Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 4, Issue 4, Article 76, 2003

CERTAIN BOUNDS FOR THE DIFFERENCES OF MEANS

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Received 17 August, 2002; accepted 3 June, 2003
Communicated by K.B. Stolarsky

ABSTRACT. Let $P_{n,r}(\mathbf{x})$ be the generalized weighted power means. We consider bounds for the differences of means in the following form:

$$\max\left\{\frac{C_{u,v,\beta}}{x_1^{2\beta-\alpha}}, \frac{C_{u,v,\beta}}{x_n^{2\beta-\alpha}}\right\}\sigma_{n,w',\beta} \geq \frac{P_{n,u}^{\alpha}-P_{n,v}^{\alpha}}{\alpha} \geq \min\left\{\frac{C_{u,v,\beta}}{x_1^{2\beta-\alpha}}, \frac{C_{u,v,\beta}}{x_n^{2\beta-\alpha}}\right\}\sigma_{n,w,\beta}.$$

Here $\beta \neq 0, \sigma_{n,t,\beta}(\mathbf{x}) = \sum_{i=1}^{n}\omega_i[x_i^{\beta}-P_{n,t}^{\beta}(\mathbf{x})]^2$ and $C_{u,v,\beta} = \frac{u-v}{2\beta^2}$. Some similar inequalities are also considered. The results are applied to inequalities of Ky Fan's type.

Key words and phrases: Ky Fan's inequality, Levinson's inequality, Generalized weighted power means, Mean value theorem.

2000 *Mathematics Subject Classification.* Primary 26D15, 26D20.

1. INTRODUCTION

Let $P_{n,r}(\mathbf{x})$ be the generalized weighted power means: $P_{n,r}(\mathbf{x}) = (\sum_{i=1}^{n}\omega_i x_i^r)^{\frac{1}{r}}$, where $\omega_i > 0, 1 \leq i \leq n$ with $\sum_{i=1}^{n}\omega_i = 1$ and $\mathbf{x} = (x_1, x_2, \dots, x_n)$. Here $P_{n,0}(\mathbf{x})$ denotes the limit of $P_{n,r}(\mathbf{x})$ as $r \to 0^+$. In this paper, we always assume that $0 < x_1 \leq x_2 \leq \cdots \leq x_n$. We write

$$\sigma_{n,t,\beta}(\mathbf{x}) = \sum_{i=1}^{n}\omega_i\left[x_i^{\beta}-P_{n,t}^{\beta}(\mathbf{x})\right]^2$$

and denote $\sigma_{n,t}$ as $\sigma_{n,t,1}$.

We let

$$A_n(\mathbf{x}) = P_{n,1}(\mathbf{x}), G_n(\mathbf{x}) = P_{n,0}(\mathbf{x}), H_n(\mathbf{x}) = P_{n,-1}(\mathbf{x})$$

and we shall write $P_{n,r}$ for $P_{n,r}(\mathbf{x})$, A_n for $A_n(\mathbf{x})$ and similarly for other means when there is no risk of confusion.

ISSN (electronic): 1443-5756

The author thanks the referees for their many valuable comments and suggestions. These greatly improved the presentation of the paper.
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We consider upper and lower bounds for the differences of the generalized weighted means in the following forms ($\beta \neq 0$):

$$\max\left\{\frac{C_{u,v,\beta}}{x_1^{2\beta-\alpha}}, \frac{C_{u,v,\beta}}{x_n^{2\beta-\alpha}}\right\}\sigma_{n,w',\beta} \geq \frac{P_{n,u}^{\alpha} - P_{n,v}^{\alpha}}{\alpha} \geq \min\left\{\frac{C_{u,v,\beta}}{x_1^{2\beta-\alpha}}, \frac{C_{u,v,\beta}}{x_n^{2\beta-\alpha}}\right\}\sigma_{n,w,\beta}, \tag{1.1}$$

where $C_{u,v,\beta} = \frac{u-v}{2\beta^2}$. If we set $x_1 = \cdots = x_{n-1} \neq x_n$, then we conclude from

$$\operatorname*{Lim}_{x_1 \to x_n} \frac{P_{n,u}^{\alpha} - P_{n,v}^{\alpha}}{\alpha\sigma_{n,w,\beta}} = \frac{u-v}{2\beta^2 x_n^{2\beta-\alpha}}$$

that $C_{u,v,\beta}$ is best possible. Here we define $(P_{n,u}^0 - P_{n,v}^0)/0 = \ln(P_{n,u}/P_{n,v})$, the limit of $(P_{n,u}^{\alpha} - P_{n,v}^{\alpha})/\alpha$ as $\alpha \to 0$.

In what follows we will refer to (1.1) as $(u, v, \alpha, \beta, w, w')$. D.I. Cartwright and M.J. Field [8] first proved the case $(1, 0, 1, 1, 1, 1)$. H. Alzer [4] proved $(1, 0, 1, 1, 1, 0)$ and [5] $(1, 0, \alpha, 1, 1, 1)$ with $\alpha \leq 1$. A.McD. Mercer [13] proved the right-hand side inequality with smaller constants for $\alpha = \beta = u = 1, v = -1, w = \pm 1$.

