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ON SUBORDINATIONS FOR CERTAIN MULTIVALENT ANALYTIC FUNCTIONS ASSOCIATED WITH THE GENERALIZED HYPERGEOMETRIC FUNCTION

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ABSTRACT. The main object of the present paper is to investigate several interesting properties of a linear operator $H_{p,q,s}(\alpha_i)$ associated with the generalized hypergeometric function.

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1. INTRODUCTION

Let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad (p \in N = \{1, 2, 3, \dots\}) \tag{1.1}$$

which are analytic in the open unit disk $U = \{z\colon z \in C \text{ and } |z| < 1\}$.

Let $f(z)$ and $g(z)$ be analytic in U. Then we say that the function $g(z)$ is subordinate to $f(z)$ if there exists an analytic function $w(z)$ in U such that $|w(z)| < 1$ (for $z \in U$) and $g(z) = f(w(z))$. This relation is denoted $g(z) \prec f(z)$. In case $f(z)$ is univalent in U we have that the subordination $g(z) \prec f(z)$ is equivalent to $g(0) = f(0)$ and $g(U) \subset f(U)$.

For analytic functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n,$$

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by $f * g$ we denote the Hadamard product or convolution of f and g, defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n = (g * f)(z). \tag{1.2}$$

Next, for real parameters A and B such that $-1 \leq B < A \leq 1$, we define the function

$$h(A, B; z) = \frac{1 + Az}{1 + Bz} \quad (z \in U). \tag{1.3}$$

It is well known that $h(A, B; z)$ for $-1 \leq B \leq 1$ is the conformal map of the unit disk onto the disk symmetrical with respect to the real axis having the center $(1 - AB)/(1 - B^2)$ and the radius $(A - B)/(1 - B^2)$ for $B \neq \mp 1$. The boundary circle cuts the real axis at the points $(1 - A)/(1 - B)$ and $(1 + A)/(1 + B)$.

For complex parameters $\alpha_1, \ldots, \alpha_q$ and $\beta_1, \ldots, \beta_s$ ($\beta_j \neq 0, -1, -2, \ldots; j = 1, \ldots, s$), we define the generalized hypergeometric function ${}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ by

$${}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_q)_n}{(\beta_1)_n \cdots (\beta_s)_n} \cdot \frac{z^n}{n!}$$
$$(q \leq s + 1; q, s \in N_0 = N \cup \{0\}; z \in U), \tag{1.4}$$

where $(x)_n$ is the Pochhammer symbol, defined, in terms of the Gamma function Γ, by

$$(x)_n = \frac{\Gamma(x + n)}{\Gamma(x)} = \begin{cases} 1 & (n = 0), \\ x(x + 1) \cdots (x + n - 1) & (n \in N). \end{cases}$$

Corresponding to a function $\mathscr{F}_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ defined by

$$\mathscr{F}_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = z^p {}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z),$$

we consider a linear operator

$$H_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) : A(p) \to A(p),$$

defined by the convolution

$$H_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) f(z) = \mathscr{F}_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) * f(z). \tag{1.5}$$

For convenience, we write

$$H_{p,q,s}(\alpha_i) = H_p(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) \quad (i = 1, 2, \ldots, q). \tag{1.6}$$

Thus, after some calculations, we have

$$z(H_{p,q,s}(\alpha_i) f(z))' = \alpha_i H_{p,q,s}(\alpha_i + 1) f(z) - (\alpha_i - p) H_{p,q,s}(\alpha_i) f(z)$$
$$(i = 1, 2, \ldots, q). \tag{1.7}$$

It should be remarked that the linear operator $H_{p,q,s}(\alpha_i)$ $(i = 1, 2, \ldots, q)$ is a generalization of many operators considered earlier. For $q = 2$ and $s = 1$ Carlson and Shaffer studied this operator under certain restrictions on the parameters α_1, α_2 and β_1 in [1]. A more general operator was studied by Ponnusamy and Rønning [13]. Also, many interesting subclasses of analytic functions, associated with the operator $H_{p,q,s}(\alpha_i)$ $(i = 1, 2, \ldots, q)$ and its many special cases, were investigated recently by (for example) Dziok and Srivastava [2, 3, 4], Gangadharan et al. [5], Liu [7], Liu and Srivastava [8, 9] and others (see also [6, 12, 15, 16, 17]).

