Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 7, Issue 4, Article 133, 2006

NOTE ON THE NORMAL FAMILY

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Received 26 March, 2006; accepted 19 June, 2006
Communicated by H.M. Srivastava

ABSTRACT. In this paper we consider the problem of normal family criteria and improve some results of I. Lihiri, S. Dewan and Y. Xu.

Key words and phrases: Normal family, Meromorphic function.

2000 *Mathematics Subject Classification.* 30D35.

1. INTRODUCTION AND RESULTS

Let $\mathbb{C}$ be the open complex plane and $\mathcal{D} \in \mathbb{C}$ be a domain. Let f be a meromorphic function in the complex plane, we assume that the reader is familiar with the notations of Nevanlinna theory (see, e.g., [5][12]).

Definition 1.1. Let k be a positive integer, for any a in the complex plane. We denote by $N_{k)}(r, 1/(f-a))$ the counting function of a-points of f with multiplicity$\leq k$, by $N_{(k}(r, 1/(f-a))$ the counting function of a-points of f with multiplicity$\geq k$, by $N_k(r, 1/(f-a))$ the counting function of a-points of f with multiplicity of k, and denote the reduced counting function by $\overline{N}_{k)}(r, 1/(f-a))$, $\overline{N}_{(k}(r, 1/(f-a))$ and $\overline{N}_k(r, 1/(f-a))$, respectively.

In 1995, Chen-Fang [3] proposed the following conjecture:

Conjecture 1.1. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain $\mathcal{D}$. If for every function $f \in \mathcal{F}$, $f^{(k)} - af^n - b$ has no zero in $\mathcal{D}$, then $\mathcal{F}$ is normal, where $a(\neq 0), b$ are two finite numbers and $k, n(\geq k+2)$ are positive integers.*

In response to this conjecture, Xu [11] proved the following result.

ISSN (electronic): 1443-5756

091-06

Theorem A. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain $\mathcal{D}$ and $a(\neq 0), b$ be two finite constants. If k and n are positive integers such that $n \geq k+2$ and for every $f \in \mathcal{F}$*

(i) *$f^{(k)} - af^n - b$ has no zero,*
(ii) *f has no simple pole,*

then F is normal.

The condition (ii) of Theorem A can be dropped if we choose $n \geq k+4$ (cf. [8][10]). If $n \geq k+3$, is condition (ii) in Theorem A necessary? We will give an answer.

Theorem 1.2. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain $\mathcal{D}$ and $a(\neq 0), b$ be two finite constants. If k and n are positive integers such that $n \geq k+3$ and for every $f \in \mathcal{F}$, $f^{(k)} - af^n$ has no zero, then $\mathcal{F}$ is normal.*

In addition, Lahiri and Dewan [6] investigated the situation when the power of f is negative in condition (i) of Theorem A.

Theorem B. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain $\mathcal{D}$ and $a(\neq 0), b$ be two finite constants. Suppose that $E_f = \{z : z \in D$ and $f^{(k)}(z) - af^{-n}(z) = b\}$, where k and $n(\geq k)$ are positive integers.*

If for every $f \in F$

(i) *f has no zero of multiplicity less than k,*
(ii) *there exists a positive number M such that for every $f \in F, |f(z)| \geq M$ whenever $z \in E_f$, then F is normal.*

I. Lahiri gave two examples to show that conditions (i) and (ii) are necessary. Naturally, we can question whether $n \geq k$ is necessary, first we note the following example.

Example 1.1. Let $\mathbb{D} : |z| < 1$ and $\mathcal{F} = \{f_n\}$, where

$$f_p(z) = \frac{z^3}{p}, \quad p = 2, 3, \dots,$$

and $n = 2, k = 3, a = 1, b = 0$. Then f_p has the zeros of multiplicity 3 and $E_{f_p} = \{z : z \in \mathbb{D} \quad \text{and} \quad 6z^6 - p^3 = 0\}$. For any $z \in E_f$, $|f_p(z)| = \sqrt{\frac{p}{6}} \to \infty$, as $p \to \infty$. But

$$|f_p^{\sharp}(z)| = \left|\frac{3pz^2}{p^2 + z^6}\right| < \left|\frac{3pz^2}{p^2 - |z|^6}\right| < \frac{3p}{p^2 - 1} < \frac{3}{p-1} \leq 3,$$

for any p. By Marty's criterion, the family $\{f_p\}$ is normal.

