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NOTE ON THE NORMAL FAMILY

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Abstract

In this paper we consider the problem of normal family criteria and improve some results of I. Lihiri, S. Dewan and Y. Xu.

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Contents

1. Introduction and Results

Let $\mathbb{C}$ be the open complex plane and $\mathcal{D} \in \mathbb{C}$ be a domain. Let f be a meromorphic function in the complex plane, we assume that the reader is familiar with the notations of Nevanlinna theory (see, e.g., [5][12]).

Definition 1.1. *Let k be a positive integer, for any a in the complex plane. We denote by $N_{k)}(r, 1/(f-a))$ the counting function of a-points of f with multiplicity $\leq k$, by $N_{(k}(r, 1/(f-a))$ the counting function of a-points of f with multiplicity $\geq k$, by $N_k(r, 1/(f-a))$ the counting function of a-points of f with multiplicity of k, and denote the reduced counting function by $\overline{N}_{k)}(r, 1/(f-a))$, $\overline{N}_{(k}(r, 1/(f-a))$ and $\overline{N}_k(r, 1/(f-a))$, respectively.*

In 1995, Chen-Fang [3] proposed the following conjecture:

Conjecture 1.1. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain $\mathcal{D}$. If for every function $f \in \mathcal{F}$, $f^{(k)} - af^n - b$ has no zero in $\mathcal{D}$, then $\mathcal{F}$ is normal, where $a(\neq 0), b$ are two finite numbers and $k, n(\geq k+2)$ are positive integers.*

In response to this conjecture, Xu [11] proved the following result.

Theorem A. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain $\mathcal{D}$ and $a(\neq 0), b$ be two finite constants. If k and n are positive integers such that $n \geq k+2$ and for every $f \in \mathcal{F}$*

(i) $f^{(k)} - af^n - b$ has no zero,

(ii) f has no simple pole,

then F is normal.

The condition (ii) of Theorem A can be dropped if we choose $n \geq k+4$ (cf. [8][10]). If $n \geq k+3$, is condition (ii) in Theorem A necessary? We will give an answer.

Theorem 1.2. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain $\mathcal{D}$ and $a(\neq 0), b$ be two finite constants. If k and n are positive integers such that $n \geq k+3$ and for every $f \in \mathcal{F}$, $f^{(k)} - af^n$ has no zero, then $\mathcal{F}$ is normal.*

In addition, Lahiri and Dewan [6] investigated the situation when the power of f is negative in condition (i) of Theorem A.

Theorem B. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain $\mathcal{D}$ and $a(\neq 0), b$ be two finite constants. Suppose that $E_f = \{z : z \in D$ and $f^{(k)}(z) - af^{-n}(z) = b\}$, where k and $n(\geq k)$ are positive integers.*

If for every $f \in F$

(i) f has no zero of multiplicity less than k,

(ii) there exists a positive number M such that for every $f \in F, |f(z)| \geq M$ whenever $z \in E_f$, then F is normal.

I. Lahiri gave two examples to show that conditions (i) and (ii) are necessary. Naturally, we can question whether $n \geq k$ is necessary, first we note the following example.

Example 1.1. *Let $\mathbb{D} : |z| < 1$ and $\mathcal{F} = \{f_n\}$, where*

$$f_p(z) = \frac{z^3}{p}, \quad p = 2, 3, \dots,$$

and $n = 2, k = 3, a = 1, b = 0$. *Then* f_p *has the zeros of multiplicity 3 and* $E_{f_p} = \{z : z \in \mathbb{D}$ *and* $6z^6 - p^3 = 0\}$. *For any* $z \in E_f$, $|f_p(z)| = \sqrt{\frac{p}{6}} \to \infty$, *as* $p \to \infty$. *But*

$$|f_p^\sharp(z)| = \left|\frac{3pz^2}{p^2 + z^6}\right| < \left|\frac{3pz^2}{p^2 - |z|^6}\right| < \frac{3p}{p^2 - 1} < \frac{3}{p-1} \leq 3,$$

for any p. *By Marty's criterion, the family* $\{f_p\}$ *is normal.*

Hence we can give some answers. In fact, we can prove the following theorem:

Theorem 1.3. *Let* $\mathcal{F}$ *be a family of meromorphic functions in a domain* $\mathcal{D}$ *and* $a(\neq 0), b$ *be two finite constants. Suppose that* $E_f = \{z : z \in \mathbb{D}$ *and* $f^{(k)} - af^{-n} = b\}$,*where* k *and* n *are positive integers,*

If for every $f \in \mathcal{F}$

(i) f *has the zero of multiplicity at least* k,

(ii) there exists a positive number M *such that for every* $f \in \mathcal{F}, |f(z)| \geq M$ *whenever* $z \in E_f$.

