SOME PROPERTIES OF A NEW CLASS OF ANALYTIC FUNCTIONS DEFINED IN TERMS OF A HADAMARD PRODUCT

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Received: 15 April, 2007
Accepted: 12 January, 2008
Communicated by: H.M. Srivastava
2000 AMS Sub. Class.: Primary 30C45.

Key words: Starlike function, Convex function, Close-to-convex function, Hadamard prod-

uct, Ruscheweyh operator, Carlson and Schaffer operator, Subordination factor

sequence, Schwarz function.

Abstract: In this paper we introduce a new class $\mathcal{H}(\phi, \alpha, \beta)$ of analytic functions which

is defined by means of a Hadamard product (or convolution) of two suitably normalized analytic functions. Several properties like, the coefficient bounds, growth and distortion theorems, radii of starlikeness, convexity and close-to-convexity are investigated. We further consider a subordination theorem, certain boundedness properties associated with partial sums, an integral transform of a certain class of functions, and some integral means inequalities. Several interest-

ing consequnces of our main results are also pointed out.

Acknowledgements: The authors thank the referee for his comments.



Properties of a New Class of Analytic Functions

R.K. Raina and Deepak Bansal

vol. 9, iss. 1, art. 22, 2008

Title Page

Contents







Page 1 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

Contents

l	Introduction and Preliminaries	3
2	Coefficient Estimates	7
3	Growth and Distortion Theorems	9
	Integral Transform of the Class $\mathcal{H}_{\mathcal{T}}(\phi, \alpha, \beta)$	10
5	Subordination Theorem	12
6	Partial Sums	15
7	Integral Means Inequalities	18

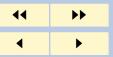


Properties of a New Class of Analytic Functions

R.K. Raina and Deepak Bansal vol. 9, iss. 1, art. 22, 2008

Title Page

Contents



Page 2 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

1. Introduction and Preliminaries

Let \mathcal{A} denote the class of functions f(z) normalized by f(0) = f'(0) - 1 = 0, and analytic in the open unit disk $\mathcal{U} = \{z; \ z \in \mathbb{C} : |z| < 1\}$, then f(z) can be expressed as

(1.1)
$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

Consider the subclass \mathcal{T} of the class \mathcal{A} consisting of functions of the form

(1.2)
$$f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k,$$

then a function $f(z) \in \mathcal{A}$ is said to be in the class of uniformly β - starlike functions of order α (denoted by $USF(\alpha, \beta)$), if

$$(1.3) \quad \Re\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} > \beta \left|\frac{zf'(z)}{f(z)} - 1\right| \qquad (-1 \le \alpha < 1, \ \beta \ge 0; z \in \mathcal{U}).$$

For $\alpha = 0$ in (1.3), we obtain the class of uniformly β -starlike functions which is denoted by $USF(\beta)$. Similarly, if $f(z) \in \mathcal{A}$ satisfies

$$(1.4) \quad \Re\left\{1 + \frac{zf''(z)}{f'(z)} - \alpha\right\} > \beta \left| \frac{zf''(z)}{f'(z)} \right| \qquad (-1 \le \alpha < 1, \ \beta \ge 0; z \in \mathcal{U}),$$

then f(z) is said to be in the class of uniformly β -convex functions of order α , and is denoted by $UCF(\alpha,\beta)$. When $\alpha=0$ in (1.4), we obtain the class of uniformly β -convex functions which is denoted by $UCF(\beta)$.

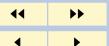


Properties of a New Class of Analytic Functions

R.K. Raina and Deepak Bansal vol. 9, iss. 1, art. 22, 2008

Title Page

Contents



Page 3 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

The classes of uniformly convex and uniformly starlike functions have been extensively studied by Goodman ([2], [3]), Kanas and Srivastava [4], Kanas and Wisniowska [5], Ma and Minda [8] and Ronning [10].