Hence we can give some answers. In fact, we can prove the following theorem:

Theorem 1.3. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain $\mathcal{D}$ and $a(\neq 0), b$ be two finite constants. Suppose that $E_f = \{z : z \in \mathbb{D}$ and $f^{(k)} - af^{-n} = b\}$,where k and n are positive integers,*

If for every $f \in \mathcal{F}$

(i) *f has the zero of multiplicity at least k,*
(ii) *there exists a positive number M such that for every $f \in \mathcal{F}, |f(z)| \geq M$ whenever $z \in E_f$.*

Then $\mathcal{F}$ is normal in $\mathcal{D}$ so long as

(A) *$n \geq 2$; or*
(B) *$n = 1$ and $\overline{N}_k\left(r, \frac{1}{f}\right) = S(r, f)$.*

Especially, if $f(z)$ is an entire function, we can obtain the complete answer.

Theorem 1.4. *Let $\mathcal{F}$ be a family of entire functions in a domain $\mathcal{D}$ and $a(\neq 0), b$ be two finite constants. Suppose that $E_f = \{z : z \in \mathbb{D}$ and $f^{(k)} - af^{-n} = b\}$, where k and n are positive integers,*

If for every $f \in \mathcal{F}$

(i) *f has no zero of multiplicity less than k,*
(ii) *there exists a positive number M such that for every $f \in \mathcal{F}, |f(z)| \geq M$ whenever $z \in E_f$, then $\mathcal{F}$ is normal.*

2. PRELIMINARIES

Lemma 2.1. [13] *Let f be nonconstant meromorphic in the complex plane, $L[f] = a_k f^{(k)} + a_{k-1} f^{(k-1)} + \cdots + a_0 f$, where $a_0, a_1, \ldots, a_k$ are small functions, for $a \neq 0, \infty$, let $F = f^n L[f] - a$, where $n \geq 2$ is a positive integer. Then*

$$\limsup_{r \to +\infty} \frac{\overline{N}(r, a; F)}{T(r, F)} > 0 \, .$$

Lemma 2.2. *Let f be nonconstant meromorphic in the complex plane, $L[f]$ is given as in Lemma 2.1 and $F = fL[f] - a$. Then*

$$T(r, f) \leq \left(6 + \frac{6}{k}\right) \left(\overline{N}\left(r, \frac{1}{F}\right) + \overline{N}_{k)}\left(r, \frac{1}{f}\right)\right) + S(r, f).$$

Proof. For the simplification, we prove the case of $L[f] = f^{(k)}$, the general case is similar. Without loss of generality, let $a = 1$, then

$$F = fL[f] - 1. \tag{2.1}$$

By differentiating the equation (2.1), we get

$$f\beta = -\frac{F'}{F}, \tag{2.2}$$

where

$$\beta = \frac{f'}{f} f^{(k)} + f^{(k+1)} - f^{(k)} \frac{F'}{F}. \tag{2.3}$$

Obviously $F \not\equiv$ constant, $\beta \not\equiv 0$. By the Clunie Lemma ([1] or [4])

$$m(r, \beta) = S(r, f). \tag{2.4}$$

Let z_0 be a pole of f of order q. Then z_0 is the simple pole of $\frac{F'}{F}$, and the poles of f of order $q(\geq 2)$ are the zeros of β of order $q - 1$ from (2.2), the simple pole of f is the non-zero analytic point of β, therefore

$$N_{(2}(r, f) \leq N\left(r, \frac{1}{\beta}\right) + \overline{N}\left(r, \frac{1}{\beta}\right) \leq 2N\left(r, \frac{1}{\beta}\right). \tag{2.5}$$

By (2.3), we know the zeros of f of order $q > k$ are not the poles of β. From (2.3), we get

$$N(r, \beta) \leq \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}_{k)}\left(r, \frac{1}{f}\right) + S(r, f). \tag{2.6}$$

Then, by (2.4) and (2.6), we have

$$T(r, \beta) \leq \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}_{k)}\left(r, \frac{1}{f}\right) + S(r, f). \tag{2.7}$$

Next with (2.5) and (2.7), we obtain

$$N_{(2}(r,f) \le 2\overline{N}\left(r,\frac{1}{F}\right) + 2\overline{N}_{k)}\left(r,\frac{1}{f}\right) + S(r,f). \tag{2.8}$$

Noting by (2.2), (2.7) and the first fundamental theorem, we obtain

$$\begin{aligned} m(r,f) &\le m\left(r,\frac{1}{\beta}\right) + m\left(r,\frac{F'}{F}\right) \\ &\le T(r,\beta) - N\left(r,\frac{1}{\beta}\right) + S(r,f) \\ &= \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}_{k)}\left(r,\frac{1}{f}\right) + S(r,f). \end{aligned} \tag{2.9}$$

If f only have finitely many simple poles, we get Lemma 2.2 by (2.5) and (2.9).