Then $\mathcal{F}$ *is normal in* $\mathcal{D}$ *so long as*

(A) $n \geq 2$; *or*

(B) $n = 1$ *and* $\overline{N}_k\left(r, \frac{1}{f}\right) = S(r, f)$.

Especially, if $f(z)$ *is an entire function, we can obtain the complete answer.*

Theorem 1.4. *Let $\mathcal{F}$ be a family of entire functions in a domain $\mathcal{D}$ and $a(\neq 0), b$ be two finite constants. Suppose that $E_f = \{z : z \in \mathbb{D}$ and $f^{(k)} - af^{-n} = b\}$, where k and n are positive integers,*

If for every $f \in \mathcal{F}$

(i) f has no zero of multiplicity less than k,

(ii) there exists a positive number M such that for every $f \in \mathcal{F}, |f(z)| \geq M$ whenever $z \in E_f$, then $\mathcal{F}$ is normal.

2. Preliminaries

Lemma 2.1. *[13] Let f be nonconstant meromorphic in the complex plane, $L[f] = a_k f^{(k)} + a_{k-1} f^{(k-1)} + \cdots + a_0 f$, where $a_0, a_1, \ldots, a_k$ are small functions, for $a \neq 0, \infty$, let $F = f^n L[f] - a$, where $n \geq 2$ is a positive integer. Then*

$$\limsup_{r\to+\infty} \frac{\overline{N}(r, a; F)}{T(r, F)} > 0\,.$$

Lemma 2.2. *Let f be nonconstant meromorphic in the complex plane, $L[f]$ is given as in Lemma 2.1 and $F = fL[f] - a$. Then*

$$T(r, f) \leq \left(6 + \frac{6}{k}\right)\left(\overline{N}\left(r, \frac{1}{F}\right) + \overline{N}_{k)}\left(r, \frac{1}{f}\right)\right) + S(r, f).$$

Proof. For the simplification, we prove the case of $L[f] = f^{(k)}$, the general case is similar. Without loss of generality, let $a = 1$, then

$$F = fL[f] - 1. \tag{2.1}$$

By differentiating the equation (2.1), we get

$$f\beta = -\frac{F'}{F}, \tag{2.2}$$

where

$$\beta = \frac{f'}{f} f^{(k)} + f^{(k+1)} - f^{(k)} \frac{F'}{F}. \tag{2.3}$$

Obviously $F \not\equiv$ constant, $\beta \not\equiv 0$. By the Clunie Lemma ([1] or [4])

$$m(r, \beta) = S(r, f). \tag{2.4}$$

Let z_0 be a pole of f of order q. Then z_0 is the simple pole of $\frac{F'}{F}$, and the poles of f of order $q(\geq 2)$ are the zeros of β of order $q-1$ from (2.2), the simple pole of f is the non-zero analytic point of β, therefore

$$N_{(2}(r, f) \leq N\left(r, \frac{1}{\beta}\right) + \overline{N}\left(r, \frac{1}{\beta}\right) \leq 2N\left(r, \frac{1}{\beta}\right). \tag{2.5}$$

By (2.3), we know the zeros of f of order $q > k$ are not the poles of β. From (2.3), we get

$$N(r, \beta) \leq \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}_{k)}\left(r, \frac{1}{f}\right) + S(r, f). \tag{2.6}$$

Then, by (2.4) and (2.6), we have

$$T(r, \beta) \leq \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}_{k)}\left(r, \frac{1}{f}\right) + S(r, f). \tag{2.7}$$

Next with (2.5) and (2.7), we obtain

$$N_{(2}(r, f) \leq 2\overline{N}\left(r, \frac{1}{F}\right) + 2\overline{N}_{k)}\left(r, \frac{1}{f}\right) + S(r, f). \tag{2.8}$$

Noting by (2.2), (2.7) and the first fundamental theorem, we obtain

$$m(r,f) \le m\left(r,\frac{1}{\beta}\right) + m\left(r,\frac{F'}{F}\right) \le T(r,\beta) - N\left(r,\frac{1}{\beta}\right) + S(r,f) = \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}_{k)}\left(r,\frac{1}{f}\right) + S(r,f). \tag{2.9}$$

If f only have finitely many simple poles, we get Lemma 2.2 by (2.5) and (2.9).