There is a close relationship between (1.1) and the following Ky Fan inequality, first published in the monograph *Inequalities* by Beckenbach and Bellman [7].(In this section, we set $A_n' = 1 - A_n, G_n' = \prod_{i=1}^n (1 - x_i)^{\omega_i}$. For general definitions, see the beginning of Section 3.)

Theorem 1.1. *For* $x_i \in \left[0, \frac{1}{2}\right]$,

$$\frac{A_n'}{G_n'} \leq \frac{A_n}{G_n} \tag{1.2}$$

with equality holding if and only if $x_1 = \cdots = x_n$.

P. Mercer [15] observed that the validity of $(1, 0, 1, 1, 1, 1)$ leads to the following refinement of the additive Ky Fan inequality:

Theorem 1.2. *Let* $0 < a \leq x_i \leq b < 1$ $(1 \leq i \leq n, n \geq 2)$. *For* $a \neq b$ *we have*

$$\frac{a}{1-a} < \frac{A_n' - G_n'}{A_n - G_n} < \frac{b}{1-b}. \tag{1.3}$$

Thus, by a result of P. Gao [9], it yields the following refinement of Ky Fan's inequality, first proved by Alzer [6]:

$$\left(\frac{A_n}{G_n}\right)^{\left(\frac{a}{1-a}\right)^2} \leq \frac{A_n'}{G_n'} \leq \left(\frac{A_n}{G_n}\right)^{\left(\frac{b}{1-b}\right)^2}.$$

For an account of Ky Fan's inequality, we refer the reader to the survey article [2] and the references therein.

The additive Ky Fan's inequality for generalized weighted means is a consequence of (1.1). Since it does not always hold (see [9]), it follows that (1.1) does not hold for arbitrary $(u, v, \alpha, \beta, w, w')$.

Our main result is a theorem that shows the validity of (1.1) for some $\alpha, \beta, u, v, w, w'$. We apply it in Section 3 to obtain further refinements and generalizations of inequalities of Ky Fan's type.

One can obtain further refinements of (1.1). Recently, A.McD. Mercer proved the following theorem [14]:

Theorem 1.3. *If* $x_1 \neq x_n, n \geq 2$, *then*

$$\frac{G_n - x_1}{2x_1(A_n - x_1)}\sigma_{n,1} > A_n - G_n > \frac{x_n - G_n}{2x_n(x_n - A_n)}\sigma_{n,1}. \tag{1.4}$$

We generalize this in Section 2.

2. THE MAIN THEOREM

Theorem 2.1. $\left(1, \frac{s}{r}, 1, \frac{\gamma}{r}, \frac{t}{r}, \frac{t'}{r}\right), r \neq s, r \neq 0, \gamma \neq 0$ *holds for the following three cases:*

(1) $\frac{s}{\gamma} \leq \frac{r}{\gamma} \leq 2, 1 \geq \frac{t}{\gamma}, \frac{t'}{\gamma} \geq \frac{s}{\gamma} \geq \frac{r}{\gamma} - 1$;
(2) $\frac{r}{\gamma} \geq 2, \frac{r}{\gamma} - 1 \geq \frac{s}{\gamma} \geq \frac{t}{\gamma}, \frac{t'}{\gamma} \geq 1$;
(3) $\frac{r}{\gamma} \leq \frac{s}{\gamma} \leq \frac{t}{\gamma}, \frac{t'}{\gamma} \leq 1$,

with equality holding if and only if $x_1 = \cdots = x_n$ for all the cases.

Proof. Let $\gamma = 1$ and $r \neq s$. We will show that (1.1) holds for the following three cases:

(1) $s \leq r \leq 2, 1 \geq t, t' \geq s \geq r - 1$;
(2) $r \geq 2, r - 1 \geq s \geq t, t' \geq 1$;
(3) $r \leq s \leq t, t' \leq 1$.

For case (1), consider the right-hand side inequality of (1.1) and let

$$D_n(\mathbf{x}) = A_n - P_{n,\frac{s}{r}} - \frac{r(r-s)}{2x_n^{\frac{2}{r}-1}} \sum_{i=1}^{n} \omega_i \left(x_i^{\frac{1}{r}} - P_{n,\frac{t}{r}}^{\frac{1}{r}}\right)^2. \tag{2.1}$$

We want to show that $D_n \geq 0$ here. We can assume that $x_1 < x_2 < \cdots < x_n$ and use induction. The case $n = 1$ is clear, so assume that the inequality holds for $n - 1$ variables. Then

$$\frac{1}{\omega_n} \cdot \frac{\partial D_n}{\partial x_n} = 1 - \left[\left(\frac{P_{n,\frac{s}{r}}}{x_n}\right)^{\frac{1}{r}}\right]^{r-s} - (r-s)\left(1 - \left(\frac{P_{n,\frac{t}{r}}}{x_n}\right)^{\frac{1}{r}}\right) + S, \tag{2.2}$$

where

$$S = \frac{(2-r)(r-s)}{2\omega_n x_n^{\frac{2}{r}-2}} \sum_{i=1}^{n} \omega_i \left(x_i^{\frac{1}{r}} - P_{n,\frac{t}{r}}^{\frac{1}{r}}\right)^2 + (r-s)\frac{P_{n,\frac{t}{r}}^{\frac{1-t}{r}}}{x_n^{\frac{2-t}{r}}}\left(P_{n,\frac{1}{r}}^{\frac{1}{r}} - P_{n,\frac{t}{r}}^{\frac{1}{r}}\right).$$

Thus, when $s \leq r \leq 2, t \leq 1$, $S \geq 0$.

