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ON DIFFERENTIABILITY WITH RESPECT TO PARAMETERS OF THE LEBESGUE INTEGRAL

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ABSTRACT. The aim of this paper is to extend the results of [7] and [8] to a Lebesgue integral whose integrand and limits of integration depend on parameters.

Key words and phrases: Lebesgue integral, Contingent cones, Contingent derivatives.

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1. INTRODUCTION

As it is known, the main instrument used in the dynamic programming method in optimal control theory to get necessary and/or sufficient optimality conditions is the value function, whose monotonicity plays an important role ([3], [4], [6], [9], [10], etc.).

In general, since the value function of an optimal control problem is not differentiable, the monotonicity properties lead to differentiable inequalities which can be expressed with the help of the generalized derivatives.

The particular structure of the value function and the necessity of the estimation of its generalized derivatives imply an estimation of the generalized derivatives with respect to parameters of the Lebesgue integral ([3], [7], [8]).

2. PRELIMINARY NOTATIONS AND RESULTS

The most known result regarding the differentiability in a classical meaning of the Lebesgue integral with respect to parameters retaken and used by L. Cesari ([3, Lemma 2.3.1]) is Theorem 3.9.2 from E.I. McShane [7]. A recent generalization of McShane's theorem is that of I. Mirică [8].

In the results to follow we shall use the concepts of tangent cones and corresponding generalized derivatives (e.g. [1], [2], [4], etc.).

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Let $\mathbb{R}^n$ be the n-dimensional euclidian space with norm $\|\cdot\|$. For $x \in \mathbb{R}^n$ and $\varepsilon > 0$ let $B_\varepsilon(x) = \{y \in \mathbb{R}^n : \|y - x\| < \varepsilon\}$ be the open ball centered at x and with radius ϵ.

If $X \subset \mathbb{R}^n$ is a nonempty set and if $x \in X$ the *contingent* (Bouligand-Severi) *cone* to a subset $X \subset \mathbb{R}^n$ at the point x is defined by:

$$K_x^{\pm}X := \{u \in \mathbb{R}^n; (\exists)\,(\theta_m, u_m) \to (0_\pm, u) : \; x + \theta_m u_m \in X \;\; (\forall)\, m \in \mathbb{N}\}. \tag{2.1}$$

For $g(\cdot) : X \subset \mathbb{R}^n \to \mathbb{R}$, we shall use *the extreme contingent derivatives at a point* $x \in X$ *in direction* $u \in K_x^{\pm}X$ defined by:

$$\overline{D}_K^{\pm} g(x; u) := \limsup_{(\theta,v)\to(0_\pm,u)} \frac{g(x+\theta.v) - g(x)}{\theta}, \; u \in K_x^{\pm}X, \tag{2.2}$$

$$\underline{D}_K^{\pm} g(x; u) := \liminf_{(\theta,v)\to(0_\pm,u)} \frac{g(x+\theta.v) - g(x)}{\theta}, \; u \in K_x^{\pm}X,$$

which coincide with the well known Dini derivatives if $X \subset \mathbb{R}$ is an interval and $u = 1 \in \mathbb{R}$.

We recall that a mapping $f(\cdot) : X \subset \mathbb{R}^n \to \mathbb{R}^m$ is said to be (Fréchet) *differentiable* at $x \in int(X)$ if there exists a linear mapping, $A \in L(\mathbb{R}^n, \mathbb{R}^m)$, such that

$$\lim_{h \to 0} \frac{(\|f(x+h) - f(x) - Ah\|)}{\|x\|} = 0.$$

In this case A is said to be the (Fréchet) *derivative* of $f(\cdot)$ at x and is denoted by: $A = Df(x)$; in the case $n = 1$ we denote: $f'(x) = Df(x) \cdot 1$.

As it is well known, the classical (Fréchet) differentiability is generalized by the "*contingent differentiability*" of a mapping $f(\cdot) : X \subset \mathbb{R}^n \to \mathbb{R}^m$ assuming the existence of the *contingent directional derivative*:

$$f_K^{\pm}(x; u) := \lim_{(\theta,v)\to(0_\pm,u)} \frac{f(x+\theta.v) - f(x)}{\theta}, \; u \in K_x^{\pm}X. \tag{2.3}$$

We recall that a real-valued function, $g(\cdot) : X \subset \mathbb{R}^n \to \mathbb{R}$ is a differentiable mapping in $x \in X$ if and only if

$$\underline{D}_K^{\pm} g(x; u) = \overline{D}_K^{\pm} g(x; u) = g_K^{\pm}(x; u), \;\; (\forall)\, u \in \mathbb{R}^n \tag{2.4}$$

and $g_K^{\pm}(x; \cdot) : \mathbb{R}^n \to \mathbb{R}$ is a linear mapping.