Next we discuss that f have infinity simple poles. Let z_0 be any simple pole of f. Then z_0 is the non-zero analytic point of β. In a neighborhood of z_0, we have

$$f(z) = \frac{d_1(z_0)}{z-z_0} + d_0(z_0) + O(z-z_0) \tag{2.10}$$

and

$$\beta(z) = \beta(z_0) + \beta'(z_0)(z-z_0) + O((z-z_0)^2), \tag{2.11}$$

where $d_1(z_0) \ne 0, \beta(z_0) \ne 0$. By differentiating (2.10), we get

$$f^{(j)}(z) = (-1)^j \frac{j!d_1(z_0)}{(z-z_0)^{j+1}} + \cdots, \qquad j = 1,2,\ldots,k. \tag{2.12}$$

with (2.3) and (2.5) we have

$$f\beta = f'f^{(k)} + ff^{(k+1)} - f^2 f^{(k)}\beta. \tag{2.13}$$

Substituting (2.10)-(2.12) into (2.13), we obtain that the coefficients have the forms

$$d_1(z_0) = \frac{k+2}{\beta(z_0)}, \tag{2.14}$$

$$d_0(z_0) = -\frac{(k+2)^2}{k+3}\frac{\beta'(z_0)}{(\beta(z_0))^2}, \tag{2.15}$$

so that

$$\frac{d_0(z_0)}{d_1(z_0)} = -\frac{k+2}{k+3}\frac{\beta'(z_0)}{\beta(z_0)}. \tag{2.16}$$

Through the calculating from (2.10) and (2.12), we get

$$\frac{f'(z)}{f(z)} = -\frac{1}{z-z_0} + \frac{d_0(z_0)}{d_1(z_0)} + O(z-z_0), \tag{2.17}$$

$$\frac{F'(z)}{F(z)} = -\frac{k+2}{z-z_0} + \frac{d_0(z_0)}{d_1(z_0)} + O(z-z_0). \tag{2.18}$$

Let

$$h(z) = \frac{F'(z)}{F(z)} - (k+2)\frac{f'(z)}{f(z)} - \frac{(k+1)(k+2)}{(k+3)}\frac{\beta'(z)}{\beta(z)}. \tag{2.19}$$

Then, by (2.17)-(2.19), clearly, $h(z_0) = 0$. Therefore the simple pole of f is the zero of $h(z)$. From (2.19), we have

$$m(r, h) = S(r, f). \tag{2.20}$$

If f only has finitely many zeros. By (2.3) and the lemma of logarithmic derivatives, we get

$$\begin{aligned} m(\left(r, \frac{1}{f}\right) &\leq m\left(r, \frac{1}{\beta}\left(\frac{f'}{f}\frac{f^{(k)}}{f} + \frac{f^{(k+1)}}{f} - \frac{f^{(k)}}{f}\frac{F'}{F}\right)\right) \\ &\leq m\left(r, \frac{1}{\beta}\right) + S(r, f). \end{aligned}$$

It follows by (2.4) and (2.5) that

$$m\left(r, \frac{1}{f}\right) \leq \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}_{k)}\left(r, \frac{1}{f}\right) + S(r, f).$$

Using Nevanlinna's first fundamental theorem and f only has finitely many zeros, we obtain

$$\begin{aligned} T(r, f) &= T\left(r, \frac{1}{f}\right) + O(1) \\ &= m\left(r, \frac{1}{f}\right) + S(r, f) \\ &\leq \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}_{k)}\left(r, \frac{1}{f}\right) + S(r, f). \end{aligned}$$

Hence the conclusion of Lemma 2.2 holds.

If f only has infinitely many zeros. We assert that $h(z) \not\equiv 0$. Otherwise $h(z) \equiv 0$, then

$$\frac{F'}{F} = (k+2)\frac{f'}{f} + \frac{(k+1)(k+2)}{(k+3)}\frac{\beta'(z_0)}{\beta(z_0)}.$$

By integrating, we have

$$F^{(k+3)} = cf^{(k+2)(k+3)}\beta^{(k+1)(k+2)}, \tag{2.21}$$

where $c \neq 0$ is a constant. Any zeros of f of order q are not the zeros and poles of F by (2.1), and any zeros of f must be the poles of β by (2.21). Suppose that $q > k$, (otherwise, the conclusion of Lemma 2.2 holds by above) it contradicts (2.6), hence $h(z) \not\equiv 0$.