Now by the mean value theorem

$$1 - \left[\left(\frac{P_{n,\frac{s}{r}}}{x_n}\right)^{\frac{1}{r}}\right]^{r-s} = (r-s)\eta^{r-s-1}\left(1 - \left(\frac{P_{n,\frac{s}{r}}}{x_n}\right)^{\frac{1}{r}}\right) \geq (r-s)\left(1 - \left(\frac{P_{n,\frac{s}{r}}}{x_n}\right)^{\frac{1}{r}}\right)$$

for $r \geq s \geq r - 1$ with

$$\min\left\{1, \left(\frac{P_{n,\frac{s}{r}}}{x_n}\right)^{\frac{1}{r}}\right\} \leq \eta \leq \max\left\{1, \left(\frac{P_{n,\frac{s}{r}}}{x_n}\right)^{\frac{1}{r}}\right\}.$$

This implies

$$1 - \left[\left(\frac{P_{n,\frac{s}{r}}}{x_n}\right)^{\frac{1}{r}}\right]^{r-s} - (r-s)\left(1 - \left(\frac{P_{n,\frac{t}{r}}}{x_n}\right)^{\frac{1}{r}}\right) \geq (r-s)\left[\left(\frac{P_{n,\frac{t}{r}}}{x_n}\right)^{\frac{1}{r}} - \left(\frac{P_{n,\frac{s}{r}}}{x_n}\right)^{\frac{1}{r}}\right],$$

which is positive if $s \leq t$.

Thus for $s \leq r \leq 2,\ 1 \geq t \geq s \geq r - 1$, we have $\frac{\partial D_n}{\partial x_n} \geq 0$. By letting x_n tend to x_{n-1}, we have $D_n \geq D_{n-1}$(with weights $\omega_1, \ldots, \omega_{n-2}, \omega_{n-1} + \omega_n$) and thus the right-hand side inequality of (1.1) holds by induction. It is also easy to see that equality holds if and only if $x_1 = \cdots = x_n$.

Now consider the left-hand side inequality of (1.1) and write

$$E_n(\mathbf{x}) = A_n - P_{n,\frac{s}{r}} - \frac{r(r-s)}{2x_1^{\frac{2}{r}-1}} \sum_{i=1}^{n} \omega_i \left(x_i^{\frac{1}{r}} - P_{n,\frac{t'}{r}}^{\frac{1}{r}}\right)^2. \tag{2.3}$$

Now $\frac{1}{\omega_1}\frac{\partial E_n}{\partial x_1}$ has an expression similar to (2.2) with $x_n \leftrightarrow x_1, \omega_n \leftrightarrow \omega_1, t \leftrightarrow t'$. It is then easy to see under the same condition, $\frac{\partial E_n}{\partial x_1} \geq 0$. Thus the left-hand side inequality of (1.1) holds by a similar induction process with the equality holding if and only if $x_1 = \cdots = x_n$.

Similarly, we can show $D_n(\mathbf{x}) \leq 0, E_n(\mathbf{x}) \geq 0$ for cases (2) and (3) with equality holding if and only if $x_1 = \cdots = x_n$ for all the cases.

Now for an arbitrary γ, a change of variables $y \to y/\gamma$ for $y = r, s, t, t'$ in the above cases leads to the desired conclusion. □

In what follows our results often include the cases $r = 0$ or $s = 0$ and we will leave the proofs of these special cases to the reader since they are similar to what we give in the paper.

Corollary 2.2. *For $r > s$, $\min\{1, r-1\} \leq s \leq \max\{1, r-1\}$ and $\min\{1, s\} \leq t, t' \leq \max\{1, s\}$, $(r, s, r, 1, t, t')$ holds. For $s \leq r \leq t, t' \leq 1$, $(r, s, s, 1, t, t')$ holds, with equality holding if and only if $x_1 = \cdots = x_n$ for all the cases.*

Proof. This follows from taking $\gamma = 1$ in Theorem 2.1 and another change of variables: $x_1 \to \min\{x_1^r, x_n^r\}$, $x_n \to \max\{x_1^r, x_n^r\}$ and $x_i = x_i^r$ for $2 \leq i \leq n-1$ if $n \geq 3$ and exchanging r and s for the case $s > r$. □

We remark here since $\sigma_{n,t'} = \sigma_{n,t} + (2A_n - P_{n,t} - P_{n,t'})(P_{n,t} - P_{n,t'})$, we have $\sigma_{n,1} \leq \sigma_{n,t}$ for $t \neq 1$ and $\sigma_{n,t} \leq \sigma_{n,t'}$ for $t' \leq t \leq 1$, $\sigma_{n,t} \geq \sigma_{n,t'}$ for $t \geq t' \geq 1$. Thus the optimal choices for the set $\{t, t'\}$ will be $\{1, s\}$ for the case $(r, s, r, 1, t, t')$ and $\{1, r\}$ for the case $(r, s, s, 1, t, t')$.