We denote by L_0 the family of all subsets of the null Lebseque measure from $\mathbb{R}$, and for the interval $I \subset \mathbb{R}$ we denote by $L_1(I; \mathbb{R}^n)$ and by $L_1^{loc}(I; \mathbb{R}^n)$ the space of the measurable mappings $u(\cdot) : I \to \mathbb{R}^n$, which are Lebesgue integrable and locally integrable respectively; in addition we denote by $L_\infty^{loc}(I; \mathbb{R}^n)$ the space of the locally essentially bounded maps.

The relationship of differentiability to the Riemann integral parameters has been studied in almost all the books that deal with mathematical analysis (e.g. [5]), but similar properties of the Lebesgue integral are very seldom approached.

The most famous result concerning the differentiability of the Lebesgue integral with parameters, in the classic meaning (a result used by L. Cesari [3, Lemma 2.3.1]), is Theorem 3.9.2.in E.J. Mc. Shane [7]:

Lemma 2.1. *([7]) Let $I \subset \mathbb{R}$ be an interval, $\alpha, \beta \in \mathbb{R}$ and $h(\cdot, \cdot) : I \times (\alpha, \beta) \to \mathbb{R}$ a function with the following properties:*

(1) $h(\cdot, z) \in L_1(I; \mathbb{R})$, $z \in (\alpha, \beta)$;
(2) *There is $k(\cdot) \in L_1(I; \mathbb{R}_+)$ and $I_0 \subset \mathbb{R}$, of zero Lebesgue measure, such that for each $t \in I \backslash I_0$, there is $D_2 h(t, z)$ with the following property:*

$$|D_2 h(t, z)| \le k(t), \; (\forall) t \in I \backslash I_0, \; z \in (\alpha, \beta). \tag{2.5}$$

Then $D_2 h(\cdot, z) \in L_1(I;\mathbb{R})$, $(\forall)\, z \in (\alpha,\beta)$ *and the function* $\Psi(\cdot);(\alpha,\beta) \to \mathbb{R}$ *defined by:*

$$\Psi(z) := \int_I h(t,z)dt,\ \ (\forall)\, z \in (\alpha,\beta) \tag{2.6}$$

is differentiable, and its derivative is given by:

$$\Psi'(z) = \int_I D_2 h(t,z)dt,\ \ (\forall)\, z \in (\alpha,\beta). \tag{2.7}$$

A recent generalization of E.J. Mc. Shane's result is given by I. Miriča [8]:

Lemma 2.2. *(*[8]*) Let* $I \subset \mathbb{R}$ *be an interval,* $Z_0 \subset \mathbb{R}^n$, $z_0 \in Z_0$ *and* $h(\cdot,\cdot) : I \times Z_0 \to \mathbb{R}$ *a function with the following properties:*

(1) $h(\cdot, z) \in L_1(I;\mathbb{R})$, $z \in Z_0$;
(2) *There is* $k(\cdot) \in L_1(I;\mathbb{R}_+)$ *such that:*

$$|h(t,z) - h(t,z_0)| \le k(t)\,\|z - z_0\|\,,\ (\forall) z \in Z_0,\ a.p.t.\,(I)\,. \tag{2.8}$$

Then the extreme contingential derivatives of the function $\Psi(\cdot)$ *in (2.6) at the point* $z_0 \in Z_0$*, in the direction* $\overline{z} \in K^{\pm}_{z_0} Z_0$*, verify the following inequalities:*

$$\int_I \underline{D}^{\pm}_K h(t,\cdot)(z_0;\overline{z})dt \le \underline{D}^{\pm}_K \Psi(z_0;\overline{z}) \le \overline{D}^{\pm}_K \Psi(z_0;\overline{z}) \le \int_I \overline{D}^{\pm}_K h(t,\cdot)(z_0;\overline{z})dt \tag{2.9}$$

In particular, if there is $I_0 \subset I$*, of zero Lebesgue measure, such that the functions* $h(t,\cdot)$ *are contingentially differentiable in the direction* $\overline{z} \in K^{\pm}_{z_0} Z_0$ *for each* $t \in I \backslash I_0$*, then the function* $t \to (h(t,\cdot))^{\pm}_K (z_0;\overline{z})$*, is Lebesgue integrable,* $\Psi(\cdot)$ *is contingentially differentiable in* z_0 *in the direction* $\overline{z}$ *is and its contingent derivative is given by:*

$$\Psi^{\pm}_K(z) = \int_I (h(t,\cdot))^{\pm}_K (z_0;\overline{z})dt. \tag{2.10}$$