If $f, g \in \mathcal{A}$ (where f(z) is given by (1.1)), and g(z) is defined by

(1.5)
$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k,$$

then their Hadamard product (or convolution) f * q is defined by

(1.6)
$$(f * g)(z) := z + \sum_{k=2}^{\infty} a_k b_k z^k =: (g * f)(z).$$

We introduce here a class $\mathcal{H}(\phi, \alpha, \beta)$ which is defined as follows: Suppose the function $\phi(z)$ is given by

$$\phi(z) = z + \sum_{k=2}^{\infty} \mu_k z^k,$$

where $\mu_k \ge 0 \ (\forall k \in \mathbb{N} \setminus \{1\})$. We say that $f(z) \in \mathcal{A}$ is in $\mathcal{H}(\phi, \alpha, \beta)$, provided that $(f * \phi)(z) \ne 0$, and

(1.8)
$$\Re\left\{\frac{z\left[(f*\phi)(z)\right]'}{(f*\phi)(z)}\right\} > \beta \left|\frac{z\left[(f*\phi)(z)\right]'}{(f*\phi)(z)} - 1\right| + \alpha,$$
$$(-1 \le \alpha < 1, \ \beta \ge 0; z \in \mathcal{U}).$$

Generally speaking, $\mathcal{H}(\phi, \alpha, \beta)$ consists of functions $F(z) = (f * \phi)(z)$ which are uniformly β -starlike functions of order α in \mathcal{U} . We also let

(1.9)
$$\mathcal{H}_{\mathcal{T}}(\phi, \alpha, \beta) = \mathcal{H}(\phi, \alpha, \beta) \cap \mathcal{T}.$$



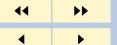
Properties of a New Class of Analytic Functions

R.K. Raina and Deepak Bansal

vol. 9, iss. 1, art. 22, 2008

Title Page

Contents



Page 4 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

Several known subclasses can be obtained from the class $\mathcal{H}(\phi, \alpha, \beta)$, by suitably choosing the values of the arbitrary function ϕ , and the parameters α and β . We mention below some of these subclasses of $\mathcal{H}(\phi, \alpha, \beta)$ consisting of functions $f(z) \in \mathcal{A}$. We observe that

(1.10)
$$\mathcal{H}\left\{\frac{z}{(1-z)^{\lambda+1}}, \alpha, \beta\right\} = \mathcal{S}_p^{\lambda}(\alpha, \beta)$$
$$(-1 \leq \alpha < 1, \ \beta \geq 0, \ \lambda > -1; z \in \mathcal{U}),$$

in which case the function $\frac{z}{(1-z)^{\lambda+1}}$ is related to the Ruscheweyh derivative operator $D^{\lambda}f(z)$ ([12]) defined by

$$D^{\lambda}f(z) = \frac{z}{(1-z)^{\lambda+1}} * f(z) \qquad (\lambda > -1).$$

The class $\mathcal{S}_p^{\lambda}(\alpha,\beta)$ was studied by Rosy *et al.* [11] and Shams *et al.* [13] and this class also reduces to $\mathcal{S}(\alpha)$ and $\mathcal{K}(\alpha)$ which are, respectively, the familiar classes of starlike functions of order α $(0 \le \alpha < 1)$ and convex functions of order α $(0 \le \alpha < 1)$ (see [15]).

Also

(1.11)
$$\mathcal{H}\left\{\phi(a,c,z),\alpha,\beta\right\} = S(\alpha,\beta),$$

and in this case the function $\phi(a,c,z)$ is related to the Carlson and Shaffer operator $\mathcal{L}(a,c)f(z)$ ([1]) defined by

$$\mathcal{L}(a,c)f(z) = \phi(a,c,z) * f(z).$$

The class $S(\alpha, \beta)$ was studied by Murugusundaramoorthy and Magesh [9]. Further, we let

(1.12)
$$\mathcal{T}S_p^{\lambda}(\alpha,\beta) = S_p^{\lambda}(\alpha,\beta) \cap \mathcal{T}; \qquad \mathcal{T}S(\alpha,\beta) = S(\alpha,\beta) \cap \mathcal{T}.$$

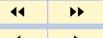


Properties of a New Class of Analytic Functions

R.K. Raina and Deepak Bansal vol. 9, iss. 1, art. 22, 2008

Title Page

Contents



Page 5 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

The object of the present paper is to investigate the coefficient estimates, distortion properties and the radii of starlikeness, convexity and close-to-convexity for the class of functions $\mathcal{H}(\phi,\alpha,\beta)$. Further (for this class of functions), we obtain a subordination theorem, boundedness properties involving partial sums, properties relating to an integral transform and some integral mean inequalities. Several corollaries depicting interesting consequences of the main results are also mentioned.



