

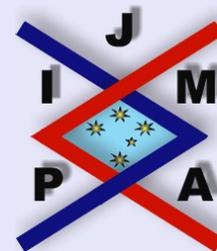
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## NOTE ON THE NORMAL FAMILY

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## Abstract

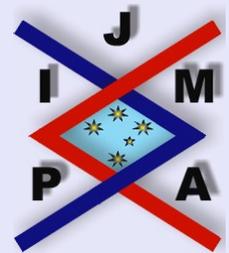
In this paper we consider the problem of normal family criteria and improve some results of I. Lihiri, S. Dewan and Y. Xu.

*2000 Mathematics Subject Classification:* 30D35.

*Key words:* Normal family, Meromorphic function.

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# 1. Introduction and Results

Let  $\mathbb{C}$  be the open complex plane and  $\mathcal{D} \in \mathbb{C}$  be a domain. Let  $f$  be a meromorphic function in the complex plane, we assume that the reader is familiar with the notations of Nevanlinna theory (see, e.g., [5][12]).

**Definition 1.1.** Let  $k$  be a positive integer, for any  $a$  in the complex plane. We denote by  $N_{(k)}(r, 1/(f - a))$  the counting function of  $a$ -points of  $f$  with multiplicity  $\leq k$ , by  $N_{(\geq k)}(r, 1/(f - a))$  the counting function of  $a$ -points of  $f$  with multiplicity  $\geq k$ , by  $N_k(r, 1/(f - a))$  the counting function of  $a$ -points of  $f$  with multiplicity of  $k$ , and denote the reduced counting function by  $\overline{N}_{(k)}(r, 1/(f - a))$ ,  $\overline{N}_{(\geq k)}(r, 1/(f - a))$  and  $\overline{N}_k(r, 1/(f - a))$ , respectively.

In 1995, Chen-Fang [3] proposed the following conjecture:

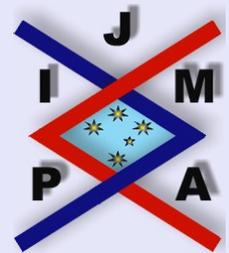
**Conjecture 1.1.** Let  $\mathcal{F}$  be a family of meromorphic functions in a domain  $\mathcal{D}$ . If for every function  $f \in \mathcal{F}$ ,  $f^{(k)} - af^n - b$  has no zero in  $\mathcal{D}$ , then  $\mathcal{F}$  is normal, where  $a(\neq 0)$ ,  $b$  are two finite numbers and  $k, n(\geq k + 2)$  are positive integers.

In response to this conjecture, Xu [11] proved the following result.

**Theorem A.** Let  $\mathcal{F}$  be a family of meromorphic functions in a domain  $\mathcal{D}$  and  $a(\neq 0)$ ,  $b$  be two finite constants. If  $k$  and  $n$  are positive integers such that  $n \geq k + 2$  and for every  $f \in \mathcal{F}$

- (i)  $f^{(k)} - af^n - b$  has no zero,
- (ii)  $f$  has no simple pole,

then  $\mathcal{F}$  is normal.



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The condition (ii) of Theorem A can be dropped if we choose  $n \geq k + 4$  (cf. [8][10]). If  $n \geq k + 3$ , is condition (ii) in Theorem A necessary? We will give an answer.

**Theorem 1.2.** *Let  $\mathcal{F}$  be a family of meromorphic functions in a domain  $\mathcal{D}$  and  $a(\neq 0), b$  be two finite constants. If  $k$  and  $n$  are positive integers such that  $n \geq k + 3$  and for every  $f \in \mathcal{F}$ ,  $f^{(k)} - af^n$  has no zero, then  $\mathcal{F}$  is normal.*

In addition, Lahiri and Dewan [6] investigated the situation when the power of  $f$  is negative in condition (i) of Theorem A.