Proof. The essential instrument for the theorem's proof is the generalization of Fatou's Lemma (e.g. [11, Ex. 8.18]) according to which if $g_m(\cdot) : I \to \mathbb{R}$ is a sequence of measurable functions, such that there is $g(\cdot) \in L_1(I;\mathbb{R}_+)$ which fulfills the following:

$$g_m(t) \le g\,(t)\ \ a.p.t.\,(I)\,,\ (\forall) m \in \mathbb{N}, \tag{2.11}$$

then we have that:

$$\int_I \liminf_{m\to\infty} g_m(t)dt \le \liminf_{m\to\infty} \int_I g_m(t)dt \le \limsup_{m\to\infty} \int_I g_m(t)dt \le \int_I \limsup_{m\to\infty} g_m(t)dt. \tag{2.12}$$

In order to be able to apply this result to our case, we consider a sequence $(\phi_m, \overline{z}_m) \to (0_\pm, \overline{z})$ with the following property:

$$z_0 + \theta_m \overline{z}_m \in Z_0,\ \ (\forall)\, m \in \mathbb{N} \tag{2.13}$$

and we define the sequence of measurable functions $g_m(t)$ as follows:

$$g_m(t) := \frac{h(t, z_0 + \theta_m \overline{z}_m) - h(t,z_0)}{\theta_m},\ m \in \mathbb{N},\ t \in I. \tag{2.14}$$

Then, from the definitions (2.3) of the extreme contingent derivatives, we have that:

$$\underline{D}^{\pm}_K(t,.)(z_0;\overline{z}) \le \liminf_{m\to\infty} g_m(t) \le \limsup_{m\to\infty} g_m(t) \le \overline{D}^{\pm}_K(t,.)(z_0;\overline{z}), \tag{2.15}$$

the extreme equalities taking place for the inferior and superior limits with respect to the set of the sequence $(\theta_m, \overline{z}_m) \to (0_\pm, \overline{z})$, with the property in (2.13). Moreover, from the Lipschitz property (2.8) and from (2.14) it follows that:

$$|g_m(t)| \le k(t)\,\|\overline{z}_m\|\ \ a.p.t.(I),\ (\forall)\, m \in \mathbb{N}. \tag{2.16}$$

From the fact that $\overline{z}_m \to \overline{z}$ it results that $(\forall)\,\varepsilon > 0$ there is $m_\varepsilon \in \mathbb{N}$ such that $\|\overline{z}_m - \overline{z}\| \leq \varepsilon,\ (\forall)\, m \geq m_\varepsilon$ and, as a consequence, the inequality (2.16) implies:

$$\begin{aligned} |g_m(t)| &\leq k(t)\,\|\overline{z}_m\| \\ &\leq k(t)[\|\overline{z}_m - \overline{z}\| + \|\overline{z}\|] \\ &\leq g_\varepsilon(t) := k(t)\,[\|\overline{z}\| + \varepsilon]\,,\ m \geq m_\varepsilon, \end{aligned} \tag{2.17}$$

which shows that the subsequence $\{g_m(\cdot); m \geq m_\varepsilon\}$ has the property in (2.11) because the function, $g_\varepsilon(\cdot)$, defined in (2.17) is obviously integrable. As a consequence, from the inequalities (2.15) and from the monotonicity property of the Lebesgue integral ([11, Corollary 8.2.4]) we deduce that the following inequalities are true:

$$\begin{aligned} \int_I \underline{D}^{\pm}_K h(t,\cdot)(z_0;\overline{z})dt &\leq \int_I \liminf_{m\to\infty} g_m(t)dt \\ &\leq \int_I \limsup_{m\to\infty} g_m(t)dt \\ &\leq \int_I \overline{D}^{\pm}_K h(t,\cdot)(z_0;\overline{z})dt. \end{aligned} \tag{2.18}$$

The inequalities (2.9) in the theorem's text now follow from (2.18), (2.12) and from the fact that for each sequence $(\theta_m, \overline{z}_m) \to (0_\pm, \overline{z})$ with the property (2.13) the extreme contingent derivatives of the function $\Psi(\cdot)$ in (2.6) verify the following inequalities:

$$\begin{aligned} \underline{D}^{\pm}_K \Psi(z_0;\overline{z}) &\leq \liminf_{m\to\infty} \frac{\Psi(z_0 + \theta_m \overline{z}_m) - \Psi(z_0)}{\theta_m} \\ &= \liminf_{m\to\infty} \int_I g_m(t)dt \\ &\leq \limsup_{m\to\infty} \int_I g_m(t)dt \\ &\leq \overline{D}^{\pm}_K \Psi(z_0;\overline{z}). \end{aligned} \tag{2.19}$$

In the case when the functions $h\,(t,\cdot)\,, t \in I\backslash I_0$ are contingentially differentiable in z_0 in the direction $\overline{z} \in K^{\pm}_{z_0} Z_0$, the contingent differentiability of the function $\Psi(\cdot)$ and the formula (2.10) result from the inequalities (2.9) and from the fact that, according to the property in (2.4), we have:

$$(h(t,\cdot))^{\pm}_K(z_0;\overline{z}) = \underline{D}^{\pm}_K h(t,\cdot)(z_0;\overline{z}) = \overline{D}^{\pm}_K h(t,\cdot)(z_0;\overline{z}),\ \ (\forall)\, t \in I\backslash I_0;$$

a similar equality taking place for the function $\Psi(\cdot)$ as well.