Since $h(z) \not\equiv 0$, and the simple pole of f is the zeros of h, we know the poles of $h(z)$ occur only at the zeros of F, the zeros of f, the multiple poles of f, the zeros and poles of β, all are the simple pole of $h(z)$. At the same time, we note $F' = f'f^{(k)} + ff^{(k+1)}$, hence the zeros of f of the order of $q(\geq k+2)$ at least are the zeros of F' of $2q - (k+1)$, and also at least are the zeros of β of order $q - (k+1)$ by (2.2), hence,

$$\begin{aligned} \overline{N}_{(k+2}\left(r, \frac{1}{f}\right) &\leq \frac{1}{k+2}N_{(k+2}\left(r, \frac{1}{f}\right) \\ &\leq \frac{1}{k+2}\left(N\left(r, \frac{1}{\beta}\right) + (k+1)\overline{N}\left(r, \frac{1}{\beta}\right)\right) \\ &\leq N\left(r, \frac{1}{\beta}\right). \end{aligned}$$

It follows from (2.8),(2.12) and (2.19), we have

$$N(r,h) \le \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}_{k+1)}\left(r,\frac{1}{f}\right) + \overline{N}(r,\beta) + N\left(r,\frac{1}{\beta}\right) \tag{2.22}$$
$$\le \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}_{k)}\left(r,\frac{1}{f}\right) + \overline{N}_{k+1}\left(r,\frac{1}{f}\right) + 2T(r,\beta) + S(r,f)$$
$$\le 3\left(\overline{N}\left(r,\frac{1}{F}\right) + \overline{N}_{k)}\left(r,\frac{1}{f}\right)\right) + \overline{N}_{k+1}\left(r,\frac{1}{f}\right) + S(r,f).$$

Using (2.20), we get

$$N_{1)}(r,f) \le N\left(r,\frac{1}{h}\right) \tag{2.23}$$
$$\le N(r,h) + S(r,f)$$
$$\le 3\left(\overline{N}\left(r,\frac{1}{F}\right) + \overline{N}_{k)}\left(r,\frac{1}{f}\right)\right) + \overline{N}_{k+1}\left(r,\frac{1}{f}\right) + S(r,f).$$

Note

$$\overline{N}_{k+1}\left(r,\frac{1}{f}\right) = \frac{1}{k+1}N_{k+1}\left(r,\frac{1}{f}\right) \le \frac{1}{k+1}T(r,f) + S(r,f).$$

By (2.14),(2.16) and (2.23), we deduce

$$T(r,f) \le \left(6+\frac{6}{k}\right)\left(\overline{N}\left(r,\frac{1}{F}\right) + \overline{N}_{k)}\left(r,\frac{1}{f}\right)\right) + S(r,f).$$

□

Lemma 2.3. *Let f be a nonconstant meromorphic function in the complex plane such that the zeros of $f(z)$ are of multiplicity at least $\ge k$ and $a(\ne 0)$ be a finite constant. Then*

(i) *If $n \ge 2$, $f^{(k)} - af^{-n}$ must have some zero, where k and n are positive integers.*

(ii) *If $n = 1$, and $\overline{N}_k\left(r,\frac{1}{f}\right) = S(r,f)$, $f^{(k)} - af^{-n}$ must have some zero, where k is a positive integer.*

Proof. First we assume that $n \ge 2$, by Lemma 2.1, we know $f^n f^{(k)} - a$ must have some zero. Since a zero of $f^n f^{(k)} - a$ is a zero of $f^{(k)} - af^{-n}$, then $f^{(k)} - af^{-n}$ must have some zero.