Properties of a New Class of Analytic Functions

R.K. Raina and Deepak Bansal vol. 9, iss. 1, art. 22, 2008

Title Page

Contents

Page 6 of 20

Go Back

Full Screen

journal of inequalities in pure and applied mathematics

Close

issn: 1443-5756

2. Coefficient Estimates

We first mention a sufficient condition for the function f(z) of the form (1.1) to belong to the class $\mathcal{H}(\phi, \alpha, \beta)$ given by the following result which can be established easily.

Theorem 2.1. If $f(z) \in A$ of the form (1.1) satisfies

(2.1)
$$\sum_{k=2}^{\infty} \mathcal{B}_k(\mu_k; \alpha, \beta) |a_k| \leq 1,$$

where

(2.2)
$$\mathcal{B}_k(\mu_k; \alpha, \beta) = \frac{\{k(\beta+1) - (\alpha+\beta)\} \mu_k}{1 - \alpha},$$

for some $\alpha(-1 \leq \alpha < 1)$, $\beta(\beta \geq 0)$ and $\mu_k \geq 0 \ (\forall k \in \mathbb{N} \setminus \{1\})$, then $f(z) \in \mathcal{H}(\phi, \alpha, \beta)$.

Our next result shows that the condition (2.1) is necessary as well for functions of the form (1.2) to belong to the class of functions $\mathcal{H}_{\mathcal{T}}(\phi, \alpha, \beta)$.

Indeed, by using (1.2), (1.6) to (1.8), and in the process letting $z \to 1^-$ along the real axis, we arrive at the following:

Theorem 2.2. A necessary and sufficient condition for f(z) of the form (1.2) to be in $\mathcal{H}_{\mathcal{T}}(\phi, \alpha, \beta)$, $-1 \leq \alpha < 1$, $\beta \geq 0$, $\mu_k \geq 0$ ($\forall k \in \mathbb{N} \setminus \{1\}$) is that

(2.3)
$$\sum_{k=2}^{\infty} \mathcal{B}_k(\mu_k; \alpha, \beta) |a_k| \leq 1,$$

where

(2.4)
$$\mathcal{B}_k(\mu_k; \alpha, \beta) = \frac{\{k(\beta+1) - (\alpha+\beta)\} \mu_k}{1 - \alpha}.$$



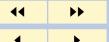
Properties of a New Class of Analytic Functions

R.K. Raina and Deepak Bansal

vol. 9, iss. 1, art. 22, 2008

Title Page

Contents



Page 7 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

Corollary 2.3. Let f(z) defined by (1.2) belong to the class $\mathcal{H}_{\mathcal{T}}(\phi, \alpha, \beta)$, then

(2.5)
$$|a_k| \le \frac{1}{\mathcal{B}_k(\mu_k; \alpha, \beta)} \qquad (k \ge 2).$$

The result is sharp (for each k), for functions of the form

(2.6)
$$f_k(z) = z - \frac{1}{\mathcal{B}_k(\mu_k; \alpha, \beta)} z^k \qquad (k = 2, 3, ...),$$

where $\mathcal{B}_k(\mu_k; \alpha, \beta)$ is given by (2.4).

Remark 1. It is clear from (2.4) that if $\{\mu_k\}_{k=2}^{\infty}$ is a non-decreasing positive sequence, then $\{\mathcal{B}_k(\mu_k;\alpha,\beta)\}_{k=2}^{\infty}$ and $\left\{\frac{\mathcal{B}_k(\mu_k;\alpha,\beta)}{k}\right\}_{k=2}^{\infty}$ would also be non-decreasing positive sequences (being the product of two non-decreasing positive sequences).

Remark 2. By appealing to (1.10), we find that Theorems 2.1, 2.2 and Corollary 2.3 correspond, respectively, to the results due to Rosy *et al.* [11, Theorems 2.1, 2.2 and Corollary 2.3]. Similarly, making use of (1.11), then Theorems 2.1, 2.2 and Corollary 2.3, respectively, give the known theorems of Murugursundarmoorthy *et al.* [9, Theorems 2.1, 2.2 and Corollary 2.3].