Finally, the integrability of the function $t \to (h\,(t,\cdot))^{\pm}_k\,(z_0,\overline{z})$ results from the fact that for each sequence $g_m(\cdot)$ having the form in (2.14) the following relation is true:

$$(h\,(t,\cdot))^{\pm}_k\,(z_0;\overline{z}) = \lim_{m\to\infty} g_m(t),\ \ (\forall)\, t \in I\backslash I_0,$$

and from Lebesgue's Theorem of Dominated Convergence ([11, Theorem 8.2.16]) according to which the inequalities (2.17) and the integrability of the functions $g_\varepsilon(\cdot)$, defined in (2.17) imply the integrability of the following limit:$\liminf_{m\to\infty} g_m(t)$ when this one exists a.p.t.(I). □

Remark 2.3. Except for some very particular cases, the property of "integral lipschitzianity" in (2.8) seems to be compulsory in order to get the results in Lemma 2.2; principally, the property (2.8) assures the fulfillment of the conditions in the form of (2.11) which, in their turn, imply the inequalities in (2.10). On the other hand, as one may easily check, property (2.8) is implied in more restrictive hypotheses, but it may be more easily checked, as in the case of condition (2.5) in Lemma 2.1: there is $r > 0$, $k(\cdot) \in L_1\,(I;\mathbb{R}_+)$ and $I_0 \subset I$, of zero Lebesgue measure, such that $Z_0 = B_r(z_0) \subset \mathbb{R}^n$and for each $t \in I\backslash I_0$, $z \in Z_0$, the function $h(t,\cdot)$ is differentiable in the point z and its derivative verifies the following inequality:

$$\|D_2 h(t,z)\| \le k(t),\ \ (\forall)\, t \in I\backslash I_0,\ z \in Z_0 := B_r(z_0). \tag{2.20}$$

In this case, by using the measurability property of the derivative and the evident relations:

$$h(t,z) - h(t,z_0) = \int_0^1 D_2 h(t, z_0 + s(z - z_0))\,(z - z_0)\,,\ \ (\forall)\, z \in Z_0,$$

by (2.20) it results the radial lipschitzianity property in (2.8) follows. As a consequence, Lemma 2.2 represents a generalization of Lemma 2.1 from the following points of view: the set of the parameters, Z_0, is not necessarily scalar and open $(Z_0 = (\alpha,\beta)) \subset \mathbb{R}$ in Lemma 2.1, and the functions $h(t,\cdot)\,, t \in I$, are not necessarily differentiable, not even at the fixed point $z_0 \in Z_0$.

3. THE MAIN RESULTS

We are going to extend the result in Lemma 2.2 to the case of the Lebesgue integral whose integrand and integrability interval depend on parameters. We need some notions and preliminary results for this particular purpose.

Let L_0 be the family of the subsets of zero Lebesgue measure in $\mathbb{R}$ and $h(\cdot) \in L_\infty^{loc}(I,\mathbb{R})$ a given function. Then we define the *essential right superior limit* and *essential left inferior limit* of the function $h(\cdot)$ at the point $t \in I$, by following relations:

$$\begin{aligned} ess \limsup_{s\to t_+} h(t) &:= \inf_{\varepsilon>0} \inf_{J\in L_0} \sup_{s\in[t,t+\varepsilon)\backslash J} h(s); \\ ess \liminf_{s\to t_-} h(t) &:= \sup_{\varepsilon>0} \sup_{J\in L_0} \inf_{s\in[t,t+\varepsilon)\backslash J} h(s). \end{aligned} \tag{3.1}$$

Similarly, we can define: $ess \limsup_{s\to t_-} h(s)$, $ess \liminf_{s\to t_+} h(s)$.