If $n = 1$, the zeros of $f(z)$ are of multiplicity at least $\ge k$, so $\overline{N}_{k-1)}\left(r,\frac{1}{f}\right) = S(r,f)$. With the condition of $\overline{N}_k\left(r,\frac{1}{f}\right) = S(r,f)$, we have $\overline{N}_{k)}\left(r,\frac{1}{f}\right) = S(r,f)$. By Lemma 2.2, we know $ff^{(k)} - a$ must have some zero. As the preceding paragraph a zero of $ff^{(k)} - a$ is a zero of $f^{(k)} - af^{-1}$, the lemma is proved. □

Lemma 2.4 ([13]). *Let $f(z)$ be a transcendental meromorphic function in the complex plane, and $a \ne 0$ be a constant. If $n \ge k+3$, then $f^{(k)} - af^n$ assumes zeros infinitely often.*

Remark 2.5. In fact, E. Mues's [7, Theorem 1(b)] gave a counterexample to show that $f' - f^4 = c$ has no solution. We know $f^{(k)} - af^n$ cannot assume non-zero values for any positive integer n, k and $n = k+3$. Hence Theorem 1.2 may be best when we drop the condition (ii) in Theorem A.

Lemma 2.6. *Let f be a meromorphic function in the complex plane, and $a \ne 0$ be a constant. If $n \ge k+3$, and $f^{(k)} - af^n \ne 0$, then $f \equiv$ constant.*

Proof. By Lemma 2.4, we know $f(z)$ is not a transcendental meromorphic function. If $f(z)$ is a rational function. Let $f(z) = p(z)/q(z)$, where $p(z), q(z)$ are two co-prime polynomials with $\deg p(z) = p, \deg q(z) = q$.

Then $f^{(k)} = \left(\frac{p(z)}{q(z)}\right)^{(k)} = \frac{p_k(z)}{q_k(z)}$, where $p_k(z), q_k(z)$ are two co-prime polynomials, it is easily seen by induction that $\deg p_k(z) = p_k = p$, $\deg q_k(z) = q_k = q + k$, and $f^n(z) = \frac{p^n(z)}{q^n(z)}$, where $\deg p^n(z) = pn, \deg q(z) = qn$. Since

$$f^{(k)} - af^n = \frac{p_k(z)}{q_k(z)} - a\frac{p^n(z)}{q^n(z)} = \frac{p_k(z)q^n(z) - aq_k(z)p^n(z)}{q_k(z)q^n(z)},$$

and the degree of the term $p_k(z)q^n(z) - aq_k(z)p^n(z)$ is $\max\{p + nq, q + k + np\}$. If $p + nq = q + k + np$, we have

$$n - 1 = \frac{k}{q - p} \geq k + 2.$$

It is impossible. Hence $p_k(z)q^n(z) - aq_k(z)p^n(z)$ is a polynomial with degree=$\max\{p_k + nq, q_k + np\} > 0$, Obviously, $f^{(k)} - af^n$ can assume zeros. It is a contradiction. Thus we have $f \equiv$ constant. □

Lemma 2.7. *Let f be meromorphic in the complex plane, and $a \neq 0$ be a constant. For any positive integer n, k, satisfy $n \geq k + 3$. If $f^{(k)} - af^n \equiv 0$, then $f \equiv$ is the constant.*

Proof. If f is not the constant, by the condition we know f is an integer function. Otherwise, if z_0 is the pole of $p(\geq 1)$ order of f, then $np = p + k$ contradicts with $n \geq k + 3$. With the identity $f^n \equiv -af^{(k)}$, or $(f)^{n-1} \equiv \frac{1}{a}\frac{f^{(k)}}{f}$, we can get

$$(n-1)T(r, f) = (n-1)m(r, f) \leq \log^+ \frac{1}{|a|} + m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f), \quad \text{if} \quad r \to \infty$$

and $r \notin E$ with E being a set of r values of finite linear measure. It is impossible. This proves the lemma. □

Lemma 2.8 ([9]). *Let $\mathcal{F}$ be a family of meromorphic functions on the unit disc $\triangle$ such that all zeros of functions in $\mathcal{F}$ have multiplicity at least k. If $\mathcal{F}$ is not normal at a point z_0, then for $0 \leq \alpha < k$, there exist a sequence of functions $f_k \in \mathcal{F}$, a sequence of complex numbers $z_k \to z_0$ and a sequence of positive numbers $\rho_k \to 0$, such that*

$$\rho_k^{-\alpha} g_k(z_k + \rho_k \xi) \to g(\xi)$$

spherically uniformly on compact subsets of $\mathcal{C}$, where g is a nonconstant meromorphic function. Moreover, g is of order at most two, and g has only zeros of multiplicity at least k.

Lemma 2.9 ([2]). *Let f be a transcendental entire function all of whose zeros have multiplicity at least k, and let n be a positive integer. Then $f^n f^{(k)}$ takes on each nonzero value $a \in \mathbb{C}$ infinitely often.*

Lemma 2.10. *Let f be a polynomial all of whose zeros have multiplicity at least k, and let n be a positive integer. Then $f^n f^{(k)}$ can assume each nonzero value $a \in \mathbb{C}$.*

The proof is trivial, we omit it here.