Properties of a New Class of Analytic Functions

R.K. Raina and Deepak Bansal vol. 9, iss. 1, art. 22, 2008

Title Page

Contents



Page 8 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

3. Growth and Distortion Theorems

In this section we state the following growth and distortion theorems for the class $\mathcal{H}_{\mathcal{T}}(\phi, \alpha, \beta)$. The results follow easily, therefore, we omit the proof details.

Theorem 3.1. Let the function f(z) defined by (1.2) be in the class $\mathcal{H}_{\mathcal{T}}(\phi, \alpha, \beta)$. If $\{\mu_k\}_{k=2}^{\infty}$ is a positive non-decreasing sequence, then

(3.1)
$$|z| - \frac{(1-\alpha)}{(2+\beta-\alpha)\mu_2} |z|^2 \le |f(z)| \le |z| + \frac{(1-\alpha)}{(2+\beta-\alpha)\mu_2} |z|^2.$$

The equality in (3.1) is attained for the function f(z) given by

(3.2)
$$f(z) = z - \frac{1 - \alpha}{(2 + \beta - \alpha)\mu_2} z^2.$$

Theorem 3.2. Let the function f(z) defined by (1.2) be in the class $\mathcal{H}_{\mathcal{T}}(\phi, \alpha, \beta)$. If $\{\mu_k\}_{k=2}^{\infty}$ is a positive non-decreasing sequence, then

(3.3)
$$1 - \frac{2(1-\alpha)}{(2+\beta-\alpha)\mu_2}|z| \le |f'(z)| \le 1 + \frac{2(1-\alpha)}{(2+\beta-\alpha)\mu_2}|z|.$$

The equality in (3.3) is attained for the function f(z) given by (3.2).

In view of the relationships (1.10) and (1.11), Theorems 3.1 and 3.2 would yield the corresponding distortion properties for the classes $\mathcal{TS}_p^{\lambda}(\alpha,\beta)$ and $\mathcal{TS}(\alpha,\beta)$.



Properties of a New Class of Analytic Functions

R.K. Raina and Deepak Bansal

vol. 9, iss. 1, art. 22, 2008

Title Page

Contents



Page 9 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

4. Integral Transform of the Class $\mathcal{H}_T(\phi, \alpha, \beta)$

For $f(z) \in \mathcal{A}$, we define the integral transform

(4.1)
$$V_{\mu}(f(z)) = \int_{0}^{1} \mu(t) \frac{f(tz)}{t} dt,$$

where μ is a real-valued non-negative weight function normalized, so that $\int_0^1 \mu(t)dt = 1$. In particular, when $\mu(t) = (1+\eta)t^{\eta}$, $\eta > -1$ then V_{μ} is a known Bernardi integral operator. On the other hand, if

(4.2)
$$\mu(t) = \frac{(1+\eta)^{\delta}}{\Gamma(\delta)} t^{\eta} \left(\log \frac{1}{t}\right)^{\delta-1} \qquad (\eta > -1, \ \delta > 0),$$

then V_{μ} becomes the Komatu integral operator (see [6]).

We first show that the class $\mathcal{H}_{\mathcal{T}}(\phi, \alpha, \beta)$ is closed under $V_{\mu}(f)$. By applying (1.2), (4.1) and (4.2), we straightforwardly arrive at the following result.

Theorem 4.1. Let
$$f(z) \in \mathcal{H}_{\mathcal{T}}(\phi, \alpha, \beta)$$
, then $V_{\mu}(f(z)) \in \mathcal{H}_{\mathcal{T}}(\phi, \alpha, \beta)$.

Following the usual methods of derivation, we can prove the following results:

Theorem 4.2. Let the function f(z) defined by (1.2) be in the class $\mathcal{H}_{\mathcal{T}}(\phi, \alpha, \beta)$. Then $V_{\mu}(f(z))$ is starlike of order $\sigma(0 \le \sigma < 1)$ in $|z| < r_1$, where

and $\mathcal{B}_k(\mu_k; \alpha, \beta)$ is given by (2.4). The result is sharp for the function f(z) given by (3.2).