In the present sequel to these earlier works, we shall use the method of differential subordination to derive several interesting properties and characteristics of the operator $H_{p,q,s}(\alpha_i)$ $(i = 1, 2, \ldots, q)$.

2. MAIN RESULTS

We begin by recalling each of the following lemmas which will be required in our present investigation.

Lemma 2.1 (see [10]). *Let $h(z)$ be analytic and convex univalent in U, $h(0) = 1$ and let $g(z) = 1 + b_1 z + b_2 z^2 + \cdots$ be analytic in U. If*

$$g(z) + zg'(z)/c \prec h(z) \quad (z \in U; c \neq 0), \tag{2.1}$$

then for $\operatorname{Re} c \geq 0$,

$$g(z) \prec \frac{c}{z^c} \int_0^z t^{c-1} h(t) dt. \tag{2.2}$$

Lemma 2.2 (see [14]). *The function $(1 - z)^\gamma \equiv e^{\gamma \log(1-z)}$, $\gamma \neq 0$, is univalent in U if and only if γ is either in the closed disk $|\gamma - 1| \leq 1$ or in the closed disk $|\gamma + 1| \leq 1$.*

Lemma 2.3 (see [11]). *Let $q(z)$ be univalent in U and let $\theta(w)$ and $\phi(w)$ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$ and suppose that*

(1) *$Q(z)$ is starlike (univalent) in U;*
(2) $\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right) = \operatorname{Re}\left(\frac{\theta'(q(z))}{\phi(q(z))} + \frac{zQ'(z)}{Q(z)}\right) > 0 \quad (z \in U).$

If $p(z)$ is analytic in U, with $p(0) = q(0)$, $p(U) \subset D$, and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) = h(z),$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

We now prove our first result given by Theorem 2.4 below.

Theorem 2.4. *Let $\alpha_i > 0$ $(i = 1, 2, \ldots, q)$, $\lambda > 0$, and $-1 \leq B < A \leq 1$. If $f(z) \in A(p)$ satisfies*

$$(1 - \lambda)\frac{H_{p,q,s}(\alpha_i)f(z)}{z^p} + \lambda\frac{H_{p,q,s}(\alpha_i + 1)f(z)}{z^p} \prec h(A, B; z), \tag{2.3}$$

then

$$\operatorname{Re}\left(\left(\frac{H_{p,q,s}(\alpha_i)f(z)}{z^p}\right)^{\frac{1}{m}}\right) > \left(\frac{\alpha_i}{\lambda}\int_0^1 u^{\frac{\alpha_i}{\lambda}-1}\left(\frac{1 - Au}{1 - Bu}\right)du\right)^{\frac{1}{m}} \quad (m \geq 1). \tag{2.4}$$

The result is sharp.

Proof. Let

$$g(z) = \frac{H_{p,q,s}(\alpha_i)f(z)}{z^p} \tag{2.5}$$

for $f(z) \in A(p)$. Then the function $g(z) = 1 + b_1 z + \cdots$ is analytic in U. By making use of (1.7) and (2.5), we obtain

$$\frac{H_{p,q,s}(\alpha_i + 1)f(z)}{z^p} = g(z) + \frac{zg'(z)}{\alpha_i}. \tag{2.6}$$

From (2.3), (2.5) and (2.6) we get

$$g(z) + \frac{\lambda}{\alpha_i}zg'(z) \prec h(A, B; z). \tag{2.7}$$

Now an application of Lemma 2.1 leads to

$$g(z) \prec \frac{\alpha_i}{\lambda} z^{-\frac{\alpha_i}{\lambda}} \int_0^1 t^{\frac{\alpha_i}{\lambda}-1} \left(\frac{1+At}{1+Bt}\right) dt \tag{2.8}$$

or

$$\frac{H_{p,q,s}(\alpha_i)f(z)}{z^p} = \frac{\alpha_i}{\lambda} \int_0^1 u^{\frac{\alpha_i}{\lambda}-1} \left(\frac{1+Auw(z)}{1+Buw(z)}\right) du, \tag{2.9}$$

where $w(z)$ is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ ($z \in U$).

In view of $-1 \leq B < A \leq 1$ and $\alpha_i > 0$, it follows from (2.9) that

$$\operatorname{Re}\left(\frac{H_{p,q,s}(\alpha_i)f(z)}{z^p}\right) > \frac{\alpha_i}{\lambda} \int_0^1 u^{\frac{\alpha_i}{\lambda}-1} \left(\frac{1-Au}{1-Bu}\right) du \quad (z \in U). \tag{2.10}$$

Therefore, with the aid of the elementary inequality $\operatorname{Re}(w^{1/m}) \geq (\operatorname{Re} w)^{1/m}$ for $\operatorname{Re} w > 0$ and $m \geq 1$, the inequality (2.4) follows directly from (2.10).