Next we discuss that f have infinity simple poles. Let z_0 be any simple pole of f. Then z_0 is the non-zero analytic point of β. In a neighborhood of z_0, we have

$$f(z) = \frac{d_1(z_0)}{z - z_0} + d_0(z_0) + O(z - z_0) \tag{2.10}$$

and

$$\beta(z) = \beta(z_0) + \beta'(z_0)(z - z_0) + O((z - z_0)^2), \tag{2.11}$$

where $d_1(z_0) \neq 0, \beta(z_0) \neq 0$. By differentiating (2.10), we get

$$f^{(j)}(z) = (-1)^j \frac{j!d_1(z_0)}{(z - z_0)^{j+1}} + \cdots, \qquad j = 1, 2, \ldots, k. \tag{2.12}$$

with (2.3) and (2.5) we have

$$f\beta = f'f^{(k)} + ff^{(k+1)} - f^2f^{(k)}\beta. \tag{2.13}$$

Substituting (2.10)-(2.12) into (2.13), we obtain that the coefficients have the forms

$$d_1(z_0) = \frac{k+2}{\beta(z_0)}, \tag{2.14}$$

$$d_0(z_0) = -\frac{(k+2)^2}{k+3}\frac{\beta'(z_0)}{(\beta(z_0))^2}, \tag{2.15}$$

so that

$$\frac{d_0(z_0)}{d_1(z_0)} = -\frac{k+2}{k+3}\frac{\beta'(z_0)}{\beta(z_0)}. \tag{2.16}$$

Through the calculating from (2.10) and (2.12), we get

$$\frac{f'(z)}{f(z)} = -\frac{1}{z-z_0} + \frac{d_0(z_0)}{d_1(z_0)} + O(z-z_0), \tag{2.17}$$

$$\frac{F'(z)}{F(z)} = -\frac{k+2}{z-z_0} + \frac{d_0(z_0)}{d_1(z_0)} + O(z-z_0). \tag{2.18}$$

Let

$$h(z) = \frac{F'(z)}{F(z)} - (k+2)\frac{f'(z)}{f(z)} - \frac{(k+1)(k+2)}{(k+3)}\frac{\beta'(z)}{\beta(z)}. \tag{2.19}$$

Then, by (2.17)-(2.19), clearly, $h(z_0) = 0$. Therefore the simple pole of f is the zero of $h(z)$. From (2.19), we have

$$m(r, h) = S(r, f). \tag{2.20}$$

If f only has finitely many zeros. By (2.3) and the lemma of logarithmic derivatives, we get

$$\begin{aligned} m\left(\left(r, \frac{1}{f}\right) \right. &\leq m\left(r, \frac{1}{\beta}\left(\frac{f'}{f}\frac{f^{(k)}}{f} + \frac{f^{(k+1)}}{f} - \frac{f^{(k)}}{f}\frac{F'}{F}\right)\right) \\ &\leq m\left(r, \frac{1}{\beta}\right) + S(r, f). \end{aligned}$$

It follows by (2.4) and (2.5) that

$$m\left(r, \frac{1}{f}\right) \leq \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}_{k)}\left(r, \frac{1}{f}\right) + S(r, f).$$

Using Nevanlinna's first fundamental theorem and f only has finitely many zeros, we obtain

$$\begin{aligned} T(r, f) &= T\left(r, \frac{1}{f}\right) + O(1) \\ &= m\left(r, \frac{1}{f}\right) + S(r, f) \\ &\leq \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}_{k)}\left(r, \frac{1}{f}\right) + S(r, f). \end{aligned}$$

Hence the conclusion of Lemma 2.2 holds.

If f only has infinitely many zeros. We assert that $h(z) \not\equiv 0$. Otherwise $h(z) \equiv 0$, then

$$\frac{F'}{F} = (k+2)\frac{f'}{f} + \frac{(k+1)(k+2)}{(k+3)}\frac{\beta'(z_0)}{\beta(z_0)}.$$

By integrating, we have

$$F^{(k+3)} = cf^{(k+2)(k+3)}\beta^{(k+1)(k+2)}, \tag{2.21}$$

where $c \neq 0$ is a constant. Any zeros of f of order q are not the zeros and poles of F by (2.1), and any zeros of f must be the poles of β by (2.21). Suppose that $q > k$, (otherwise, the conclusion of Lemma 2.2 holds by above) it contradicts (2.6), hence $h(z) \not\equiv 0$.