Our next two propositions give relations between differences of means with different powers:

Proposition 2.3. *For $l - r \geq t - s \geq 0,\ l \neq t,\ x_i \in [a, b],\ a > 0$,*

$$\left|\frac{(r-s)}{(l-t)}\right| \frac{1}{a^{l-r}} \geq \left|\frac{(P_{n,r}^r - P_{n,s}^r)/r}{(P_{n,l}^l - P_{n,t}^l)/l}\right| \geq \left|\frac{(r-s)}{(l-t)}\right| \frac{1}{b^{l-r}}. \tag{2.4}$$

Except for the trivial cases $r = s$ or $(l, t) = (r, s)$, the equality holds if and only if $x_1 = \cdots = x_n$, where we define $0/0 = x_i^{r-l}$ for any i.

Proof. This is a generalization of a result A.McD. Mercer [12]. We may assume that $x_1 = a, x_n = b$ and consider

$$D(\mathbf{x}) = P_{n,r}^r - P_{n,s}^r - \frac{r(r-s)}{l(l-t)x_n^{l-r}}(P_{n,l}^l - P_{n,t}^l),$$

$$E(\mathbf{x}) = P_{n,r}^r - P_{n,s}^r - \frac{r(r-s)}{l(l-t)x_1^{l-r}}(P_{n,l}^l - P_{n,t}^l).$$

We will show that $D_n \cdot E_n \leq 0$. Suppose $r - s \geq 0$ here; the case $r - s \leq 0$ is similar. We have

$$\frac{x_n^{1-r}}{r\omega_n} \cdot \frac{\partial D_n}{\partial x_n} = 1 - \left(\frac{P_{n,s}}{x_n}\right)^{r-s} - \frac{r-s}{l-t}\left(1 - \left[\left(\frac{P_{n,t}}{x_n}\right)^{r-s}\right]^{\frac{l-t}{r-s}}\right) + S,$$

where

$$S = \frac{(r-s)(l-r)}{l(l-t)x_n^{l-2}\omega_n}(P_{n,l}^l - P_{n,t}^l) \geq 0.$$

Now by the mean value theorem

$$1 - \left[\left(\frac{P_{n,t}}{x_n}\right)^{r-s}\right]^{\frac{l-t}{r-s}} = \frac{l-t}{r-s}\eta^{l-t-r+s}\left(1 - \left(\frac{P_{n,t}}{x_n}\right)^{r-s}\right),$$

where $\frac{P_{n,t}}{x_n} < \eta < 1$ and

$$\frac{x_n^{1-r}}{r\omega_n}\frac{\partial D_n}{\partial x_n} \geq 1 - \left(\frac{P_{n,s}}{x_n}\right)^{r-s} - \left(1 - \left(\frac{P_{n,t}}{x_n}\right)^{r-s}\right) \geq 0$$

since $t \geq s$.

Similarly, we have $\frac{x_1^{1-r}}{r\omega_1}\frac{\partial E_n}{\partial x_1} \geq 0$ and by a similar induction process as the one in the proof of Theorem 2.1, we have $D_n \cdot E_n \leq 0$. This completes the proof. □

By taking $l = 2, t = 0, r = 1, s = -1$ in the proposition, we get the following inequality:

$$\frac{1}{2x_1}(P_{n,2}^2 - G_n^2) \geq A_n - H_n \geq \frac{1}{2x_n}(P_{n,2}^2 - G_n^2) \tag{2.5}$$

and the right-hand side inequality above gives a refinement of a result of A.McD. Mercer [13].

Proposition 2.4. *For $r > s, \alpha > \beta$,*

$$x_1^{\beta-\alpha} \geq P_{n,s}^{\beta-\alpha} \geq \frac{(P_{n,r}^{\beta} - P_{n,s}^{\beta})/\beta}{(P_{n,r}^{\alpha} - P_{n,s}^{\alpha})/\alpha} \geq P_{n,r}^{\beta-\alpha} \geq x_n^{\beta-\alpha} \tag{2.6}$$

with equality holding if and only if $x_1 = \cdots = x_n$, where we define $0/0 = x_i^{\beta-\alpha}$ for any i.

Proof. By the mean value theorem,

$$P_{n,r}^{\beta} - P_{n,s}^{\beta} = (P_{n,r}^{\alpha})^{\beta/\alpha} - (P_{n,s}^{\alpha})^{\beta/\alpha} = \frac{\beta}{\alpha}\eta^{\beta-\alpha}(P_{n,r}^{\alpha} - P_{n,s}^{\alpha}),$$

where $P_{n,s} < \eta < P_{n,r}$ and (2.6) follows. □

We apply (2.6) to the case $(1, 0, 1, 1, 1, 1)$ to see that $(1, 0, \alpha, 1, 1, 1)$ holds with $\alpha \leq 1$, a result of Alzer [5]. We end this section with a generalization of (1.4) and leave the formulation of similar refinements to the reader.