To show the sharpness of (2.4), we take $f(z) \in A(p)$ defined by

$$\frac{H_{p,q,s}(\alpha_i)f(z)}{z^p} = \frac{\alpha_i}{\lambda} \int_0^1 u^{\frac{\alpha_i}{\lambda}-1} \left(\frac{1+Auz}{1+Buz}\right) du.$$

For this function, we find that

$$(1-\lambda)\frac{H_{p,q,s}(\alpha_i)f(z)}{z^p} + \lambda\frac{H_{p,q,s}(\alpha_i+1)f(z)}{z^p} = \frac{1+Az}{1+Bz}$$

and

$$\frac{H_{p,q,s}(\alpha_i)f(z)}{z^p} \to \frac{\alpha_i}{\lambda} \int_0^1 u^{\frac{\alpha_i}{\lambda}-1} \left(\frac{1-Au}{1-Bu}\right) du \quad \text{as } z \to -1.$$

Hence the proof of the theorem is complete. □

Next we prove our second theorem.

Theorem 2.5. *Let $\alpha_i > 0$ $(i = 1, 2, \ldots, q)$, and $0 \leq \rho < 1$. Let γ be a complex number with $\gamma \neq 0$ and satisfy either $|2\gamma(1-\rho)\alpha_i - 1| \leq 1$ or $|2\gamma(1-\rho)\alpha_i + 1| \leq 1$ $(i = 1, 2, \ldots, q)$. If $f(z) \in A(p)$ satisfies the condition*

$$\operatorname{Re}\left(\frac{H_{p,q,s}(\alpha_i+1)f(z)}{H_{p,q,s}(\alpha_i)f(z)}\right) > \rho \quad (z \in U; i = 1, 2, \ldots, q), \tag{2.11}$$

then

$$\left(\frac{H_{p,q,s}(\alpha_i)f(z)}{z^p}\right)^{\gamma} \prec \frac{1}{(1-z)^{2\gamma(1-\rho)\alpha_i}} = q(z) \quad (z \in U; i = 1, 2, \ldots, q), \tag{2.12}$$

where $q(z)$ is the best dominant.

Proof. Let

$$p(z) = \left(\frac{H_{p,q,s}(\alpha_i)f(z)}{z^p}\right)^{\gamma} \quad (z \in U; i = 1, 2, \ldots, q). \tag{2.13}$$

Then, by making use of (1.7), (2.11) and (2.13), we have

$$1 + \frac{zp'(z)}{\gamma\alpha_i p(z)} \prec \frac{1+(1-2\rho)z}{1-z} \quad (z \in U). \tag{2.14}$$

If we take

$$q(z) = \frac{1}{(1-z)^{2\gamma(1-\rho)\alpha_i}}, \quad \theta(w) = 1 \quad \text{and} \quad \phi(w) = \frac{1}{\gamma\alpha_i w},$$

then $q(z)$ is univalent by the condition of the theorem and Lemma 2.2. Further, it is easy to show that $q(z), \theta(w)$ and $\phi(w)$ satisfy the conditions of Lemma 2.3. Since

$$Q(z) = zq'(z)\phi(q(z)) = \frac{2(1-\rho)z}{1-z}$$

is univalent starlike in U and

$$h(z) = \theta(q(z)) + Q(z) = \frac{1+(1-2\rho)z}{1-z}.$$

It may be readily checked that the conditions (1) and (2) of Lemma 2.3 are satisfied. Thus the result follows from (2.14) immediately. The proof is complete. □

Corollary 2.6. *Let* $\alpha_i > 0$ $(i = 1, 2, \ldots, q)$ *and* $0 \leq \rho < 1$. *Let* γ *be a real number and* $\gamma \geq 1$. *If* $f(z) \in A(p)$ *satisfies the condition (2.11), then*

$$\operatorname{Re}\left(\frac{H_{p,q,s}(\alpha_i)f(z)}{z^p}\right)^{\frac{1}{2\gamma(1-\rho)\alpha_i}} > 2^{-1/\gamma} \quad (z \in U; i = 1, 2, \ldots, q).$$

The bound $2^{-1/\gamma}$ *is the best possible.*

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