Since $h(z) \not\equiv 0$, and the simple pole of f is the zeros of h, we know the poles of $h(z)$ occur only at the zeros of F, the zeros of f, the multiple poles of f, the zeros and poles of β, all are the simple pole of $h(z)$. At the same time, we note $F' = f'f^{(k)} + ff^{(k+1)}$, hence the zeros of f of the order of $q(\geq k+2)$ at least are the zeros of F' of $2q-(k+1)$, and also at least are the zeros of β of order $q-(k+1)$ by (2.2), hence,

$$\begin{aligned}
\overline{N}_{(k+2}\left(r, \frac{1}{f}\right) &\leq \frac{1}{k+2} N_{(k+2}\left(r, \frac{1}{f}\right) \\
&\leq \frac{1}{k+2}\left(N\left(r, \frac{1}{\beta}\right) + (k+1)\overline{N}\left(r, \frac{1}{\beta}\right)\right) \\
&\leq N\left(r, \frac{1}{\beta}\right).
\end{aligned}$$

It follows from (2.8),(2.12) and (2.19), we have

$$(2.22)\quad \begin{aligned} N(r,h) &\le \overline{N}\left(r,\frac{1}{F}\right)+\overline{N}_{k+1)}\left(r,\frac{1}{f}\right)+\overline{N}(r,\beta)+N\left(r,\frac{1}{\beta}\right)\\ &\le \overline{N}\left(r,\frac{1}{F}\right)+\overline{N}_{k)}\left(r,\frac{1}{f}\right)+\overline{N}_{k+1}\left(r,\frac{1}{f}\right)+2T(r,\beta)+S(r,f)\\ &\le 3\left(\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}_{k)}\left(r,\frac{1}{f}\right)\right)+\overline{N}_{k+1}\left(r,\frac{1}{f}\right)+S(r,f). \end{aligned}$$

Using (2.20), we get

$$(2.23)\quad \begin{aligned} N_{1)}(r,f) &\le N\left(r,\frac{1}{h}\right)\le N(r,h)+S(r,f)\\ &\le 3\left(\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}_{k)}\left(r,\frac{1}{f}\right)\right)+\overline{N}_{k+1}\left(r,\frac{1}{f}\right)+S(r,f). \end{aligned}$$

Note

$$\overline{N}_{k+1}\left(r,\frac{1}{f}\right)=\frac{1}{k+1}N_{k+1}\left(r,\frac{1}{f}\right)\le\frac{1}{k+1}T(r,f)+S(r,f).$$

By (2.14),(2.16) and (2.23), we deduce

$$T(r,f)\le\left(6+\frac{6}{k}\right)\left(\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}_{k)}\left(r,\frac{1}{f}\right)\right)+S(r,f).$$

□

Lemma 2.3. *Let f be a nonconstant meromorphic function in the complex plane such that the zeros of $f(z)$ are of multiplicity at least $\geq k$ and $a(\neq 0)$ be a finite constant. Then*

(i) *If $n \geq 2$, $f^{(k)} - af^{-n}$ must have some zero, where k and n are positive integers.*

(ii) *If $n = 1$, and $\overline{N}_k\left(r, \frac{1}{f}\right) = S(r, f)$, $f^{(k)} - af^{-n}$ must have some zero, where k is a positive integer.*

Proof. First we assume that $n \geq 2$, by Lemma 2.1, we know $f^n f^{(k)} - a$ must have some zero. Since a zero of $f^n f^{(k)} - a$ is a zero of $f^{(k)} - af^{-n}$, then $f^{(k)} - af^{-n}$ must have some zero.

If $n = 1$, the zeros of $f(z)$ are of multiplicity at least $\geq k$, so $\overline{N}_{k-1)}\left(r, \frac{1}{f}\right) = S(r, f)$. With the condition of $\overline{N}_k\left(r, \frac{1}{f}\right) = S(r, f)$, we have $\overline{N}_{k)}\left(r, \frac{1}{f}\right) = S(r, f)$. By Lemma 2.2, we know $ff^{(k)} - a$ must have some zero. As the preceding paragraph a zero of $ff^{(k)} - a$ is a zero of $f^{(k)} - af^{-1}$, the lemma is proved. □