Lemma 3.1. *If $I \subset \mathbb{R}$ is an interval and $h(\cdot) \in L_\infty^{loc}(I,\mathbb{R})$ then the following inequalities are true:*

$$ess \liminf_{s\to t_\pm} h(t) \le \liminf_{\delta\to 0_\pm} \frac{1}{\delta}\int_t^{t+\delta} h(s)ds \le \limsup_{\delta\to 0_\pm} \frac{1}{\delta}\int_t^{t+\delta} h(s)ds \le ess \limsup_{s\to t_\pm} h(t). \tag{3.2}$$

Proof. Let $t \in I$ and $\varepsilon > 0$ such that $[t,t+\varepsilon) \subset I$, let $J \in L_0$ such that $J \subset I$ and let $\alpha_{\varepsilon,J} := \sup_{s\in[t,t+\varepsilon)\backslash J} h(s)$, $\alpha_\varepsilon := \inf_{J\in L_0} \alpha_{\varepsilon,J}$, $\alpha := \inf_{\varepsilon>0} \alpha_\varepsilon$. Since $h(s) \le \alpha_{\varepsilon,J}$, $(\forall)s \in [t,t+\varepsilon)\backslash J$, we have that:

$$\frac{1}{\varepsilon}\int_t^{t+\varepsilon} h(s)ds \le \inf_{J\in L_0} \alpha_{\varepsilon,J} = \alpha_\varepsilon. \tag{3.3}$$

Since $h(\cdot)$ is essentially bounded, then α_ε is finite; hence by the fact that the function $\varepsilon \to \alpha_\varepsilon$ is increasing (which means that $\sup_{\varepsilon\in(0,\delta)} \alpha_\varepsilon = \alpha_\delta$), and by (3.3), we have that:

$$\begin{aligned} \limsup_{\delta\to 0_+} \frac{1}{\delta}\int_t^{t+\delta} h(s)ds &:= \inf_{\delta>0} \sup_{\varepsilon\in(0,\delta)} \frac{1}{\delta}\int_t^{t+\delta} h(s)ds \\ &\le \inf_{\delta>0} \sup_{\varepsilon\in(0,\delta)} \alpha_\varepsilon \\ &= \inf_{\varepsilon\in(0,\delta)} \alpha_\varepsilon = \alpha, \end{aligned}$$

thus, we have established the following inequality:

$$\limsup_{\delta\to 0_+}\frac{1}{\delta}\int_t^{t+\delta}h(s)ds \le ess\limsup_{s\to t_+}h(t).$$

We can similarly establish other inequalities in the text. □

Lemma 3.2. *Let* $\alpha(\cdot):Z\subset\mathbb{R}^n\to\mathbb{R}$*,* $\beta(\cdot):Z\subset\mathbb{R}^n\to\mathbb{R}$ *be the given functions and* $z_0\in Z$ *such that there is* $\lim_{z\to z_0}\alpha(\cdot)=\alpha_0\in\mathbb{R}$ *and* $\beta(\cdot)$ *is bounded on a neighborhood of the point* z_0*. Then the following relations are true:*

$$\limsup_{z\to z_0}\left[\alpha(z)\beta(z)\right]=\begin{cases}\alpha_0\limsup\limits_{z\to z_0}\beta(z)\,,\ \textit{if}\ \alpha\ge 0\\ \alpha_0\liminf\limits_{z\to z_0}\beta(z)\,,\ \textit{if}\ \alpha<0,\end{cases}\tag{3.4}$$

$$\liminf_{z\to z_0}\left[\alpha(z)\beta(z)\right]=\begin{cases}\alpha_0\liminf\limits_{z\to z_0}\beta(z)\,,\ \textit{if}\ \alpha\ge 0\\ \alpha_0\limsup\limits_{z\to z_0}\beta(z)\,,\ \textit{if}\ \alpha<0.\end{cases}\tag{3.5}$$

Proof. Some more steps are necessary in order to prove the relations (3.4) and (3.5).

(1) If $a\in\mathbb{R}$ and $B\subset\mathbb{R}$ is nonempty bounded set, then we have that:

$$\inf(aB)=\begin{cases}a\inf B,\ \text{if}\ a\ge 0\\ \\ a\sup B,\ \text{if}\ a<0,\end{cases}\qquad \sup(aB)=\begin{cases}a\sup B,\ \text{if}\ a\ge 0\\ \\ a\inf B,\ \text{if}\ a<0.\end{cases}\tag{3.6}$$

Indeed, if $a=0$ the equalities in (3.6) are obvious. In the case $a>0$, from the fact that $\inf B\le b$, $(\forall)\,b\in B$, it follows that $a\inf B\le ab$, $(\forall)\,b\in B$, which shows that $a\inf B$ is a minorant of the set aB, hence $a\inf B\le\inf(aB)$.

On the other hand, for each $\varepsilon>0$, there is $b_\varepsilon\in B$, such that $b_\varepsilon<\inf B+\varepsilon/a$; it results: $ab_\varepsilon<a\inf B+\varepsilon$, hence $\inf(aB)\le ab_\varepsilon<a\inf B+\varepsilon$. Since $\varepsilon>0$ is arbitrary it follows that $\inf(aB)\le a\inf B$. Therefore, for $a>0$, from the inequalities $a\inf B\le\inf(aB)$, $\inf(aB)\le a\inf B$, we deduce the fact that $\inf(aB)=a\inf B$.