**Theorem B.** *Let  $\mathcal{F}$  be a family of meromorphic functions in a domain  $\mathcal{D}$  and  $a(\neq 0), b$  be two finite constants. Suppose that  $E_f = \{z : z \in D \text{ and } f^{(k)}(z) - af^{-n}(z) = b\}$ , where  $k$  and  $n(\geq k)$  are positive integers.*

*If for every  $f \in F$*

- (i)  *$f$  has no zero of multiplicity less than  $k$ ,*
- (ii) *there exists a positive number  $M$  such that for every  $f \in F, |f(z)| \geq M$  whenever  $z \in E_f$ , then  $F$  is normal.*

I. Lahiri gave two examples to show that conditions (i) and (ii) are necessary. Naturally, we can question whether  $n \geq k$  is necessary, first we note the following example.

**Example 1.1.** *Let  $\mathbb{D} : |z| < 1$  and  $\mathcal{F} = \{f_n\}$ , where*

$$f_p(z) = \frac{z^3}{p}, \quad p = 2, 3, \dots,$$



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and  $n = 2, k = 3, a = 1, b = 0$ . Then  $f_p$  has the zeros of multiplicity 3 and  $E_{f_p} = \{z : z \in \mathbb{D} \text{ and } 6z^6 - p^3 = 0\}$ . For any  $z \in E_f, |f_p(z)| = \sqrt{\frac{p}{6}} \rightarrow \infty$ , as  $p \rightarrow \infty$ . But

$$|f_p^\sharp(z)| = \left| \frac{3pz^2}{p^2 + z^6} \right| < \left| \frac{3pz^2}{p^2 - |z|^6} \right| < \frac{3p}{p^2 - 1} < \frac{3}{p - 1} \leq 3,$$

for any  $p$ . By Marty's criterion, the family  $\{f_p\}$  is normal.

Hence we can give some answers. In fact, we can prove the following theorem:

**Theorem 1.3.** Let  $\mathcal{F}$  be a family of meromorphic functions in a domain  $\mathcal{D}$  and  $a (\neq 0), b$  be two finite constants. Suppose that  $E_f = \{z : z \in \mathbb{D} \text{ and } f^{(k)} - af^{-n} = b\}$ , where  $k$  and  $n$  are positive integers,

If for every  $f \in \mathcal{F}$

- (i)  $f$  has the zero of multiplicity at least  $k$ ,
- (ii) there exists a positive number  $M$  such that for every  $f \in \mathcal{F}, |f(z)| \geq M$  whenever  $z \in E_f$ .

Then  $\mathcal{F}$  is normal in  $\mathcal{D}$  so long as

(A)  $n \geq 2$ ; or

(B)  $n = 1$  and  $\overline{N}_k \left( r, \frac{1}{f} \right) = S(r, f)$ .

*Epecially, if  $f(z)$  is an entire function, we can obtain the complete answer.*



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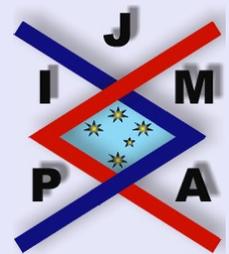
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**Theorem 1.4.** Let  $\mathcal{F}$  be a family of entire functions in a domain  $\mathcal{D}$  and  $a(\neq 0)$ ,  $b$  be two finite constants. Suppose that  $E_f = \{z : z \in \mathbb{D} \text{ and } f^{(k)} - af^{-n} = b\}$ , where  $k$  and  $n$  are positive integers,

If for every  $f \in \mathcal{F}$

- (i)  $f$  has no zero of multiplicity less than  $k$ ,
- (ii) there exists a positive number  $M$  such that for every  $f \in \mathcal{F}$ ,  $|f(z)| \geq M$  whenever  $z \in E_f$ , then  $\mathcal{F}$  is normal.



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## 2. Preliminaries

**Lemma 2.1.** [13] Let  $f$  be nonconstant meromorphic in the complex plane,  $L[f] = a_k f^{(k)} + a_{k-1} f^{(k-1)} + \dots + a_0 f$ , where  $a_0, a_1, \dots, a_k$  are small functions, for  $a \neq 0, \infty$ , let  $F = f^n L[f] - a$ , where  $n \geq 2$  is a positive integer. Then

$$\limsup_{r \rightarrow +\infty} \frac{\overline{N}(r, a; F)}{T(r, F)} > 0.$$

**Lemma 2.2.** Let  $f$  be nonconstant meromorphic in the complex plane,  $L[f]$  is given as in Lemma 2.1 and  $F = fL[f] - a$ . Then

$$T(r, f) \leq \left(6 + \frac{6}{k}\right) \left(\overline{N}\left(r, \frac{1}{F}\right) + \overline{N}_k\left(r, \frac{1}{f}\right)\right) + S(r, f).$$