Theorem 2.5. *If $x_1 \neq x_n,\ n \geq 2$, then for $1 > s \geq 0$*

$$\frac{P_{n,s}^{1-s} - x_1^{1-s}}{2x_1^{1-s}(A_n - x_1)}\sigma_{n,1} > A_n - P_{n,s} > \frac{x_n^{1-s} - P_{n,s}^{1-s}}{2x_n^{1-s}(x_n - A_n)}\sigma_{n,1}. \tag{2.7}$$

Proof. We prove the right-hand inequality; the left-hand side inequality is similar. Let

$$D_n(\mathbf{x}) = (x_n - A_n)(A_n - P_{n,s}) - \frac{x_n^{1-s} - P_{n,s}^{1-s}}{2x_n^{1-s}}\sigma_{n,1}.$$

We show by induction that $D_n \geq 0$. We have

$$\begin{aligned}\frac{\partial D_n}{\partial x_n} &= (1 - \omega_n)(A_n - P_{n,s}) - \frac{1-s}{2x_n}\left(\frac{P_{n,s}}{x_n}\right)^{1-s}\left(1 - \left(\frac{x_n}{P_{n,s}}\right)^{s}\omega_n\right)\sigma_{n,1} \\ &\geq (1 - \omega_n)\left(A_n - P_{n,s} - \frac{1-s}{2x_n}\sigma_{n,1}\right) \geq 0,\end{aligned}$$

where the last inequality holds by Theorem 2.1. By an induction process similar to the one in the proof of Theorem 2.1, we have $D_n \geq 0$. Since not all the x_i's are equal, we get the desired result. □

Corollary 2.6. *For $1 > s \geq 0$,*

$$\frac{1-s}{2x_1} \cdot \frac{P_{n,s}}{A_n}\sigma_{n,1} \geq A_n - P_{n,s} \geq \frac{1-s}{2x_n}\sigma_{n,s}, \tag{2.8}$$

with equality holding if and only if $x_1 = \cdots = x_n$.

Proof. By Theorem 2.5, we only need to show $\frac{P_{n,s}^{1-s}-x_1^{1-s}}{2x_1^{1-s}(A_n-x_1)} \leq \frac{1-s}{2x_1}\frac{P_{n,s}}{A_n}$ and this is easily verified by using the mean value theorem. □

3. Applications to Inequalities of Ky Fan's Type

Let $f(x,y)$ be a real function. We regard y as an implicit function defined by $f(x,y)=0$ and for $\mathbf{y}=(y_1,\dots,y_n)$, let $f(x_i,y_i)=0$, $1\leq i\leq n$. We write $P'_{n,r}=P_{n,r}(\mathbf{y})$ with $A'_n=P'_{n,1}$, $G'_n=P'_{n,0}$, $H'_n=P'_{n,-1}$. Furthermore, we write $x_1=a>0$ and $x_n=b$ so that $x_i\in[a,b]$ with $y_i\in[a',b']$, $a'>0$ and require that f'_x, f'_y exist for $x_i\in[a,b]$, $y_i\in[a',b']$.

To simplify expressions, we define:

$$\Delta_{r,s,\alpha}=\frac{P_{n,r}^{\alpha}(\mathbf{y})-P_{n,s}^{\alpha}(\mathbf{y})}{P_{n,r}^{\alpha}(\mathbf{x})-P_{n,s}^{\alpha}(\mathbf{x})} \tag{3.1}$$

with $\Delta_{r,s,0}=\left(\ln\frac{P_{n,r}(\mathbf{y})}{P_{n,s}(\mathbf{y})}\right)\Big/\left(\ln\frac{P_{n,r}(\mathbf{x})}{P_{n,s}(\mathbf{x})}\right)$ and, in order to include the case of equality for various inequalities in our discussion, we define $0/0=1$ from now on.

In this section, we apply our results above to inequalities of Ky Fan's type. Let $f(x,y)$ be any function satisfying the conditions in the first paragraph of this section. We now show how to get inequalities of Ky Fan's type in general.

Suppose (1.1) holds for some $\alpha>0,\ r>s,\ \beta=1$ and $t=t'=1$, write $\sigma_{n,1}(\mathbf{y})=\sigma'_{n,1}$, apply (1.1) to sequences $\mathbf{x},\mathbf{y}$ and then take their quotients to get

$$\frac{a\sigma'_{n,1}}{b'\sigma_{n,1}}\leq\Delta_{r,s,\alpha}\leq\frac{b\sigma'_{n,1}}{a'\sigma_{n,1}}.$$

Since $\sigma'_{n,1}=\sum_{i=1}^n w_i(\sum_{k=1}^n w_k(y_i-y_k))^2$, the mean value theorem yields

$$y_i-y_k=-\frac{f'_x}{f'_y}(\xi,y(\xi))(x_i-x_k)$$

for some $\xi\in(a,b)$. Thus

$$\min_{a\leq x\leq b}\left|\frac{f'_x}{f'_y}\right|^2\sigma_{n,1}\leq\sigma'_{n,1}\leq\max_{a\leq x\leq b}\left|\frac{f'_x}{f'_y}\right|^2\sigma_{n,1},$$

which implies

$$\frac{a}{b'}\min_{a\leq x\leq b}\left|\frac{f'_x}{f'_y}\right|^2\leq\Delta_{r,s,\alpha}\leq\frac{b}{a'}\max_{a\leq x\leq b}\left|\frac{f'_x}{f'_y}\right|^2.$$

We next apply the above argument to a special case.