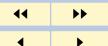


Properties of a New Class of Analytic Functions

R.K. Raina and Deepak Bansal vol. 9, iss. 1, art. 22, 2008

Title Page

Contents



Page 10 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

Theorem 4.3. Let the function f(z) defined by (1.2) be in the class $\mathcal{H}_{\mathcal{T}}(\phi, \alpha, \beta)$. Then $V_{\mu}(f(z))$ is convex of order $\sigma(0 \le \sigma < 1)$ in $|z| < r_2$, where

(4.4)
$$|z| < r_2 = \inf_{k} \left[\frac{\mathcal{B}_k(\mu_k; \alpha, \beta)(1 - \sigma)}{k(k - \sigma) \left(\frac{\eta + 1}{\eta + k}\right)^{\delta}} \right]^{\frac{1}{k - 1}} \quad (k \ge 2, \ \eta > -1, \delta > 0; \ z \in \mathcal{U}),$$

and $\mathcal{B}_k(\mu_k; \alpha, \beta)$ is given by (2.4).

Theorem 4.4. Let the function f(z) defined by (1.2) be in the class $\mathcal{H}_T(\phi, \alpha, \beta)$. Then $V_{\mu}(f(z))$ is close-to-convex of order $\sigma(0 \le \sigma < 1)$ in $|z| < r_3$, where

and $\mathcal{B}_k(\mu_k; \alpha, \beta)$ is given by (2.4).

Remark 3. On choosing the arbitrary function $\phi(z)$, suitably in accordance with the subclass defined by (1.10), Theorems 4.1, 4.2 and 4.3, respectively, give the results due to Shams *et al.* [13, Theorems 1, 2 and 3]. Also, making use of (1.11), Theorems 4.1, 4.2, 4.3 and 4.4 yield the corresponding results for the class $\mathcal{T}S(\alpha,\beta)$.

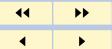


Properties of a New Class of Analytic Functions

R.K. Raina and Deepak Bansal vol. 9, iss. 1, art. 22, 2008

Title Page

Contents



Page 11 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

5. Subordination Theorem

Before stating and proving our subordination theorem for the class $\mathcal{H}(\phi, \alpha, \beta)$, we need the following definitions and a lemma due to Wilf [16].

Definition 5.1. For two functions f and g analytic in \mathcal{U} , we say that the function f is subordinate to g in \mathcal{U} (denoted by $f \prec g$), if there exists a Schwarz function w(z), analytic in \mathcal{U} with w(0) = 0 and |w(z)| < |z| < 1 ($z \in \mathcal{U}$), such that f(z) = g(w(z)).

Definition 5.2. A sequence $\{b_k\}_{k=1}^{\infty}$ of complex numbers is called a subordination factor sequence if whenever f(z) is analytic, univalent and convex in \mathcal{U} , then

(5.1)
$$\sum_{k=1}^{\infty} b_k a_k z^k \prec f(z) \qquad (z \in \mathcal{U}, \ a_1 = 1).$$

Lemma 5.3. The sequence $\{b_k\}_{k=1}^{\infty}$ is a subordinating factor sequence if and if only

(5.2)
$$\Re\left\{1 + 2\sum_{k=1}^{\infty} b_k z^k\right\} > 0, \quad (z \in \mathcal{U}).$$

Theorem 5.4. Let f(z) of the form (1.1) satisfy the coefficient inequality (2.1), and $\langle \mu_k \rangle_{k=2}^{\infty}$ be a non-decreasing sequence, then

(5.3)
$$\frac{(2+\beta-\alpha)\,\mu_2}{2\,\{(2+\beta-\alpha)\,\mu_2+(1-\alpha)\}}(f*g)(z) \prec g(z),\\ (-1 \le \alpha < 1, \ \beta \ge 0, \ z \in \mathcal{U}, \ \mu_k \ge 0 \ (\forall k \in \mathbb{N} \setminus \{1\}))$$



Properties of a New Class of Analytic Functions

R.K. Raina and Deepak Bansal

vol. 9, iss. 1, art. 22, 2008

Title Page

Contents

44 **)**

Page 12 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

for every function $g(z) \in \mathcal{K}$ (class of convex functions). In particular:

(5.4)
$$\Re\{f(z)\} > \frac{-\{(2+\beta-\alpha)\,\mu_2 + (1-\alpha)\}}{(2+\beta-\alpha)\,\mu_2} \ (z\in\mathcal{U}).$$

The constant factor

(5.5)
$$\frac{(2+\beta-\alpha)\,\mu_2}{2\,\{(2+\beta-\alpha)\,\mu_2+(1-\alpha)\}},$$

in the subordination result (5.3) cannot be replaced by any larger one.