*Proof.* For the simplification, we prove the case of  $L[f] = f^{(k)}$ , the general case is similar. Without loss of generality, let  $a = 1$ , then

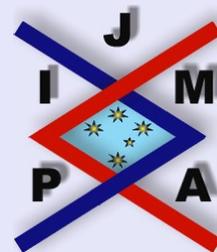
$$(2.1) \quad F = fL[f] - 1.$$

By differentiating the equation (2.1), we get

$$(2.2) \quad f\beta = -\frac{F'}{F},$$

where

$$(2.3) \quad \beta = \frac{f'}{f} f^{(k)} + f^{(k+1)} - f^{(k)} \frac{F'}{F}.$$



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Obviously  $F \not\equiv \text{constant}$ ,  $\beta \not\equiv 0$ . By the Clunie Lemma ([1] or [4])

$$(2.4) \quad m(r, \beta) = S(r, f).$$

Let  $z_0$  be a pole of  $f$  of order  $q$ . Then  $z_0$  is the simple pole of  $\frac{F'}{F}$ , and the poles of  $f$  of order  $q (\geq 2)$  are the zeros of  $\beta$  of order  $q - 1$  from (2.2), the simple pole of  $f$  is the non-zero analytic point of  $\beta$ , therefore

$$(2.5) \quad N_{(2)}(r, f) \leq N\left(r, \frac{1}{\beta}\right) + \overline{N}\left(r, \frac{1}{\beta}\right) \leq 2N\left(r, \frac{1}{\beta}\right).$$

By (2.3), we know the zeros of  $f$  of order  $q > k$  are not the poles of  $\beta$ . From (2.3), we get

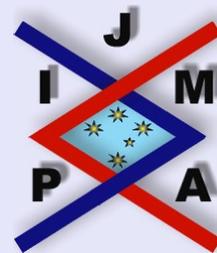
$$(2.6) \quad N(r, \beta) \leq \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}_k\left(r, \frac{1}{f}\right) + S(r, f).$$

Then, by (2.4) and (2.6), we have

$$(2.7) \quad T(r, \beta) \leq \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}_k\left(r, \frac{1}{f}\right) + S(r, f).$$

Next with (2.5) and (2.7), we obtain

$$(2.8) \quad N_{(2)}(r, f) \leq 2\overline{N}\left(r, \frac{1}{F}\right) + 2\overline{N}_k\left(r, \frac{1}{f}\right) + S(r, f).$$



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Noting by (2.2), (2.7) and the first fundamental theorem, we obtain

$$\begin{aligned}
 (2.9) \quad m(r, f) &\leq m\left(r, \frac{1}{\beta}\right) + m\left(r, \frac{F'}{F}\right) \\
 &\leq T(r, \beta) - N\left(r, \frac{1}{\beta}\right) + S(r, f) \\
 &= \bar{N}\left(r, \frac{1}{F}\right) + \bar{N}_k\left(r, \frac{1}{f}\right) + S(r, f).
 \end{aligned}$$

If  $f$  only have finitely many simple poles, we get Lemma 2.2 by (2.5) and (2.9).

Next we discuss that  $f$  have infinity simple poles. Let  $z_0$  be any simple pole of  $f$ . Then  $z_0$  is the non-zero analytic point of  $\beta$ . In a neighborhood of  $z_0$ , we have

$$(2.10) \quad f(z) = \frac{d_1(z_0)}{z - z_0} + d_0(z_0) + O(z - z_0)$$

and

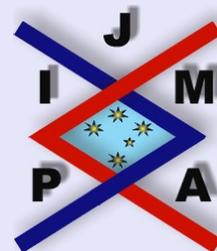
$$(2.11) \quad \beta(z) = \beta(z_0) + \beta'(z_0)(z - z_0) + O((z - z_0)^2),$$

where  $d_1(z_0) \neq 0, \beta(z_0) \neq 0$ . By differentiating (2.10), we get

$$(2.12) \quad f^{(j)}(z) = (-1)^j \frac{j! d_1(z_0)}{(z - z_0)^{j+1}} + \dots, \quad j = 1, 2, \dots, k.$$