In the case $a<0$, we have:

$$\inf(aB)=\inf[(-a)(-B)]=-a\inf(-B)=-a(-\sup B)=a\sup B;$$

hence, the first equality in (3.6) is established. The second equality may be similarly established.

(2) If for a sequence of real numbers $(u_m)_m$ we denote by $L(u_m)$ the set of its limit points, defined by:

$$L(u_m)=\{u\in\overline{\mathbb{R}};(\exists)\,(u_{m_k})_k\ \text{subsequence of the sequence}\ (u_m)_m:u_{m_k}\to u\},\tag{3.7}$$

then for every two sequences of real numbers $(x_m)_m,(y_m)_m$ such that $x_m\to x\in\mathbb{R}$ and $(y_m)_m$ is bounded, the following equality is true:

$$L(x_my_m)=xL(y_m)\,.\tag{3.8}$$

Indeed, if $x_m\to 0$, then from the fact that the sequence $(y_m)_m$ is bounded, it results that $x_my_m\to 0$, hence $L(x_my_m)=\{0\}=0.L(y_m)$. Therefore, we suppose that $x_m\to x\in\mathbb{R}\backslash\{0\}$ and we consider an element $u\in L(x_my_m)$; then, according to the definition (3.7), there is the subsequence $x_{m_k}.y_{m_k}\to u$, as a consequence, since $x_{m_k}\to x$, it results that $y_{m_k}\to\frac{u}{x}\in L(y_m)$, hence $u=x\frac{u}{x}\in xL(y_m)$; thus, we have established the inclusion $L(x_my_m)\subseteq xL(y_m)$. Vice versa, if $u\in xL(y_m)$, then, according to the definition (3.7), there is a subsequence $(y_{m_k})_k$ of the sequence $(y_m)_m$ such that $y_{m_k}\to\frac{u}{x}\in L(y_m)$. If for the subsequence $(y_{m_k})_k$ we choose a corresponding subsequence $(x_{m_k})_k$ of the sequence $(x_m)_m$, then $x_{m_k}.y_{m_k} \to x\frac{u}{x} = u$ which shows that $u \in L(x_m y_m)$. Therefore, it has established the inclusion $L(x_m y_m) \supseteq xL(y_m)$ and thus the relation (3.8) is proved.

(3) If $(x_m)_m$, $(y_m)_m$ are sequences of real numbers such that $x_m \to x \in \mathbb{R}$ and $(y_m)_m$ is bounded, then the following relations are true:

$$\limsup (x_m y_m) = \begin{cases} x \limsup\limits_{m\to\infty} (y_m), \text{ if } x \geq 0 \\ x \liminf\limits_{m\to\infty} (y_m), \text{ if } x < 0, \end{cases} \tag{3.9}$$

$$\liminf_{m\to\infty} (x_m y_m) = \begin{cases} x \liminf\limits_{m\to\infty} (y_m), \text{ if } x \geq 0 \\ x \limsup\limits_{m\to\infty} (y_m), \text{ if } x < 0, \end{cases} \tag{3.10}$$

Indeed, if $x_m \to 0$, then, since $(y_m)_m$ is bounded sequence, it results that $x_m y_m \to 0$ and thus the equalities (3.9) and (3.10) are obvious. Therefore, we suppose that $x_m \to x \in \mathbb{R}\backslash\{0\}$; then for $x > 0$, from (3.6) and from (3.8), we have that:

$$\begin{aligned} \limsup_{m\to\infty} (x_m y_m) &:= \sup L(x_m y_m) \\ &= \sup[xL(y_m)] \\ &= x \sup L(y_m) = x \limsup_{m\to\infty} y_m, \end{aligned}$$

and if $x < 0$ we have that:

$$\begin{aligned} \limsup_{m\to\infty} (x_m y_m) &:= \sup L(x_m y_m) \\ &= \sup[xL(y_m)] \\ &= x \inf L(y_m) = x \liminf_{m\to\infty} y_m, \end{aligned}$$

hence the equality (3.9); we get to the equality (3.10) by a similar procedure.

