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INEQUALITIES RELATED TO THE UNITARY ANALOGUE OF LEHMER PROBLEM

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ABSTRACT. Observing that $\phi(n)$ divides $n-1$ if n is a prime, where $\phi(n)$ is the well known Euler function, Lehmer has asked whether there is any composite number n with this property. For this unsolved problem, partial answers were given by several researchers. Considering the unitary analogue $\phi^*(n)$ of $\phi(n)$, Subbarao noted that $\phi^*(n)$ divides $n-1$, if n is the power of a prime; and sought for integers n other than prime powers which satisfy this condition. In this paper we improve two inequalities, established by Subbarao and Siva Rama Prasad [5], to be satisfied by n for $\phi^*(n)$ which divides $n-1$.

[5] M.V. Subbarao and V. Siva Rama Prasad, Some analogues of a Lehmer problem on the totient function, Rocky Mountain Journal of Mathematics; Vol. 15, Number 2: Spring 1985, 609-619.

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1. INTRODUCTION

Let $\phi(n)$ denote, as usual the number of positive integers not exceeding n that are relatively prime to n. Noting that $\phi(n) \mid n-1$ if n is a prime, Lehmer [2] asked, in 1932, whether there is a composite number n for which $\phi(n) \mid n-1$.

Equivalently, if

$$S_M = \{n : M\phi(n) = n-1\} \quad \text{for} \quad M = 1, 2, 3, \dots, \tag{1.1}$$

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then the Lehmer problem seeks composite numbers in $S = \bigcup_{M>1} S_M$. For this problem, which has not been settled so far, several partial answers were provided, the details of which can be found in [5]. Lehmer [2] has shown that

$$\text{If } n \in S, \text{ then } n \text{ is square free.} \tag{1.2}$$

It is well known that a divisor $d > 0$ of a positive integer n for which $(d, n/d) = 1$ is called a *unitary divisor* of n. For positive integers a and b, the greatest divisor of a which is a unitary divisor of b is denoted by $(a, b)^*$.

E. Cohen [1] has defined $\phi^*(n)$, the unitary analogue of the Euler totient function, as the number of integers a with $1 \leq a \leq n$ and $(a, n)^* = 1$. It can be seen that $\phi^*(1) = 1$ and if $n > 1$ with $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdot \cdots \cdot p_r^{\alpha_r}$, then

$$\phi^*(n) = (p_1^{\alpha_1} - 1)(p_2^{\alpha_2} - 1) \cdots (p_r^{\alpha_r} - 1) \tag{1.3}$$

Noting that $\phi^*(n) \mid n - 1$ whenever n is a prime power, Subbarao [3] has asked whether non-prime powers n exist with this property and this is the unitary analogue of the Lehmer problem. If

$$S_M^* = \{n : M\phi^*(n) = n - 1\} \quad \text{for} \quad M = 1, 2, 3, \ldots, \tag{1.4}$$

$$\text{the problem seeks non-prime powers in } S_M^* = \bigcup_{M>1} S_M^*.$$

For excellent information on the Lehmer problem, its generalizations and extensions, we refer readers to the book of J. Sandor and B. Crstici ([3, p. 212-215]).

Let Q denote the set of all square free numbers. Since $\phi^*(n) = \phi(n)$ for $n \in Q$, it follows that $S_M^* \cap Q = S_M$ for each $M > 1$ and therefore $S^* \cap Q = S$, showing $S \subset S^*$ and hence a separate study of S^* is meaningful.

In a study of certain analogues of the Lehmer problem, Subbarao and Siva Rama Prasad [5] have proved, among other things, that if $\omega(n) = r$ is the number of distinct prime factors of $n \in S^*$ then

$$\omega(n) \geq 11 \tag{1.5}$$

and that

$$n < (r - 1)^{2^r - 1} \tag{1.6}$$

The purpose of this paper is to prove Theorems A and B (see Section 3) which improve (1.5) and (1.6) respectively.