with (2.3) and (2.5) we have

$$(2.13) \quad f\beta = f'f^{(k)} + f f^{(k+1)} - f^2 f^{(k)} \beta.$$



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Substituting (2.10)-(2.12) into (2.13), we obtain that the coefficients have the forms

$$(2.14) \quad d_1(z_0) = \frac{k+2}{\beta(z_0)},$$

$$(2.15) \quad d_0(z_0) = -\frac{(k+2)^2}{k+3} \frac{\beta'(z_0)}{(\beta(z_0))^2},$$

so that

$$(2.16) \quad \frac{d_0(z_0)}{d_1(z_0)} = -\frac{k+2}{k+3} \frac{\beta'(z_0)}{\beta(z_0)}.$$

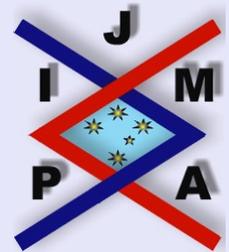
Through the calculating from (2.10) and (2.12), we get

$$(2.17) \quad \frac{f'(z)}{f(z)} = -\frac{1}{z-z_0} + \frac{d_0(z_0)}{d_1(z_0)} + O(z-z_0),$$

$$(2.18) \quad \frac{F'(z)}{F(z)} = -\frac{k+2}{z-z_0} + \frac{d_0(z_0)}{d_1(z_0)} + O(z-z_0).$$

Let

$$(2.19) \quad h(z) = \frac{F'(z)}{F(z)} - (k+2) \frac{f'(z)}{f(z)} - \frac{(k+1)(k+2)}{(k+3)} \frac{\beta'(z)}{\beta(z)}.$$




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Then, by (2.17)-(2.19), clearly,  $h(z_0) = 0$ . Therefore the simple pole of  $f$  is the zero of  $h(z)$ . From (2.19), we have

$$(2.20) \quad m(r, h) = S(r, f).$$

If  $f$  only has finitely many zeros. By (2.3) and the lemma of logarithmic derivatives, we get

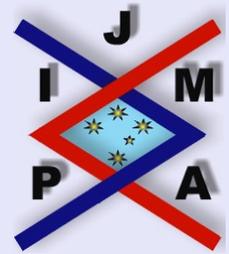
$$\begin{aligned} m\left(r, \frac{1}{f}\right) &\leq m\left(r, \frac{1}{\beta} \left(\frac{f'}{f} \frac{f^{(k)}}{f} + \frac{f^{(k+1)}}{f} - \frac{f^{(k)}}{f} \frac{F'}{F}\right)\right) \\ &\leq m\left(r, \frac{1}{\beta}\right) + S(r, f). \end{aligned}$$

It follows by (2.4) and (2.5) that

$$m\left(r, \frac{1}{f}\right) \leq \bar{N}\left(r, \frac{1}{F}\right) + \bar{N}_k\left(r, \frac{1}{f}\right) + S(r, f).$$

Using Nevanlinna's first fundamental theorem and  $f$  only has finitely many zeros, we obtain

$$\begin{aligned} T(r, f) &= T\left(r, \frac{1}{f}\right) + O(1) \\ &= m\left(r, \frac{1}{f}\right) + S(r, f) \\ &\leq \bar{N}\left(r, \frac{1}{F}\right) + \bar{N}_k\left(r, \frac{1}{f}\right) + S(r, f). \end{aligned}$$




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Hence the conclusion of Lemma 2.2 holds.

If  $f$  only has infinitely many zeros. We assert that  $h(z) \not\equiv 0$ . Otherwise  $h(z) \equiv 0$ , then

$$\frac{F'}{F} = (k+2)\frac{f'}{f} + \frac{(k+1)(k+2)}{(k+3)}\frac{\beta'(z_0)}{\beta(z_0)}.$$

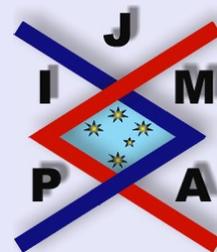
By integrating, we have

$$(2.21) \quad F^{(k+3)} = cf^{(k+2)(k+3)}\beta^{(k+1)(k+2)},$$

where  $c \neq 0$  is a constant. Any zeros of  $f$  of order  $q$  are not the zeros and poles of  $F$  by (2.1), and any zeros of  $f$  must be the poles of  $\beta$  by (2.21). Suppose that  $q > k$ , (otherwise, the conclusion of Lemma 2.2 holds by above) it contradicts (2.6), hence  $h(z) \not\equiv 0$ .