Corollary 3.1. *Let* $f(x,y)=cx^p+dy^p-1$, $0<c\leq d$, $p\geq1$, $x_i\in[0,(c+d)^{-\frac{1}{p}}]$. *For* $s\in[0,2]$ *and* $\alpha=\max\{s,1\}$ *we have*

$$\Delta_{1,s,\alpha}\leq1 \tag{3.2}$$

with equality holding if and only if $x_1=\cdots=x_n$.

Proof. This follows from Corollary 2.2 by the appropriate choice of r and s. □

From now on we will concentrate on the case $f(x,y)=x+y-1$. Extensions to the case of general functions $f(x,y)$ are left to the reader.

Corollary 3.2. *Let* $f(x,y) = x + y - 1$, $0 < a < b < 1$ *and* $x_i \in [a,b]$ $(i = 1,\dots,n)$, $n \geq 2$. *Then for* $r > s$, $\min\{1, r-1\} \leq s \leq \max\{1, r-1\}$

$$\max\left\{\left(\frac{b}{1-b}\right)^{2-r}, \left(\frac{a}{1-a}\right)^{2-r}\right\} > \Delta_{r,s,r} > \min\left\{\left(\frac{b}{1-b}\right)^{2-r}, \left(\frac{a}{1-a}\right)^{2-r}\right\}. \tag{3.3}$$

For $s < r \leq 1$,

$$\max\left\{\left(\frac{b}{1-b}\right)^{2-s}, \left(\frac{a}{1-a}\right)^{2-s}\right\} > \Delta_{r,s,s} > \min\left\{\left(\frac{b}{1-b}\right)^{2-s}, \left(\frac{a}{1-a}\right)^{2-s}\right\}. \tag{3.4}$$

Proof. Apply Corollary 2.2 to sequences $\mathbf{x}, \mathbf{y}$ with $t = t' = 1$ and take their quotients, by noticing $\sigma_{n,1}(\mathbf{x}) = \sigma_{n,1}(\mathbf{y})$. □

As a special case of the above corollary, by taking $r = 0,\ s = -1$, we get the following refinement of the Wang-Wang inequality [17]:

$$\left(\frac{G_n}{H_n}\right)^{\left(\frac{a}{1-a}\right)^2} \leq \frac{G'_n}{H'_n} \leq \left(\frac{G_n}{H_n}\right)^{\left(\frac{b}{1-b}\right)^2}. \tag{3.5}$$

We can use Corollary 2.6 to get further refinements of inequalities of Ky Fan's type. Since $\sigma_{n,s} = \sigma_{n,1} + (A_n - P_{n,s})^2$, we can rewrite the right-hand side inequality in (2.8) as

$$(P_{n,1}(\mathbf{x}) - P_{n,s}(\mathbf{x}))\left(1 - \frac{1-s}{2b}(P_{n,1}(\mathbf{x}) - P_{n,s}(\mathbf{x}))\right) \geq \frac{1-s}{2b}\sigma_{n,1}. \tag{3.6}$$

Apply (2.8) to $\mathbf{y}$ and taking the quotient with (3.6), we get

$$\frac{P_{n,1}(\mathbf{y}) - P_{n,s}(\mathbf{y})}{(P_{n,1}(\mathbf{x}) - P_{n,s}(\mathbf{x}))\left(1 - \frac{1-s}{2b}(P_{n,1}(\mathbf{x}) - P_{n,s}(\mathbf{x}))\right)} \leq \frac{b\sigma'_{n,1}}{a'\sigma_{n,1}}\frac{P'_{n,s}}{A'_n} = \frac{b}{a'}\frac{P'_{n,s}}{A'_n}.$$

Similarly,

$$\frac{(P_{n,1}(\mathbf{y}) - P_{n,s}(\mathbf{y}))\left(1 - \frac{1-s}{2a'}(P_{n,1}(\mathbf{y}) - P_{n,s}(\mathbf{y}))\right)}{P_{n,1}(\mathbf{x}) - P_{n,s}(\mathbf{x})} \geq \frac{a}{b'}\frac{A_n}{P_{n,s}}.$$

Combining these with a result in [9], we obtain the following refinement of Ky Fan's inequality:

Corollary 3.3. *Let* $0 < a < b < 1$ *and* $x_i \in [a,b]$ $(i = 1,\dots,n)$, $n \geq 2$. *Then for* $\alpha \leq 1$, $0 \leq s < 1$

$$\left(\frac{b}{1-b}\right)^{2-\alpha}\frac{P'_{n,s}}{A'_n}B > \Delta_{1,s,\alpha} > \left(\frac{a}{1-a}\right)^{2-\alpha}\frac{A_n}{P_{n,s}}A, \tag{3.7}$$

where

$$A = \left(1 - \frac{1-s}{2a'}(P_{n,1}(\mathbf{y}) - P_{n,s}(\mathbf{y}))\right)^{-1},$$

$$B = 1 - \frac{1-s}{2b}(P_{n,1}(\mathbf{x}) - P_{n,s}(\mathbf{x})).$$

We note here when $\alpha = 1,\ s = 0,\ b \leq \frac{1}{2}$, the left-hand side inequality of (3.7) yields

$$\frac{A'_n - G'_n}{A_n - G_n} < \frac{b}{1-b}\frac{G'_n}{A'_n}(A'_n + G_n) \tag{3.8}$$

a refinement of the following two results of H. Alzer [1]: $A'_n/G'_n \leq (1-G_n)/(1-A_n)$, which is equivalent to $(A'_n - G'_n)/(A_n - G_n) < G'_n/A'_n$ and [3]: $A'_n - G'_n \leq (A_n - G_n)(A'_n + G_n)$.