2. PRELIMINARIES

We state below the results proved in [4] which are needed for our purpose.

$$\text{If } n \in S^*, \text{ then } n \text{ is odd and is not a powerful number.} \tag{2.1}$$

A number is said to be powerful if each prime dividing it is of multiplicity at least 2.

$$\text{If } n \in S^* \text{ and } p, q \text{ are primes such that } p \text{ divides } n \text{ and } q^\beta \equiv 1 (\mathrm{mod}\, p), \tag{2.2}$$

$$\text{then } q^\beta \text{cannot be a unitary divisor of } n.$$

$$\text{If } n \in S^* \text{ and } 3 | n \text{ then } \omega(n) \geq 1850. \tag{2.3}$$

$$\text{If } n \in S^*,\ 3 \nmid n \text{ and } 5 \mid n \text{ then } \omega(n) \geq 11. \tag{2.4}$$

(2.5) If $n \in S^*,\ 3 \nmid n$ and $5 \nmid n$ then $\omega(n) \geq 17$.

(2.6) If $n \in S^*$, with $2 < \omega(n) \leq 16$ then $n \in S_2^*,\ 3 \nmid n,\ 5 \mid n$ and $7 \mid n$.

Suppose $n \in S_M^*$ for some $M > 1$. Then $\frac{n}{\phi^*(n)} > M \geq 2$, which gives

(2.7) $$2 < \frac{n}{\phi^*(n)} \text{ for all } n \in S^*.$$

Also if $n \in S^*$ is of the form

(2.8) $$n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_r^{\alpha_r} \text{ with } p_1 < p_2 < \cdots < p_r,$$

then by (2.1) at least one $\alpha_i = 1$

(2.9) ([5, Lemma 5.3]): If $n \in S_M^*$ and $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots\cdots p_r^{\alpha_r}$, with

$$p_1^{\alpha_1} < p_2^{\alpha_2} < \cdots < p_r^{\alpha_r}, \text{ then } p_i^{\alpha_i} < (r-i+1) \prod_{j=1}^{i-1} p_j^{\alpha_j} \text{ for } i = 2, 3, \ldots, r.$$

(2.10) ([5, Lemma 5.3]): If $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_r^{\alpha_r}$, with

$$p_1^{\alpha_1} < p_2^{\alpha_2} < \cdots < p_r^{\alpha_r} \text{ is such that } \frac{n}{\phi^*(n)} > 2, \text{ then } p_1^{\alpha_1} < 2 + 2\binom{r}{3}.$$

3. MAIN RESULTS

Theorem A. *If $n \in S^*$ and 455 is not a unitary divisor of n then $\omega(n) \geq 17$.*

Proof. (2.3) and (2.5) respectively prove the theorem in the cases $3|n$ and $15 \nmid n$.

Therefore we assume that $3 \nmid n$ and $5 \mid n$.

Let n be of the form (2.8) with $\omega(n) \leq 16$ then by (2.6), $n \in S_2^*$, $5|n$ and $7|n$. That is $p_1 = 5, p_2 = 7$ and so $n = 5^{\alpha_1} 7^{\alpha_2} p_3^{\alpha_3} \cdots p_r^{\alpha_r}$, where $p_i \not\equiv 1 (\bmod 5)$ and $p_i \not\equiv 1 (\bmod 7)$ for $i \geq 3$, in view of (2.2).

Suppose A is a set of primes (in increasing order) containing 5 and 7; and those primes p with $p \not\equiv 1 (\bmod 5)$ and $p \not\equiv 1 (\bmod 7)$. Denote the i^{th} element of A by a_i so that $a_1 = 5,\ a_2 = 7,\ a_3 = 13, a_4 = 17, a_5 = 19, a_6 = 23, a_7 = 37, \ldots$.

Now since

$$\frac{n}{\phi^*(n)} = \prod_{i=1}^{r} \frac{p_i^{\alpha_i}}{p_i^{\alpha_i} - 1}$$

increases with r and $r \leq 16$, we consider the case $r = 16$ and prove that the product on the right is < 2 in this case, which contradicts (2.7).