Since  $h(z) \not\equiv 0$ , and the simple pole of  $f$  is the zeros of  $h$ , we know the poles of  $h(z)$  occur only at the zeros of  $F$ , the zeros of  $f$ , the multiple poles of  $f$ , the zeros and poles of  $\beta$ , all are the simple pole of  $h(z)$ . At the same time, we note  $F' = f'f^{(k)} + ff^{(k+1)}$ , hence the zeros of  $f$  of the order of  $q (\geq k+2)$  at least are the zeros of  $F'$  of  $2q - (k+1)$ , and also at least are the zeros of  $\beta$  of order  $q - (k+1)$  by (2.2), hence,

$$\begin{aligned} \bar{N}_{(k+2)}\left(r, \frac{1}{f}\right) &\leq \frac{1}{k+2}N_{(k+2)}\left(r, \frac{1}{f}\right) \\ &\leq \frac{1}{k+2}\left(N\left(r, \frac{1}{\beta}\right) + (k+1)\bar{N}\left(r, \frac{1}{\beta}\right)\right) \\ &\leq N\left(r, \frac{1}{\beta}\right). \end{aligned}$$



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It follows from (2.8),(2.12) and (2.19), we have

$$\begin{aligned}
 (2.22) \quad & N(r, h) \\
 & \leq \bar{N}\left(r, \frac{1}{F}\right) + \bar{N}_{k+1}\left(r, \frac{1}{f}\right) + \bar{N}(r, \beta) + N\left(r, \frac{1}{\beta}\right) \\
 & \leq \bar{N}\left(r, \frac{1}{F}\right) + \bar{N}_k\left(r, \frac{1}{f}\right) + \bar{N}_{k+1}\left(r, \frac{1}{f}\right) + 2T(r, \beta) + S(r, f) \\
 & \leq 3\left(\bar{N}\left(r, \frac{1}{F}\right) + \bar{N}_k\left(r, \frac{1}{f}\right)\right) + \bar{N}_{k+1}\left(r, \frac{1}{f}\right) + S(r, f).
 \end{aligned}$$

Using (2.20), we get

$$\begin{aligned}
 (2.23) \quad & N_{1)}(r, f) \\
 & \leq N\left(r, \frac{1}{h}\right) \leq N(r, h) + S(r, f) \\
 & \leq 3\left(\bar{N}\left(r, \frac{1}{F}\right) + \bar{N}_k\left(r, \frac{1}{f}\right)\right) + \bar{N}_{k+1}\left(r, \frac{1}{f}\right) + S(r, f).
 \end{aligned}$$

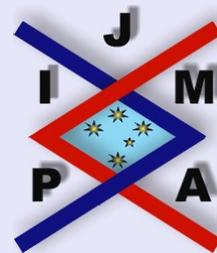
Note

$$\bar{N}_{k+1}\left(r, \frac{1}{f}\right) = \frac{1}{k+1}N_{k+1}\left(r, \frac{1}{f}\right) \leq \frac{1}{k+1}T(r, f) + S(r, f).$$

By (2.14),(2.16) and (2.23), we deduce

$$T(r, f) \leq \left(6 + \frac{6}{k}\right) \left(\bar{N}\left(r, \frac{1}{F}\right) + \bar{N}_k\left(r, \frac{1}{f}\right)\right) + S(r, f).$$

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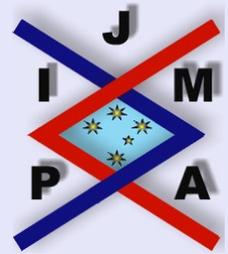


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**Lemma 2.3.** Let  $f$  be a nonconstant meromorphic function in the complex plane such that the zeros of  $f(z)$  are of multiplicity at least  $\geq k$  and  $a (\neq 0)$  be a finite constant. Then

- (i) If  $n \geq 2$ ,  $f^{(k)} - af^{-n}$  must have zero, where  $k$  and  $n$  are positive integers.
- (ii) If  $n = 1$ , and  $\overline{N}_k\left(r, \frac{1}{f}\right) = S(r, f)$ ,  $f^{(k)} - af^{-n}$  must have some zero, where  $k$  is a positive integer.