Proof. Let f(z) defined by (1.1) satisfy the coefficient inequality (2.1). In view of (1.5) and Definition 5.2, the subordination (5.3) of our theorem will hold true if the sequence

(5.6)
$$\left\{ \frac{(2+\beta-\alpha)\,\mu_2}{2\left\{ (2+\beta-\alpha)\,\mu_2 + (1-\alpha) \right\}} a_k \right\}_{k=1}^{\infty} \qquad (a_1=1),$$

is a subordinating factor sequence which by virtue Lemma 5.3 is equivalent to the inequality

(5.7)
$$\Re\left(1 + \sum_{k=1}^{\infty} \frac{(2+\beta-\alpha)\,\mu_2}{\{(2+\beta-\alpha)\,\mu_2 + (1-\alpha)\}} a_k z^k\right) > 0 \qquad (z \in \mathcal{U}).$$

In view of (2.1) and when |z| = r (0 < r < 1), we obtain

$$\Re\left(1 + \sum_{k=1}^{\infty} \frac{(2 + \beta - \alpha) \mu_2}{\{(2 + \beta - \alpha) \mu_2 + (1 - \alpha)\}} a_k z^k\right)$$

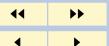


Properties of a New Class of Analytic Functions

R.K. Raina and Deepak Bansal vol. 9, iss. 1, art. 22, 2008

Title Page

Contents



Page 13 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

$$\geq 1 - \frac{(2+\beta-\alpha)\,\mu_2}{\{(2+\beta-\alpha)\,\mu_2 + (1-\alpha)\}}r - \sum_{k=2}^{\infty} \frac{(1-\alpha)}{\{(2+\beta-\alpha)\,\mu_2 + (1-\alpha)\}} |a_k|r > 0.$$

This evidently establishes the inequality (5.7), and consequently the subordination relation (5.3) of Theorem 5.4 is proved. The assertion (5.4) follows readily from (5.3) when the function g(z) is selected as

$$g(z) = \frac{z}{1-z} = z + \sum_{k=2}^{\infty} z^k.$$

The sharpness of the multiplying factor in (5.3) can be established by considering a function h(z) defined by

$$h(z) = z - \frac{1 - \alpha}{(2 + \beta - \alpha) \mu_2} z^2,$$

which belongs to the class $\mathcal{H}_{\mathcal{T}}(\phi, \alpha, \beta)$. Using (5.3), we infer that

(5.8)
$$\frac{(2+\beta-\alpha)\,\mu_2}{2\,\{(2+\beta-\alpha)\,\mu_2+(1-\alpha)\}}h(z)\prec\frac{z}{1-z},$$

and it follows that

(5.9)
$$\inf_{|z| \le 1} \left\{ \Re \left(\frac{(2+\beta-\alpha)\,\mu_2}{2\left\{ (2+\beta-\alpha)\,\mu_2 + (1-\alpha) \right\}} h(z) \right) \right\} = -\frac{1}{2},$$

which completes the proof of Theorem 5.4.

If we choose the sequence μ_k appropriately by comparing (1.7) with (1.10) and (1.11), we can deduce additional subordination results from Theorem 5.4.



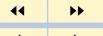
Properties of a New Class of Analytic Functions

R.K. Raina and Deepak Bansal

vol. 9, iss. 1, art. 22, 2008

Title Page

Contents



Page 14 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

6. Partial Sums

In this section we investigate the ratio of real parts of functions involving (1.2) and its sequence of partial sums defined by

(6.1)
$$f_1(z) = z;$$
 and $f_N(z) = z - \sum_{k=2}^N |a_k| z^k$ $(\forall k \in \mathbb{N} \setminus \{1\}),$

and determine sharp lower bounds for $\Re \{f(z)/f_N(z)\}$, $\Re \{f_N(z)/f(z)\}$, $\Re \{f'(z)/f'_N(z)\}$ and $\Re \{f'_N(z)/f'(z)\}$.