Therefore $r \leq 16$ cannot hold, proving the theorem.

If $r = 16$ and $p_3 \neq a_3$, then $p_i \geq a_{i+1}$ for $i \geq 3$ so that, in view of the fact that $x/(x-1)$ is decreasing, we get

$$\frac{n}{\phi^*(n)} = \frac{5^{\alpha_1}}{5^{\alpha_1} - 1} . \frac{7^{\alpha_2}}{7^{\alpha_2} - 1} . \prod_{i=3}^{16} \frac{p_i^{\alpha_i}}{p_i^{\alpha_i} - 1} < \frac{5}{4} . \frac{7}{6} \prod_{i=3}^{16} \frac{a_{i+1}}{a_{i+1} - 1} < 2$$

Hence $p_3 = a_3$. Now since $13^2 \equiv 1 (\bmod 7)$ we get, by (2.2), $2 \nmid \alpha_3$ and so $n = 5^{\alpha_1} 7^{\alpha_2} 13^{\alpha_3} \cdots p_{16}^{\alpha_{16}}$, where α_3 is odd. Further since 455 is not a unitary divisor of n, we must have $\alpha_1 \alpha_2 \alpha_3 > 1$.

If $\alpha_1\alpha_2 = 1$ or $\alpha_1\alpha_2 > 1$, we get contradiction to (2.7). In fact in case $\alpha_1\alpha_2 = 1$, we must have $\alpha_3 \geq 3$ so that

$$\frac{p_3^{\alpha_3}}{p_3^{\alpha_3}-1} \leq \frac{13^3}{13^3-1} = \frac{2197}{2196}$$

and therefore

$$\frac{n}{\phi^*(n)} < \frac{5}{4}\cdot\frac{7}{6}\cdot\frac{2197}{2196}\prod_{i=4}^{16}\frac{a_i}{a_i-1} < 2$$

and in case $\alpha_1\alpha_2 > 1$, it is enough to consider the case $\alpha_3 = 1$, so that in this case

$$\frac{n}{\phi^*(n)} < \frac{5}{4}\cdot\frac{7}{6}\cdot\frac{13}{12}\prod_{i=4}^{16}\frac{a_i}{a_i-1} < 2$$

Finally the case $\alpha_1 > 1$, $\alpha_2 > 1$, and $\alpha_3 > 1$ can be handled similarly. □

Theorem B. *If $n \in S^*$ with $\omega(n) = r$ and 455 does not divide n unitarily then $n < \left(r - \frac{23}{10}\right)^{2^r-1}$.*

Proof. Let $n = p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}\cdots p_r^{\alpha_r}$, where $p_1^{\alpha_1} < p_2^{\alpha_2} < \cdots < p_r^{\alpha_r}$. By (2.10) and Theorem A, we have

$$p_1^{\alpha_1} < 2 + 2\binom{r}{3} < r - \frac{18}{5}, \quad \text{for } r \geq 17. \tag{3.1}$$

Now by (2.9) and (3.1), we successively have

$$\begin{aligned}
p_1^{\alpha_1} &< r - \frac{18}{5} < \left(r - \frac{23}{10}\right)\\
p_2^{\alpha_2} &< (r-1)\,p_1^{\alpha_1} < (r-1)\left(r - \frac{18}{5}\right) < \left(r - \frac{23}{10}\right)^2\\
p_3^{\alpha_3} &< (r-2)\,p_1^{\alpha_1}p_2^{\alpha_2} < \left(r - \frac{23}{10}\right)^{2^2}\\
&\cdots\\
p_r^{\alpha_r} &< \left(r - \frac{23}{10}\right)^{2^{r-1}}.
\end{aligned}$$

Multiplying all these inequalities we get, $n < \left(r - \frac{23}{10}\right)^{2^r-1}$, proving the theorem. □

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