*Proof.* First we assume that  $n \geq 2$ , by Lemma 2.1, we know  $f^n f^{(k)} - a$  must have some zero. Since a zero of  $f^n f^{(k)} - a$  is a zero of  $f^{(k)} - af^{-n}$ , then  $f^{(k)} - af^{-n}$  must have some zero.

If  $n = 1$ , the zeros of  $f(z)$  are of multiplicity at least  $\geq k$ , so  $\overline{N}_{k-1}\left(r, \frac{1}{f}\right) = S(r, f)$ . With the condition of  $\overline{N}_k\left(r, \frac{1}{f}\right) = S(r, f)$ , we have  $\overline{N}_k\left(r, \frac{1}{f}\right) = S(r, f)$ . By Lemma 2.2, we know  $f f^{(k)} - a$  must have some zero. As the preceding paragraph a zero of  $f f^{(k)} - a$  is a zero of  $f^{(k)} - af^{-1}$ , the lemma is proved.  $\square$

**Lemma 2.4 ([13]).** Let  $f(z)$  be a transcendental meromorphic function in the complex plane, and  $a \neq 0$  be a constant. If  $n \geq k + 3$ , then  $f^{(k)} - af^n$  assumes zeros infinitely often.

**Remark 1.** In fact, E. Mues's [7, Theorem 1(b)] gave a counterexample to show that  $f' - f^4 = c$  has no solution. We know  $f^{(k)} - af^n$  cannot assume non-zero values for any positive integer  $n, k$  and  $n = k + 3$ . Hence Theorem 1.2 may be best when we drop the condition (ii) in Theorem A.

**Lemma 2.5.** Let  $f$  be a meromorphic function in the complex plane, and  $a \neq 0$  be a constant. If  $n \geq k + 3$ , and  $f^{(k)} - af^n \neq 0$ , then  $f \equiv \text{constant}$ .

*Proof.* By Lemma 2.4, we know  $f(z)$  is not a transcendental meromorphic function. If  $f(z)$  is a rational function. Let  $f(z) = p(z)/q(z)$ , where  $p(z), q(z)$  are two co-prime polynomials with  $\deg p(z) = p, \deg q(z) = q$ .

Then  $f^{(k)} = \left(\frac{p(z)}{q(z)}\right)^{(k)} = \frac{p_k(z)}{q_k(z)}$ , where  $p_k(z), q_k(z)$  are two co-prime polynomials, it is easily seen by induction that  $\deg p_k(z) = p_k = p, \deg q_k(z) = q_k = q + k$ , and  $f^n(z) = \frac{p^n(z)}{q^n(z)}$ , where  $\deg p^n(z) = pn, \deg q(z) = qn$ . Since

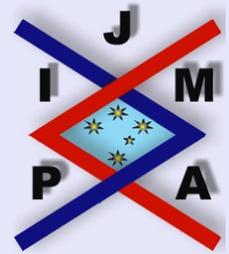
$$f^{(k)} - af^n = \frac{p_k(z)}{q_k(z)} - a\frac{p^n(z)}{q^n(z)} = \frac{p_k(z)q^n(z) - aq_k(z)p^n(z)}{q_k(z)q^n(z)},$$

and the degree of the term  $p_k(z)q^n(z) - aq_k(z)p^n(z)$  is  $\max\{p+nq, q+k+np\}$ . If  $p+nq = q+k+np$ , we have

$$n-1 = \frac{k}{q-p} \geq k+2.$$

It is impossible. Hence  $p_k(z)q^n(z) - aq_k(z)p^n(z)$  is a polynomial with degree  $= \max\{p_k+nq, q_k+np\} > 0$ , Obviously,  $f^{(k)} - af^n$  can assume zeros. It is a contradiction. Thus we have  $f \equiv \text{constant}$ .  $\square$

**Lemma 2.6.** Let  $f$  be meromorphic in the complex plane, and  $a \neq 0$  be a constant. For any positive integer  $n, k$ , satisfy  $n \geq k + 3$ . If  $f^{(k)} - af^n \equiv 0$ , then  $f \equiv \text{is the constant}$ .