Lemma 2.4 ([13]). *Let $f(z)$ be a transcendental meromorphic function in the complex plane, and $a \neq 0$ be a constant. If $n \geq k + 3$, then $f^{(k)} - af^n$ assumes zeros infinitely often.*

Remark 1. *In fact, E. Mues's [7, Theorem 1(b)] gave a counterexample to show that $f' - f^4 = c$ has no solution. We know $f^{(k)} - af^n$ cannot assume non-zero values for any positive integer n, k and $n = k + 3$. Hence Theorem 1.2 may be best when we drop the condition (ii) in Theorem A.*

Lemma 2.5. *Let f be a meromorphic function in the complex plane, and $a \neq 0$ be a constant. If $n \geq k+3$, and $f^{(k)} - af^n \neq 0$, then $f \equiv$ constant.*

Proof. By Lemma 2.4, we know $f(z)$ is not a transcendental meromorphic function. If $f(z)$ is a rational function. Let $f(z) = p(z)/q(z)$, where $p(z), q(z)$ are two co-prime polynomials with $\deg p(z) = p, \deg q(z) = q$.

Then $f^{(k)} = \left(\frac{p(z)}{q(z)}\right)^{(k)} = \frac{p_k(z)}{q_k(z)}$, where $p_k(z), q_k(z)$ are two co-prime polynomials, it is easily seen by induction that $\deg p_k(z) = p_k = p$, $\deg q_k(z) = q_k = q+k$, and $f^n(z) = \frac{p^n(z)}{q^n(z)}$, where $\deg p^n(z) = pn$, $\deg q(z) = qn$. Since

$$f^{(k)} - af^n = \frac{p_k(z)}{q_k(z)} - a\frac{p^n(z)}{q^n(z)} = \frac{p_k(z)q^n(z) - aq_k(z)p^n(z)}{q_k(z)q^n(z)},$$

and the degree of the term $p_k(z)q^n(z) - aq_k(z)p^n(z)$ is $\max\{p+nq, q+k+np\}$. If $p+nq = q+k+np$, we have

$$n - 1 = \frac{k}{q-p} \geq k+2.$$

It is impossible. Hence $p_k(z)q^n(z) - aq_k(z)p^n(z)$ is a polynomial with degree $= \max\{p_k + nq, q_k + np\} > 0$, Obviously, $f^{(k)} - af^n$ can assume zeros. It is a contradiction. Thus we have $f \equiv$ constant. □

Lemma 2.6. *Let f be meromorphic in the complex plane, and $a \neq 0$ be a constant. For any positive integer n, k, satisfy $n \geq k+3$. If $f^{(k)} - af^n \equiv 0$, then $f \equiv$ is the constant.*

Proof. If f is not the constant, by the condition we know f is an integer function. Otherwise, if z_0 is the pole of $p(\geq 1)$ order of f, then $np = p + k$ contradicts with $n \geq k+3$. With the identity $f^n \equiv -af^{(k)}$, or $(f)^{n-1} \equiv \frac{1}{a}\frac{f^{(k)}}{f}$, we can get

$$\begin{aligned}(n-1)T(r,f) &= (n-1)m(r,f)\\ &\leq \log^+ \frac{1}{|a|} + m\left(r, \frac{f^{(k)}}{f}\right) = S(r,f), \quad \text{if} \quad r \to \infty\end{aligned}$$

and $r \notin E$ with E being a set of r values of finite linear measure. It is impossible. This proves the lemma. □

Lemma 2.7 ([9]). *Let $\mathcal{F}$ be a family of meromorphic functions on the unit disc $\triangle$ such that all zeros of functions in $\mathcal{F}$ have multiplicity at least k. If $\mathcal{F}$ is not normal at a point z_0, then for $0 \leq \alpha < k$, there exist a sequence of functions $f_k \in \mathcal{F}$, a sequence of complex numbers $z_k \to z_0$ and a sequence of positive numbers $\rho_k \to 0$, such that*

$$\rho_k^{-\alpha} g_k(z_k + \rho_k \xi) \to g(\xi)$$

spherically uniformly on compact subsets of $\mathcal{C}$, where g is a nonconstant meromorphic function. Moreover, g is of order at most two, and g has only zeros of multiplicity at least k.

Lemma 2.8 ([2]). *Let f be a transcendental entire function all of whose zeros have multiplicity at least k, and let n be a positive integer. Then $f^n f^{(k)}$ takes on each nonzero value $a \in \mathbb{C}$ infinitely often.*

Lemma 2.9. *Let f be a polynomial all of whose zeros have multiplicity at least k, and let n be a positive integer. Then $f^n f^{(k)}$ can assume each nonzero value $a \in \mathbb{C}$.*

The proof is trivial, we omit it here.