Theorem 6.1. Let f(z) of the form (1.2) belong to $\mathcal{H}_{\mathcal{T}}(\phi, \alpha, \beta)$, and $\langle \mu_k \rangle_{k=2}^{\infty}$ be a non-decreasing sequence such that $\mu_2 \geq \frac{1-\alpha}{2+\beta-\alpha} \left(0 < \frac{1-\alpha}{2+\beta-\alpha} < 1; -1 \leq \alpha < 1, \beta \geq 0\right)$, then

(6.2)
$$\Re\left(\frac{f(z)}{f_N(z)}\right) \ge 1 - \frac{1}{\mathcal{B}_{N+1}(\mu_{N+1}; \alpha, \beta)}$$

and

(6.3)
$$\Re\left(\frac{f_N(z)}{f(z)}\right) \ge \frac{\mathcal{B}_{N+1}(\mu_{N+1};\alpha,\beta)}{\mathcal{B}_{N+1}(\mu_{N+1};\alpha,\beta)+1},$$

where $\mathcal{B}_{N+1}(\mu_{N+1}; \alpha, \beta)$ is given by (2.4). The results are sharp for every N, with the extremal functions given by

(6.4)
$$f(z) = z - \frac{1}{\mathcal{B}_{N+1}(\mu_{N+1}; \alpha, \beta)} z^{N+1} \qquad (N \in \mathbb{N} \setminus \{1\}).$$

Proof. In order to prove (6.2), it is sufficient to show that

(6.5)
$$\mathcal{B}_{N+1}(\mu_{N+1}; \alpha, \beta) \left[\frac{f(z)}{f_N(z)} - \left(1 - \frac{1}{\mathcal{B}_{N+1}(\mu_{N+1}; \alpha, \beta)} \right) \right] \prec \frac{1+z}{1-z} \ (z \in \mathcal{U}).$$

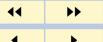


Properties of a New Class of Analytic Functions

R.K. Raina and Deepak Bansal vol. 9, iss. 1, art. 22, 2008

Title Page

Contents



Page 15 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

We can write

$$\mathcal{B}_{N+1}(\mu_{N+1}; \alpha, \beta) \left[\frac{1 - \sum_{k=2}^{\infty} |a_k| z^{k-1}}{1 - \sum_{k=2}^{N} |a_k| z^{k-1}} - \left(1 - \frac{1}{\mathcal{B}_{N+1}(\mu_{N+1}; \alpha, \beta)} \right) \right] = \frac{1 + w(z)}{1 - w(z)}.$$

Obviously w(0) = 0, and

$$|w(z)| \leq \frac{\mathcal{B}_{N+1}(\mu_{N+1}; \alpha, \beta) \sum_{k=N+1}^{\infty} |a_k|}{2 - 2 \sum_{k=2}^{N} |a_k| - \mathcal{B}_{N+1}(\mu_{N+1}; \alpha, \beta) \sum_{k=N+1}^{\infty} |a_k|},$$

which is less than one if and only if

(6.6)
$$\sum_{k=2}^{N} |a_k| + \mathcal{B}_{N+1}(\mu_{N+1}; \alpha, \beta) \sum_{k=N+1}^{\infty} |a_k| \le 1.$$

In view of (2.3), this is equivalent to showing that

(6.7)
$$\sum_{k=2}^{N} \left\{ \mathcal{B}_{k}(\mu_{k}; \alpha, \beta) - 1 \right\} |a_{k}| + \sum_{k=N+1}^{\infty} \left\{ \mathcal{B}_{k}(\mu_{k}; \alpha, \beta) - \mathcal{B}_{N+1}(\mu_{N+1}; \alpha, \beta) \right\} |a_{k}| \ge 0.$$

We observe that the first term of the first series in (6.7) is positive when $\mu_2 \ge \frac{1-\alpha}{2+\beta-\alpha}$, which is true (in view of the hypothesis). Now, since $\{\mathcal{B}_k(\mu_k; \alpha, \beta)\}_{k=2}^{\infty}$ is a non-decreasing sequence (see Remark 1), therefore, all the other terms in the first series

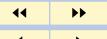


Properties of a New Class of Analytic Functions

R.K. Raina and Deepak Bansal vol. 9, iss. 1, art. 22, 2008

Title Page

Contents



Page 16 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

are positive. Also, the first term of the second series in (6.7) vanishes, and all other terms of this series also remain positive. Thus, the inequality (6.7) holds true. This completes the proof of (6.2). Finally, it can be verified that the equality in (6.2) is attained for the function given by (6.4) when, $z = re^{2\pi i/N}$ and $r \to 1^-$.