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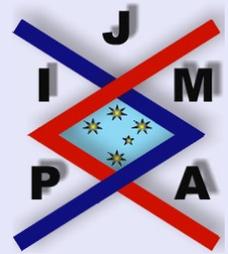


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*Proof.* If  $f$  is not the constant, by the condition we know  $f$  is an integer function. Otherwise, if  $z_0$  is the pole of  $p(\geq 1)$  order of  $f$ , then  $np = p + k$  contradicts with  $n \geq k + 3$ . With the identity  $f^n \equiv -af^{(k)}$ , or  $(f)^{n-1} \equiv \frac{1}{a} \frac{f^{(k)}}{f}$ , we can get

$$(n-1)T(r, f) = (n-1)m(r, f) \\ \leq \log^+ \frac{1}{|a|} + m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f), \quad \text{if } r \rightarrow \infty$$

and  $r \notin E$  with  $E$  being a set of  $r$  values of finite linear measure. It is impossible. This proves the lemma.  $\square$

**Lemma 2.7 ([9]).** *Let  $\mathcal{F}$  be a family of meromorphic functions on the unit disc  $\Delta$  such that all zeros of functions in  $\mathcal{F}$  have multiplicity at least  $k$ . If  $\mathcal{F}$  is not normal at a point  $z_0$ , then for  $0 \leq \alpha < k$ , there exist a sequence of functions  $f_k \in \mathcal{F}$ , a sequence of complex numbers  $z_k \rightarrow z_0$  and a sequence of positive numbers  $\rho_k \rightarrow 0$ , such that*

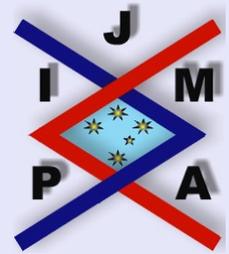
$$\rho_k^{-\alpha} g_k(z_k + \rho_k \xi) \rightarrow g(\xi)$$

*spherically uniformly on compact subsets of  $\mathcal{C}$ , where  $g$  is a nonconstant meromorphic function. Moreover,  $g$  is of order at most two, and  $g$  has only zeros of multiplicity at least  $k$ .*

**Lemma 2.8 ([2]).** *Let  $f$  be a transcendental entire function all of whose zeros have multiplicity at least  $k$ , and let  $n$  be a positive integer. Then  $f^n f^{(k)}$  takes on each nonzero value  $a \in \mathbb{C}$  infinitely often.*

**Lemma 2.9.** *Let  $f$  be a polynomial all of whose zeros have multiplicity at least  $k$ , and let  $n$  be a positive integer. Then  $f^n f^{(k)}$  can assume each nonzero value  $a \in \mathbb{C}$ .*

The proof is trivial, we omit it here.



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### 3. Proof of the Theorems

*Proof of Theorem 1.2.* We may assume that  $D = \Delta$ . Suppose that  $\mathcal{F}$  is not normal at  $z_0 \in \Delta$ . Then, taking  $\alpha = \frac{k}{n-1}$ , where  $0 < \alpha < k$ , and applying Lemma 2.7 to  $g = \{1/f : f \in F\}$ , we can find  $f_j \in F (j = 1, 2, \dots)$ ,  $z_j \rightarrow z_0$  and  $\rho_j (> 0) \rightarrow 0$  such that  $g_j(\zeta) = \rho_j^\alpha f_j(z_j + \rho_j \zeta)$ , converges locally uniformly with respect to the spherical metric to  $g(\zeta)$ , where  $g$  is a nonconstant meromorphic function on  $\mathcal{C}$ . By Lemma 2.5, there exists  $\zeta_0 \in \{|z| \leq R\}$  such that

$$(3.1) \quad g(\zeta_0)^n - a(g^{(k)}(\zeta_0)) = 0.$$

From the above equality,  $g(\zeta_0) \neq \infty$ . Through the calculation, we have

