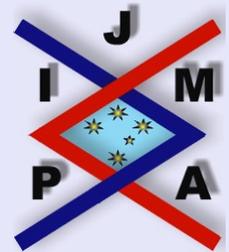
$$p_2^{\alpha_2} < (r - 1) p_1^{\alpha_1} < (r - 1) \left(r - \frac{18}{5}\right) < \left(r - \frac{23}{10}\right)^2$$

$$p_3^{\alpha_3} < (r - 2) p_1^{\alpha_1} p_2^{\alpha_2} < \left(r - \frac{23}{10}\right)^{2^2}$$

...

$$p_r^{\alpha_r} < \left(r - \frac{23}{10}\right)^{2^{r-1}}.$$

Multiplying all these inequalities we get, $n < \left(r - \frac{23}{10}\right)^{2^r - 1}$, proving the theorem. \square



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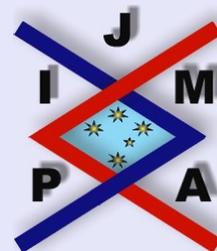
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