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A q-ANALOGUE OF AN INEQUALITY DUE TO KONRAD KNOPP

JIN-LIN LIU

Department of Mathematics
Yangzhou University
Yangzhou 225002, Jiangsu
People's Republic of China

EMail: jlliu@yzu.edu.cn

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Abstract
Contents
Home Page
Go Back
Close
Quit

Abstract

The main object of the present paper is to investigate several interesting properties of a linear operator $H_{p,q,s}(\alpha_i)$ associated with the generalized hypergeometric function.

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Contents

1. Introduction

Let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad (p \in N = \{1, 2, 3, \dots\}) \tag{1.1}$$

which are analytic in the open unit disk $U = \{z \colon z \in C \text{ and } |z| < 1\}$.

Let $f(z)$ and $g(z)$ be analytic in U. Then we say that the function $g(z)$ is subordinate to $f(z)$ if there exists an analytic function $w(z)$ in U such that $|w(z)| < 1$ (for $z \in U$) and $g(z) = f(w(z))$. This relation is denoted $g(z) \prec f(z)$. In case $f(z)$ is univalent in U we have that the subordination $g(z) \prec f(z)$ is equivalent to $g(0) = f(0)$ and $g(U) \subset f(U)$.

For analytic functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n,$$

by $f * g$ we denote the Hadamard product or convolution of f and g, defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n = (g * f)(z). \tag{1.2}$$

Next, for real parameters A and B such that $-1 \leq B < A \leq 1$, we define the function

$$h(A, B; z) = \frac{1 + Az}{1 + Bz} \quad (z \in U). \tag{1.3}$$

It is well known that $h(A,B;z)$ for $-1 \leq B \leq 1$ is the conformal map of the unit disk onto the disk symmetrical with respect to the real axis having the center $(1-AB)/(1-B^2)$ and the radius $(A-B)/(1-B^2)$ for $B \neq \mp 1$. The boundary circle cuts the real axis at the points $(1-A)/(1-B)$ and $(1+A)/(1+B)$.

For complex parameters $\alpha_1,\dots,\alpha_q$ and $\beta_1,\dots,\beta_s$ ($\beta_j \neq 0,-1,-2,\dots; j = 1,\dots,s$), we define the generalized hypergeometric function ${}_qF_s(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s;z)$ by

$$ {}_qF_s(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s;z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_q)_n}{(\beta_1)_n \cdots (\beta_s)_n} \cdot \frac{z^n}{n!} $$

(1.4) $$ (q \leq s+1; q,s \in N_0 = N \cup \{0\}; z \in U), $$

where $(x)_n$ is the Pochhammer symbol, defined, in terms of the Gamma function Γ, by

$$ (x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} 1 & (n=0), \\ x(x+1)\cdots(x+n-1) & (n \in N). \end{cases} $$

Corresponding to a function $\mathscr{F}_p(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s;z)$ defined by

$$ \mathscr{F}_p(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s;z) = z^p {}_qF_s(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s;z), $$

we consider a linear operator

$$ H_p(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s) : A(p) \to A(p), $$

defined by the convolution

(1.5) $$ H_p(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s)f(z) = \mathscr{F}_p(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s;z) * f(z). $$

For convenience, we write

$$H_{p,q,s}(\alpha_i) = H_p(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) \quad (i = 1, 2, \ldots, q). \tag{1.6}$$

Thus, after some calculations, we have

$$z(H_{p,q,s}(\alpha_i)f(z))' = \alpha_i H_{p,q,s}(\alpha_i + 1)f(z) - (\alpha_i - p)H_{p,q,s}(\alpha_i)f(z) \\ (i = 1, 2, \ldots, q). \tag{1.7}$$

It should be remarked that the linear operator $H_{p,q,s}(\alpha_i)$ $(i = 1, 2, \ldots, q)$ is a generalization of many operators considered earlier. For $q = 2$ and $s = 1$ Carlson and Shaffer studied this operator under certain restrictions on the parameters α_1, α_2 and β_1 in [1]. A more general operator was studied by Ponnusamy and Rønning [13]. Also, many interesting subclasses of analytic functions, associated with the operator $H_{p,q,s}(\alpha_i)$ $(i = 1, 2, \ldots, q)$ and its many special cases, were investigated recently by (for example) Dziok and Srivastava [2, 3, 4], Gangadharan et al. [5], Liu [7], Liu and Srivastava [8, 9] and others (see also [6, 12, 15, 16, 17]).