(4) According to the results previously established, we prove the relations (3.4), (3.5). Indeed, if $\lim_{z\to z_0} a(z) = 0$ then, from the fact that $\beta(\cdot)$ is bounded in a neighborhood of the point z_0, we have that $\lim_{z\to z_0} \alpha(z)\beta(z) = 0$ and the equalities (3.4), (3.5) are obvious. Therefore, we suppose that $\lim_{z\to z_0} \alpha(z) = \alpha_0 \in \mathbb{R}\backslash\{0\}$. In case that $\alpha_0 > 0$, according to the equalities (3.6), (3.9) we have that:

$$\begin{aligned} \limsup_{z\to z_0} [\alpha(z)\beta(z)] &= \sup\left\{\limsup_{m\to\infty} \alpha(z_m)\beta(z_m);\ z_m \to z_0,\ z_m \in Z\backslash\{z_0\},\ (\forall)\, m \in \mathbb{N}\right\} \\ &= \sup\left\{\alpha_0 \limsup_{m\to\infty} \beta(z_m);\ z_m \to z_0,\ z_m \in Z\backslash\{z_0\},\ (\forall)\, m \in \mathbb{N}\right\} \\ &= \alpha_0 \sup\left\{\limsup_{m\to\infty} \beta(z_m); z_m \to z_0, z_m \in Z\backslash\{z_0\}, (\forall)\, m \in \mathbb{N}\right\} \\ &= \alpha_0 \limsup_{z\to z_0} \beta(z), \end{aligned}$$

and if $\alpha_0 < 0$, we have that:

$$\begin{aligned}\limsup_{z\to z_0}[\alpha(z)\beta(z)] &= \sup\left\{\limsup_{m\to\infty}\alpha(z_m)\beta(z_m); z_m \to z_0,\ z_m \in Z\backslash\{z_0\},\ (\forall)\, m \in \mathbb{N}\right\}\\ &= \sup\left\{\alpha_0 \liminf_{m\to\infty}\beta(z_m);\ z_m \to z_0,\ z_m \in Z\backslash\{z_0\},\ (\forall)\, m \in \mathbb{N}\right\}\\ &= \alpha_0 \sup\left\{\liminf_{m\to\infty}\beta(z_m); z_m \to z_0, z_m \in Z\backslash\{z_0\}, (\forall)\, m \in \mathbb{N}\right\}\\ &= \alpha_0 \liminf_{z\to z_0}\beta(z),\end{aligned}$$

hence, the equality (3.4) is established. The equality (3.5) can be established in a similar procedure.

□

Now we are able to present a generalization of the result in Lemma 2.2 to the case of the Lebesgue integral whose integrand and integrability interval depend on the parameters.

Theorem 3.3. *Let $I \subset \mathbb{R}$ be an interval, $Z \subset \mathbb{R}^n$ and $g(\cdot,\cdot) : I \times Z \to \mathbb{R}$ a function with the following properties:*

(1) $g(\cdot, z) \in L^{loc}_{\infty}(I;\mathbb{R}), (\forall)\, z \in Z;$
(2) *For each $z \in Z$, there is $k_z(\cdot) \in L^{loc}_{\infty}(I;\mathbb{R}_+)$ and $r > 0$ such that:*

$$|g(s,y) - g(s,z)| \le k_z(s)\,\|y - z\|\,,\ \ (\forall)\, y \in B_r(z) \cap Z,\ a.p.t.(I). \tag{3.11}$$

Let $a(\cdot) : Z \to I$, $b(\cdot) : Z \to I$ be contingent differentiable functions at $z \in Z$ in direction $\overline{z} \in K^{\pm}_z Z$ such that $a(z) < b(z)$ and let:

$$\overline{A}^{\pm}(z;\overline{z}) := \begin{cases} a^{\pm}_K(z;\overline{z}) ess \liminf_{s\to a(z)_{\pm}} g(s,z), \text{ if } a^{\pm}_K(z;\overline{z}) \ge 0 \\ a^{\pm}_K(z;\overline{z}) ess \limsup_{s\to a(z)_{\mp}} g(s,z), \text{ if } a^{\pm}_K(z;\overline{z}) < 0, \end{cases} \tag{3.12}$$

$$\underline{A}^{\pm}(z;\overline{z}) := \begin{cases} a^{\pm}_K(z;\overline{z}) ess \limsup_{s\to a(z)_{\pm}} g(s,z), \text{ if } a^{\pm}_K(z;\overline{z}) \ge 0 \\ a^{\pm}_K(z;\overline{z}) ess \liminf_{s\to a(z)_{\mp}} g(s,z), \text{ if } a^{\pm}_K(z;\overline{z}) < 0, \end{cases} \tag{3.13}$$

$$\overline{B}^{\pm}(z;\overline{z}) := \begin{cases} b^{\pm}_K(z;\overline{z}) ess \limsup_{s\to a(z)_{\pm}} g(s,z), \text{ if } b^{\pm}_K(z;\overline{z}) \ge 0 \\ b^{\pm}_K(z;\overline{z}) ess \liminf_{s\to a(z)_{\mp}} g(s,z), \text{ if } b^{\pm}_K(z;\overline{z}) < 0, \end{cases} \tag{3.14}$$

$$\underline{B}^{\pm}(z;\overline{z}) := \begin{cases} b^{\pm}_K(z;\overline{z}) ess \liminf_{s\to a(z)_{\pm}} g(s,z), \text{ if } b^{\pm}_K(z;\overline{z}) \ge 0 \\ b^{\pm}_K(z;\overline{z}) ess \limsup_{s\to a(z)_{\mp}} g(s,z), \text{ if } b^{\pm}_K(z;\overline{z}) < 0. \end{cases} \tag{3.15}$$