Next, we give a result related to Levinson's generalization of Ky Fan's inequality. We first generalize a lemma of A.McD. Mercer [12].

Lemma 3.4. *Let $J(x)$ be the smallest closed interval that contains all of x_i and let $y \in J(x)$ and $f(x), g(x) \in C^2(J(x))$ be two twice continuously differentiable functions. Then*

$$\frac{\sum_{i=1}^n \omega_i f(x_i) - f(y) - (\sum_{i=1}^n \omega_i x_i - y) f'(y)}{\sum_{i=1}^n \omega_i g(x_i) - g(y) - (\sum_{i=1}^n \omega_i x_i - y) g'(y)} = \frac{f''(\xi)}{g''(\xi)} \tag{3.9}$$

for some $\xi \in J(x)$, provided that the denominator of the left-hand side is nonzero.

Proof. The proof is very similar to the one given in [12]. Write

$$(Qf)(t) = \sum_{i=1}^n w_i f(tx_i + (1-t)y) - f(y) - t(A-y)f'(y)$$

and consider $W(t) = (Qf)(t) - K(Qg)(t)$, where K is the left-hand side expression in (3.9). The lemma then follows by the same argument as in [12]. □

By taking $g(x) = x^2, y = P_{n,t}$ in the lemma, we get:

Corollary 3.5. *Let $f(x) \in C^2[a,b]$ with $m = \min_{a \le x \le b} f''(x)$, $M = \max_{a \le x \le b} f''(x)$. Then*

$$\frac{M}{2}\sigma_{n,t} \ge \sum_{i=1}^n \omega_i f(x_i) - f\left(\sum_{i=1}^n \omega_i x_i\right) - (A_n - P_{n,t}) f'(P_{n,t}) \ge \frac{m}{2}\sigma_{n,t}. \tag{3.10}$$

Moreover, if $f'''(x)$ exists for $x \in [a,b]$ with $f'''(x) > 0$ or $f'''(x) < 0$ for $x \in [a,b]$ then the equality holds if and only if $x_1 = \cdots = x_n$.

The case $t = 1$ in the above corollary was treated by A.McD. Mercer [11]. Note for an arbitrary $f(x)$, equality can hold even if the condition $x_1 = \cdots = x_n$ is not satisfied, for example, for $f(x) = x^2$, we have the following identity:

$$\sum_{i=1}^n \omega_i x_i^2 - \left(\sum_{i=1}^n \omega_i x_i\right)^2 = \sum_{i=1}^n \omega_i \left(x_i - \sum_{k=1}^n \omega_k x_k\right)^2.$$

Corollary 3.5 can be regarded as a refinement of Jensen's inequality and it leads to the following well-known Levinson's inequality for 3-convex functions [10]:

Corollary 3.6. *Let $x_i \in (0,a]$. If $f'''(x) \ge 0$ in $(0, 2a)$, then*

$$\sum_{i=1}^n \omega_i f(x_i) - f\left(\sum_{i=1}^n \omega_i x_i\right) \le \sum_{i=1}^n \omega_i f(2a - x_i) - f\left(\sum_{i=1}^n \omega_i (2a - x_i)\right). \tag{3.11}$$

If $f'''(x) > 0$ on $(0, 2a)$ then equality holds if and only if $x_1 = \cdots = x_n$.

Proof. Take $t = 1$ in (3.10) and apply Corollary 3.5 to $(x_1, \dots, x_n)$ and $(2a - x_1, \dots, 2a - x_n)$. Since $f'''(x) \ge 0$ in $(0, 2a)$, it follows that $\max_{0 \le x \le a} f''(x) \le \min_{a \le x \le 2a} f''(x)$ and the corollary is proved. □

Now we establish an inequality relating different $\Delta_{r,s,\alpha}$'s:

Corollary 3.7. *For $l - r \ge t - s \ge 0$, $l \ne t$, $r \ne s$, $(l,t) \ne (r,s)$, $x_i \in [a,b]$, $y_i \in [a,b]$, $n \ge 2$,*

$$\left(\frac{b}{a'}\right)^{l-r} > \left|\frac{\Delta_{r,s,r}}{\Delta_{l,t,l}}\right| > \left(\frac{a}{b'}\right)^{l-r}. \tag{3.12}$$

Proof. Apply (2.4) to both $\mathbf{x}$ and $\mathbf{y}$ and take their quotients. □

For another proof of inequality (3.5), use this corollary with $l = 1,\ t = 0,\ s = -1$ and $r = 0$.