3. Proof of the Theorems

Proof of Theorem 1.2. We may assume that $D = \triangle$. Suppose that $\mathcal{F}$ is not normal at $z_0 \in \triangle$. Then, taking $\alpha = \frac{k}{n-1}$, where $0 < \alpha < k$, and applying Lemma 2.7 to $g = \{1/f : f \in F\}$, we can find $f_j \in F (j = 1, 2, \dots)$, $z_j \to z_0$ and $\rho_j (> 0) \to 0$ such that $g_j(\zeta) = \rho_j^{\alpha} f_j(z_j + \rho_j \zeta)$, converges locally uniformly with respect to the spherical metric to $g(\zeta)$, where g is a nonconstant meromorphic function on $\mathcal{C}$. By Lemma 2.5, there exists $\zeta_0 \in \{|z| \leq R\}$ such that

$$g(\zeta_0)^n - a(g^{(k)}(\zeta_0)) = 0. \tag{3.1}$$

From the above equality, $g(\zeta_0) \neq \infty$. Through the calculation, we have

$$g_j^n(\zeta) - a(g_j^{(k)}(\zeta)) = \rho_j^{\frac{nk}{n-1}}(f_j^n(\zeta) - a(f_j^{(k)}(\zeta))) \neq 0.$$

On the other hand,

$$g_j^n(\zeta) - a(g_j^{(k)}(\zeta)) \to g^n(\zeta) - a(g^{(k)}(\zeta)).$$

By Hurwitz's theorem, we know $g^n(\zeta) - a(g^{(k)}(\zeta))$ is either identity zero or identity non-zero. From (3.1), we know $g^n(\zeta) - a(g^{(k)}(\zeta)) \equiv 0$, then by Lemma 2.6 yields $g(\zeta)$ is a constant, it is a contradiction. Hence we complete the proof of Theorem 1.2. □

Proof of Theorem 1.3. Let $\alpha = \frac{k}{n-1} < k$. If possible suppose that $\mathcal{F}$ is not normal at $z_0 \in \mathcal{D}$. Then by Lemma 2.7, there exist a sequence of functions $f_j \in$

F $(j = 1, 2, \dots)$, a sequence of complex numbers $z_j \to z_0$ and $\rho_j(> 0) \to 0$, such that

$$g_j(\zeta) = \rho_j^{-\alpha} f_j(z_j + \rho_j \zeta)$$

converges spherically and locally uniformly to a nonconstant meromorphic function $g(\zeta)$ in $\mathcal{C}$. Also the zeros of $g(z)$ are of multiplicity at least $\geq k$. So $g^{(k)} \not\equiv 0$. By the condition of Theorem 1.3 and Lemma 2.3, we get

$$g^{(k)}(\zeta_0) + \frac{a}{g(\zeta_0)^n} = 0 \tag{3.2}$$

for some $\zeta_0 \in \mathcal{C}$. Clearly ζ_0 is neither a zero nor a pole of g. So in some neighborhood of ζ_0, $g_j(\zeta)$ converges uniformly to $g(\zeta)$. Now in some neighborhood of ζ_0 we see that $g^{(k)}(\zeta) + ag(\zeta)^{-n}$ is the uniform limit of

$$g^{(k)}(\zeta_0) + ag(\zeta_0)^{-n} - \rho_j^{n\alpha} b = \rho_j^{\frac{nk}{1+n}} \left\{ f_j^{(k)}(z_j + \rho_j \zeta_j) + af_j^{-n}(z_j + \rho_j \zeta_j) - b \right\}.$$

By (3.2) and Hurwitz's theorem, there exists a sequence $\zeta_j \to \zeta_0$ such that for all large values of j

$$f_j^{(k)}(z_j + \rho_j \zeta_j) + af_j^{-n}(z_j + \rho_j \zeta_j) = b.$$

Therefore for all large values of j, it follows from the given condition $|g_j(\zeta_j)| \geq M/\rho_j^{\alpha}$ and as in the last part of the proof of Theorem 1.1 in [6], we arrive at a contradiction. This proves the theorem. □

Proof of Theorem 1.4. In a similar manner to the proof of Theorem 1.3, we can prove the theorem by Lemma 2.7, 2.8 and Lemma 2.9. □

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