3. PROOF OF THE THEOREMS

Proof of Theorem 1.2. We may assume that $D = \triangle$. Suppose that $\mathcal{F}$ is not normal at $z_0 \in \triangle$. Then, taking $\alpha = \frac{k}{n-1}$, where $0 < \alpha < k$, and applying Lemma 2.8 to $g = \{1/f : f \in F\}$, we can find $f_j \in F(j = 1, 2, \dots)$, $z_j \to z_0$ and $\rho_j(> 0) \to 0$ such that $g_j(\zeta) = \rho_j^{\alpha} f_j(z_j + \rho_j\zeta)$, converges locally uniformly with respect to the spherical metric to $g(\zeta)$, where g is a nonconstant meromorphic function on $\mathcal{C}$. By Lemma 2.6, there exists $\zeta_0 \in \{|z| \le R\}$ such that

$$g(\zeta_0)^n - a(g^{(k)}(\zeta_0)) = 0. \tag{3.1}$$

From the above equality, $g(\zeta_0) \neq \infty$. Through the calculation, we have

$$g_j^n(\zeta) - a(g_j^{(k)}(\zeta)) = \rho_j^{\frac{nk}{n-1}}(f_j^n(\zeta) - a(f_j^{(k)}(\zeta))) \neq 0.$$

On the other hand,

$$g_j^n(\zeta) - a(g_j^{(k)}(\zeta)) \to g^n(\zeta) - a(g^{(k)}(\zeta)).$$

By Hurwitz's theorem, we know $g^n(\zeta) - a(g^{(k)}(\zeta))$ is either identity zero or identity non-zero. From (3.1), we know $g^n(\zeta) - a(g^{(k)}(\zeta)) \equiv 0$, then by Lemma 2.7 yields $g(\zeta)$ is a constant, it is a contradiction. Hence we complete the proof of Theorem 1.2. □

Proof of Theorem 1.3. Let $\alpha = \frac{k}{n-1} < k$. If possible suppose that $\mathcal{F}$ is not normal at $z_0 \in \mathcal{D}$. Then by Lemma 2.8, there exist a sequence of functions $f_j \in F$ $(j = 1, 2, \dots)$, a sequence of complex numbers $z_j \to z_0$ and $\rho_j(> 0) \to 0$, such that

$$g_j(\zeta) = \rho_j^{-\alpha} f_j(z_j + \rho_j\zeta)$$

converges spherically and locally uniformly to a nonconstant meromorphic function $g(\zeta)$ in $\mathcal{C}$. Also the zeros of $g(z)$ are of multiplicity at least $\ge k$. So $g^{(k)} \not\equiv 0$. By the condition of Theorem 1.3 and Lemma 2.3, we get

$$g^{(k)}(\zeta_0) + \frac{a}{g(\zeta_0)^n} = 0 \tag{3.2}$$

for some $\zeta_0 \in \mathcal{C}$. Clearly ζ_0 is neither a zero nor a pole of g. So in some neighborhood of ζ_0, $g_j(\zeta)$ converges uniformly to $g(\zeta)$. Now in some neighborhood of ζ_0 we see that $g^{(k)}(\zeta) + ag(\zeta)^{-n}$ is the uniform limit of

$$g^{(k)}(\zeta_0) + ag(\zeta_0)^{-n} - \rho_j^{n\alpha} b = \rho_j^{\frac{nk}{1+n}} \left\{ f_j^{(k)}(z_j + \rho_j\zeta_j) + af_j^{-n}(z_j + \rho_j\zeta_j) - b \right\}.$$

By (3.2) and Hurwitz's theorem, there exists a sequence $\zeta_j \to \zeta_0$ such that for all large values of j

$$f_j^{(k)}(z_j + \rho_j\zeta_j) + af_j^{-n}(z_j + \rho_j\zeta_j) = b.$$

Therefore for all large values of j, it follows from the given condition $|g_j(\zeta_j)| \ge M/\rho_j^{\alpha}$ and as in the last part of the proof of Theorem 1.1 in [6], we arrive at a contradiction. This proves the theorem. □

Proof of Theorem 1.4. In a similar manner to the proof of Theorem 1.3, we can prove the theorem by Lemma 2.8, 2.9 and Lemma 2.10. □

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