In the present sequel to these earlier works, we shall use the method of differential subordination to derive several interesting properties and characteristics of the operator $H_{p,q,s}(\alpha_i)$ $(i = 1, 2, \ldots, q)$.

2. Main Results

We begin by recalling each of the following lemmas which will be required in our present investigation.

Lemma 2.1 (see [10]). *Let $h(z)$ be analytic and convex univalent in U, $h(0) = 1$ and let $g(z) = 1 + b_1 z + b_2 z^2 + \cdots$ be analytic in U. If*

(2.1) $$g(z) + zg'(z)/c \prec h(z) \quad (z \in U; c \neq 0),$$

then for $\operatorname{Re} c \geq 0$,

(2.2) $$g(z) \prec \frac{c}{z^c} \int_0^z t^{c-1} h(t) dt.$$

Lemma 2.2 (see [14]). *The function $(1-z)^\gamma \equiv e^{\gamma \log(1-z)}$, $\gamma \neq 0$, is univalent in U if and only if γ is either in the closed disk $|\gamma - 1| \leq 1$ or in the closed disk $|\gamma + 1| \leq 1$.*

Lemma 2.3 (see [11]). *Let $q(z)$ be univalent in U and let $\theta(w)$ and $\phi(w)$ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$ and suppose that*

1. *$Q(z)$ is starlike (univalent) in U;*
2. $\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right) = \operatorname{Re}\left(\frac{\theta'(q(z))}{\phi(q(z))} + \frac{zQ'(z)}{Q(z)}\right) > 0 \quad (z \in U).$

If $p(z)$ is analytic in U, with $p(0) = q(0)$, $p(U) \subset D$, and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) = h(z),$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

We now prove our first result given by Theorem 2.4 below.

Theorem 2.4. *Let* $\alpha_i > 0$ $(i = 1, 2, \dots, q)$, $\lambda > 0$, *and* $-1 \leq B < A \leq 1$. *If* $f(z) \in A(p)$ *satisfies*

$$(2.3) \qquad (1-\lambda)\frac{H_{p,q,s}(\alpha_i)f(z)}{z^p} + \lambda\frac{H_{p,q,s}(\alpha_i+1)f(z)}{z^p} \prec h(A,B;z),$$

then

$$(2.4) \qquad \operatorname{Re}\left(\left(\frac{H_{p,q,s}(\alpha_i)f(z)}{z^p}\right)^{\frac{1}{m}}\right) > \left(\frac{\alpha_i}{\lambda}\int_0^1 u^{\frac{\alpha_i}{\lambda}-1}\left(\frac{1-Au}{1-Bu}\right)du\right)^{\frac{1}{m}} \quad (m \geq 1).$$

The result is sharp.

Proof. Let

$$(2.5) \qquad g(z) = \frac{H_{p,q,s}(\alpha_i)f(z)}{z^p}$$

for $f(z) \in A(p)$. Then the function $g(z) = 1 + b_1 z + \cdots$ is analytic in U. By making use of (1.7) and (2.5), we obtain

$$(2.6) \qquad \frac{H_{p,q,s}(\alpha_i+1)f(z)}{z^p} = g(z) + \frac{zg'(z)}{\alpha_i}.$$

From (2.3), (2.5) and (2.6) we get

$$g(z) + \frac{\lambda}{\alpha_i} z g'(z) \prec h(A, B; z). \tag{2.7}$$

Now an application of Lemma 2.1 leads to

$$g(z) \prec \frac{\alpha_i}{\lambda} z^{-\frac{\alpha_i}{\lambda}} \int_0^1 t^{\frac{\alpha_i}{\lambda}-1} \left(\frac{1+At}{1+Bt} \right) dt \tag{2.8}$$

or

$$\frac{H_{p,q,s}(\alpha_i) f(z)}{z^p} = \frac{\alpha_i}{\lambda} \int_0^1 u^{\frac{\alpha_i}{\lambda}-1} \left(\frac{1+Auw(z)}{1+Buw(z)} \right) du, \tag{2.9}$$

where $w(z)$ is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$.

In view of $-1 \leq B < A \leq 1$ and $\alpha_i > 0$, it follows from (2.9) that

$$\operatorname{Re} \left(\frac{H_{p,q,s}(\alpha_i) f(z)}{z^p} \right) > \frac{\alpha_i}{\lambda} \int_0^1 u^{\frac{\alpha_i}{\lambda}-1} \left(\frac{1-Au}{1-Bu} \right) du \quad (z \in U). \tag{2.10}$$

Therefore, with the aid of the elementary inequality $\operatorname{Re}(w^{1/m}) \geq (\operatorname{Re} w)^{1/m}$ for $\operatorname{Re} w > 0$ and $m \geq 1$, the inequality (2.4) follows directly from (2.10).