4. A FEW COMMENTS

A variant of (1.1) is the following conjecture by A.McD. Mercer [13] ($r > s, t, t' = r, s$):

$$\max\left\{\frac{r-s}{2x_1^{2-r}}, \frac{r-s}{2x_n^{2-r}}\right\}\sigma_{n,t'} \geq \frac{P_{n,r} - P_{n,s}}{P_{n,r}^{1-r}} \geq \min\left\{\frac{r-s}{2x_1^{2-r}}, \frac{r-s}{2x_n^{2-r}}\right\}\sigma_{n,t}. \tag{4.1}$$

The conjecture presented here has been reformulated (one can compare it with the original one in [13]), since here $(r-s)/2$ is the best possible constant by the same argument as above.

Note when $r = 1$, (4.1) coincides with (1.1) and thus the conjecture in general is false.

There are many other kinds of expressions for the bounds of the difference between the arithmetic and geometric means. See Chapter II of the book *Classical and New Inequalities in Analysis* [16].

In [12], A.McD. Mercer showed

$$\frac{P_{n,2}^2 - G_n^2}{4x_1} \geq A_n - G_n \geq \frac{P_{n,2}^2 - G_n^2}{4x_n}. \tag{4.2}$$

He also pointed out that the above inequality is not comparable to either of the inequalities in (1.1) with $\alpha = \beta = u = 1$, $v = 0$, $t = t' = 0, 1$. We note that (4.2) can be obtained from (1.1) by averaging the case $\alpha = \beta = u = t = t' = 1, v = 0$ with the following trivial bound:

$$\frac{A_n^2 - G_n^2}{2x_1} \geq A_n - G_n \geq \frac{A_n^2 - G_n^2}{2x_n}.$$

Thus the incomparability of (4.2) and (4.1) with $r = 1$, $s = 0$, $t = 1$ reflects the fact that $P_{n,2}^2 - A_n^2$ and $A_n^2 - G_n^2$ are in general not comparable.

We also note when replacing $C_{u,v,\beta}$ by a smaller constant, that we sometimes get a trivial bound. For example, for $s \leq \frac{1}{2}$, the following inequality holds:

$$A_n - P_{n,s} \geq \frac{1}{2}\sum_{k=1}^{n}\omega_k\left(x_k^{1/2} - A_n^{1/2}\right)^2 \geq \frac{1}{8x_n}\sum_{k=1}^{n}\omega_k(x_k - A_n)^2.$$

The first inequality is equivalent to $P_{n,1/2}^{1/2}A_n^{1/2} \geq P_{n,s}$. For the second, simply apply the mean value theorem to

$$\left(x_k^{1/2} - A_n^{1/2}\right)^2 = \left(\frac{1}{2}\xi_k^{-1/2}(x_k - A_n)\right)^2 \geq \frac{1}{4x_n}(x_k - A_n)^2,$$

with ξ_k in between x_k and A_n.

REFERENCES

[1] H. ALZER, Refinements of Ky Fan's inequality, *Proc. Amer. Math. Soc.*, **117** (1993), 159–165.

[2] H. ALZER, The inequality of Ky Fan and related results, *Acta Appl. Math.*, **38** (1995), 305–354.

[3] H. ALZER, On Ky Fan's inequality and its additive analogue, *J. Math. Anal. Appl.*, **204** (1996), 291–297.

[4] H. ALZER, A new refinement of the arithmetic mean–geometric mean inequality, *Rocky Mountain J. Math.*, **27**(3) (1997), 663–667.

[5] H. ALZER, On an additive analogue of Ky Fan's inequality, *Indag. Math.(N.S.)*, **8** (1997), 1–6.

[6] H. ALZER, Some inequalities for arithmetic and geometric means, *Proc. Roy. Soc. Edinburgh Sect. A*, **129** (1999), 221–228.

[7] E.F. BECKENBACH AND R. BELLMAN, *Inequalities*, Springer-Verlag, Berlin-Göttingen-Heidelberg 1961.

[8] D.I. CARTWRIGHT AND M.J. FIELD, A refinement of the arithmetic mean-geometric mean inequality, *Proc. Amer. Math. Soc.*, **71** (1978), 36–38.

[9] P. GAO, A generalization of Ky Fan's inequality, *Int. J. Math. Math. Sci.*, **28** (2001), 419–425.

[10] N. LEVINSON, Generalization of an inequality of Ky Fan, *J. Math. Anal. Appl.*, **8** (1964), 133–134.

[11] A.McD. MERCER, An "error term" for the Ky Fan inequality, *J. Math. Anal. Appl.*, **220** (1998), 774–777.

[12] A.McD. MERCER, Some new inequalities involving elementary mean values, *J. Math. Anal. Appl.*, **229** (1999), 677–681.

[13] A.McD. MERCER, Bounds for A-G, A-H, G-H, and a family of inequalities of Ky Fan's type, using a general method, *J. Math. Anal. Appl.*, **243** (2000), 163–173.

[14] A.McD. MERCER, Improved upper and lower bounds for the difference $A_n - G_n$, *Rocky Mountain J. Math.*, **31** (2001), 553–560.

[15] P. MERCER, A note on Alzer's refinement of an additive Ky Fan inequality, *Math. Inequal. Appl.*, **3** (2000), 147–148.

[16] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers Group, Dordrecht, 1993.

[17] P.F. WANG AND W.L. WANG, A class of inequalities for the symmetric functions, *Acta Math. Sinica* (in Chinese), **27** (1984), 485–497.