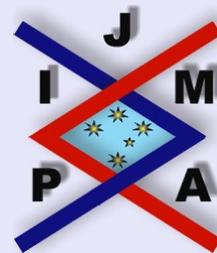
$$g_j^n(\zeta) - a(g_j^{(k)}(\zeta)) = \rho_j^{\frac{nk}{n-1}} (f_j^n(\zeta) - a(f_j^{(k)}(\zeta))) \neq 0.$$

On the other hand,

$$g_j^n(\zeta) - a(g_j^{(k)}(\zeta)) \rightarrow g^n(\zeta) - a(g^{(k)}(\zeta)).$$

By Hurwitz's theorem, we know  $g^n(\zeta) - a(g^{(k)}(\zeta))$  is either identity zero or identity non-zero. From (3.1), we know  $g^n(\zeta) - a(g^{(k)}(\zeta)) \equiv 0$ , then by Lemma 2.6 yields  $g(\zeta)$  is a constant, it is a contradiction. Hence we complete the proof of Theorem 1.2.  $\square$

*Proof of Theorem 1.3.* Let  $\alpha = \frac{k}{n-1} < k$ . If possible suppose that  $\mathcal{F}$  is not normal at  $z_0 \in \mathcal{D}$ . Then by Lemma 2.7, there exist a sequence of functions  $f_j \in$



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$F$  ( $j = 1, 2, \dots$ ), a sequence of complex numbers  $z_j \rightarrow z_0$  and  $\rho_j (> 0) \rightarrow 0$ , such that

$$g_j(\zeta) = \rho_j^{-\alpha} f_j(z_j + \rho_j \zeta)$$

converges spherically and locally uniformly to a nonconstant meromorphic function  $g(\zeta)$  in  $\mathcal{C}$ . Also the zeros of  $g(z)$  are of multiplicity at least  $\geq k$ . So  $g^{(k)} \neq 0$ . By the condition of Theorem 1.3 and Lemma 2.3, we get

$$(3.2) \quad g^{(k)}(\zeta_0) + \frac{a}{g(\zeta_0)^n} = 0$$

for some  $\zeta_0 \in \mathcal{C}$ . Clearly  $\zeta_0$  is neither a zero nor a pole of  $g$ . So in some neighborhood of  $\zeta_0$ ,  $g_j(\zeta)$  converges uniformly to  $g(\zeta)$ . Now in some neighborhood of  $\zeta_0$  we see that  $g^{(k)}(\zeta) + ag(\zeta)^{-n}$  is the uniform limit of

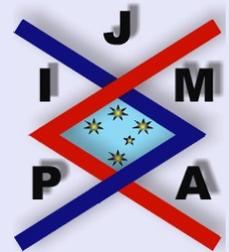
$$g^{(k)}(\zeta_0) + ag(\zeta_0)^{-n} - \rho_j^{n\alpha} b = \rho_j^{\frac{nk}{1+\alpha}} \left\{ f_j^{(k)}(z_j + \rho_j \zeta_j) + a f_j^{-n}(z_j + \rho_j \zeta_j) - b \right\}.$$

By (3.2) and Hurwitz's theorem, there exists a sequence  $\zeta_j \rightarrow \zeta_0$  such that for all large values of  $j$

$$f_j^{(k)}(z_j + \rho_j \zeta_j) + a f_j^{-n}(z_j + \rho_j \zeta_j) = b.$$

Therefore for all large values of  $j$ , it follows from the given condition  $|g_j(\zeta_j)| \geq M/\rho_j^\alpha$  and as in the last part of the proof of Theorem 1.1 in [6], we arrive at a contradiction. This proves the theorem.  $\square$

*Proof of Theorem 1.4.* In a similar manner to the proof of Theorem 1.3, we can prove the theorem by Lemma 2.7, 2.8 and Lemma 2.9.  $\square$




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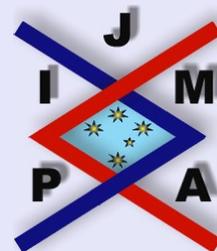
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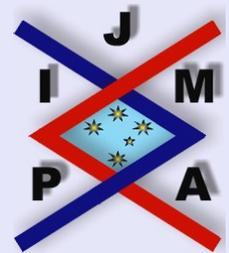
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