To show the sharpness of (2.4), we take $f(z) \in A(p)$ defined by

$$\frac{H_{p,q,s}(\alpha_i) f(z)}{z^p} = \frac{\alpha_i}{\lambda} \int_0^1 u^{\frac{\alpha_i}{\lambda}-1} \left(\frac{1+Auz}{1+Buz} \right) du.$$

For this function, we find that

$$(1-\lambda) \frac{H_{p,q,s}(\alpha_i) f(z)}{z^p} + \lambda \frac{H_{p,q,s}(\alpha_i + 1) f(z)}{z^p} = \frac{1+Az}{1+Bz}$$

and

$$\frac{H_{p,q,s}(\alpha_i)f(z)}{z^p} \to \frac{\alpha_i}{\lambda}\int_0^1 u^{\frac{\alpha_i}{\lambda}-1}\left(\frac{1-Au}{1-Bu}\right)du \quad \text{as } z \to -1.$$

Hence the proof of the theorem is complete. □

Next we prove our second theorem.

Theorem 2.5. *Let* $\alpha_i > 0$ $(i = 1, 2, \ldots, q)$, *and* $0 \leq \rho < 1$. *Let* γ *be a complex number with* $\gamma \neq 0$ *and satisfy either* $|2\gamma(1-\rho)\alpha_i - 1| \leq 1$ *or* $|2\gamma(1-\rho)\alpha_i + 1| \leq 1$ $(i = 1, 2, \ldots, q)$. *If* $f(z) \in A(p)$ *satisfies the condition*

$$\operatorname{Re}\left(\frac{H_{p,q,s}(\alpha_i+1)f(z)}{H_{p,q,s}(\alpha_i)f(z)}\right) > \rho \quad (z \in U; i = 1, 2, \ldots, q), \tag{2.11}$$

then

$$\left(\frac{H_{p,q,s}(\alpha_i)f(z)}{z^p}\right)^\gamma \prec \frac{1}{(1-z)^{2\gamma(1-\rho)\alpha_i}} = q(z) \quad (z \in U; i = 1, 2, \ldots, q), \tag{2.12}$$

where $q(z)$ *is the best dominant.*

Proof. Let

$$p(z) = \left(\frac{H_{p,q,s}(\alpha_i)f(z)}{z^p}\right)^\gamma \quad (z \in U; i = 1, 2, \ldots, q). \tag{2.13}$$

Then, by making use of (1.7), (2.11) and (2.13), we have

$$1+\frac{zp'(z)}{\gamma\alpha_i p(z)} \prec \frac{1+(1-2\rho)z}{1-z} \quad (z\in U). \tag{2.14}$$

If we take

$$q(z)=\frac{1}{(1-z)^{2\gamma(1-\rho)\alpha_i}},\quad \theta(w)=1 \quad \text{and} \quad \phi(w)=\frac{1}{\gamma\alpha_i w},$$

then $q(z)$ is univalent by the condition of the theorem and Lemma 2.2. Further, it is easy to show that $q(z),\theta(w)$ and $\phi(w)$ satisfy the conditions of Lemma 2.3. Since

$$Q(z)=zq'(z)\phi(q(z))=\frac{2(1-\rho)z}{1-z}$$

is univalent starlike in U and

$$h(z)=\theta(q(z))+Q(z)=\frac{1+(1-2\rho)z}{1-z}.$$

It may be readily checked that the conditions (1) and (2) of Lemma 2.3 are satisfied. Thus the result follows from (2.14) immediately. The proof is complete. □

Corollary 2.6. *Let $\alpha_i>0$ $(i=1,2,\dots,q)$ and $0\le\rho<1$. Let γ be a real number and $\gamma\ge 1$. If $f(z)\in A(p)$ satisfies the condition (2.11), then*

$$\operatorname{Re}\left(\frac{H_{p,q,s}(\alpha_i)f(z)}{z^p}\right)^{\frac{1}{2\gamma(1-\rho)\alpha_i}} > 2^{-1/\gamma} \quad (z\in U; i=1,2,\dots,q).$$

The bound $2^{-1/\gamma}$ is the best possible.

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