The proof of (6.3) is similar to (6.2), and is hence omitted.

Similarly, we can establish the following theorem.

Theorem 6.2. Let f(z) of the form (1.2) belong to $\mathcal{H}_T(\phi, \alpha, \beta)$, and $\langle \mu_k \rangle_{k=2}^{\infty}$ be a non-decreasing sequence such that $\mu_2 \geq \frac{2(1-\alpha)}{2+\beta-\alpha} \left(0 < \frac{1-\alpha}{2+\beta-\alpha} < 1; -1 \leq \alpha < 1, \beta \geq 0\right)$, then

(6.8)
$$\Re\left(\frac{f'(z)}{f'_N(z)}\right) \ge 1 - \frac{N+1}{\mathcal{B}_{N+1}(\mu_{N+1}; \alpha, \beta)}$$

and

(6.9)
$$\Re\left(\frac{f'_{N}(z)}{f'(z)}\right) \ge \frac{\mathcal{B}_{N+1}(\mu_{N+1}; \alpha, \beta)}{N+1+\mathcal{B}_{N+1}(\mu_{N+1}; \alpha, \beta)},$$

where $\mathcal{B}_{N+1}(\mu_{N+1}; \alpha, \beta)$ is given by (2.4). The results are sharp for every N, with the extremal functions given by (6.4).

Making use of (1.10) to (1.12), then Theorems 6.1 and 6.2 would yield the corresponding results for the classes $\mathcal{TS}_p^{\lambda}(\alpha,\beta)$ and $\mathcal{TS}(\alpha,\beta)$.



Properties of a New Class of Analytic Functions

R.K. Raina and Deepak Bansal vol. 9, iss. 1, art. 22, 2008

Title Page

Contents



Page 17 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

7. Integral Means Inequalities

The following subordination result due to Littlewood [7] will be required in our investigation.

Lemma 7.1. If f(z) and g(z) are analytic in \mathcal{U} with $f(z) \prec g(z)$, then

(7.1)
$$\int_0^{2\pi} \left| f(re^{i\theta}) \right|^{\mu} d\theta \le \int_0^{2\pi} \left| g(re^{i\theta}) \right|^{\mu} d\theta,$$

where $\mu > 0$, $z = re^{i\theta}$ and 0 < r < 1.

Application of Lemma 7.1 for functions f(z) in the class $\mathcal{H}_{\mathcal{T}}(\phi, \alpha, \beta)$ gives the following result using known procedures.

Theorem 7.2. Let $\mu > 0$. If $f(z) \in \mathcal{H}_{\mathcal{T}}(\phi, \alpha, \beta)$ is given by (1.2), and $\{\mu_k\}_{k=2}^{\infty}$, is a non-decreasing sequence, then, for $z = re^{i\theta}$ (0 < r < 1):

(7.2)
$$\int_0^{2\pi} \left| f(re^{i\theta}) \right|^{\mu} d\theta \leqq \int_0^{2\pi} \left| f_1(re^{i\theta}) \right|^{\mu} d\theta,$$

where

(7.3)
$$f_1(z) = z - \frac{(1-\alpha)}{(2+\beta-\alpha)\mu_2} z^2.$$

We conclude this paper by observing that several integral means inequalities can be deduced from Theorem 7.2 in view of the relationships (1.10) and (1.11).



Properties of a New Class of Analytic Functions

R.K. Raina and Deepak Bansal

vol. 9, iss. 1, art. 22, 2008

Title Page

Contents

44 **)**

Page 18 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

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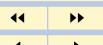
Properties of a New Class of Analytic Functions

R.K. Raina and Deepak Bansal

vol. 9, iss. 1, art. 22, 2008

Title Page

Contents



Page 19 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

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Properties of a New Class of Analytic Functions

R.K. Raina and Deepak Bansal vol. 9, iss. 1, art. 22, 2008



Go Back

Full Screen
Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756