Then, for any $z \in Z$ and $\overline{z} \in K^{\pm}_z Z$, the function $G(\cdot) : Z \to \mathbb{R}$, defined by:

$$G(z) := \int_{a(z)}^{b(z)} g(s,z)ds,\ z \in Z, \tag{3.16}$$

verifies the inequalities:

$$\begin{aligned}(3.17)\qquad &\underline{B}^{\pm}(z;\overline{z}) - \underline{A}^{\pm}(z;\overline{z}) + \int_{a(z)}^{b(z)} \underline{D}_K^{\pm} g(s,\cdot)(z;\overline{z})\,ds \\ &\leq \underline{D}_K^{\pm} G(z;\overline{z}) \\ &\leq \overline{D}_K^{\pm} G(z;\overline{z}) \\ &\leq \overline{B}^{\pm}(z;\overline{z}) - \overline{A}^{\pm}(z;\overline{z}) + \int_{a(z)}^{b(z)} \overline{D}_K^{\pm} g(s,\cdot)(z;\overline{z})\,ds.\end{aligned}$$

Proof. Let $z \in Z$ and $\overline{z} \in K_z^{\pm}Z$; from the definition (2.3) of the extreme contingent derivatives and from definition (3.16) of the function $G(\cdot)$, we have that:

$$\begin{aligned}(3.18)\qquad \overline{D}_K^{\pm} G(z;\overline{z}) &= \limsup_{(\theta,v)\to(0_+,\overline{z})} \frac{G(z+\theta v) - G(z)}{\theta} \\ &= \limsup_{(\theta,v)\to(0_+,\overline{z})} \left[\int_{a(z)}^{b(z)} \frac{g(s,z+\theta v) - g(s,z)}{\theta} ds \right.\\ &\qquad \left. + \frac{1}{\theta}\int_{b(z)}^{b(z+\theta v)} g(s,z+\theta v)ds - \frac{1}{\theta}\int_{a(z)}^{a(z+\theta v)} g(s,z+\theta v)ds\right] \\ &\leq \limsup_{(\theta,v)\to(0_+,\overline{z})} \int_{a(z)}^{b(z)} \frac{g(s,z+\theta v) - g(s,z)}{\theta} ds \\ &\qquad + \limsup_{(\theta,v)\to(0_+,\overline{z})} \frac{1}{\theta}\int_{b(z)}^{b(z+\theta v)} g(s,z+\theta v)ds \\ &\qquad - \liminf_{(\theta,v)\to(0_{\div},\overline{z})} \frac{1}{\theta}\int_{a(z)}^{a(z+\theta v)} g(s,z+\theta v)ds\end{aligned}$$

Under the conditions (1) and (2) from the hypothesis of the theorem, we can apply Lemma 2.2 to obtain that:

$$(3.19)\qquad \limsup_{(\theta,v)\to(0_+,\overline{z})} \int_{a(z)}^{b(z)} \frac{g(s,z+\theta v) - g(s,z)}{\theta} ds \leq \int_{a(z)}^{b(z)} \overline{D}_K^{+} g(s,\cdot)(z;\overline{z})\,ds.$$

We are now going to prove that:

$$(3.20)\qquad \limsup_{(\theta,v)\to(0_{\div},\overline{z})} \frac{1}{\theta}\int_{b(z)}^{b(z+\theta v)} g(s,z+\theta v)ds \leq \overline{B}^{+}(z;\overline{z}).$$

First we are going to establish the following equality:

$$(3.21)\qquad \limsup_{(\theta,v)\to(0_+,\overline{z})} \int_{b(z)}^{b(z+\theta v)} g(s,z+\theta v)ds = \limsup_{(\theta,v)\to(0_+,\overline{z})} \int_{b(z)}^{b(z+\theta v)} g(s,z)ds.$$

Indeed, if we take into account the fact that $\overline{z} + \theta v \to \overline{z}$ when $(\theta, v) \to (0_+, \overline{z})$, then, from the condition (2), from the hypothesis, we deduce that there exists $r > 0$ such that:

$$|g(s,z+\theta v) - g(s,z)| \leq k_z(s)\theta\|v\|,\ \ (\forall)\, v \in B_r(\overline{z}) \cap Z,\ 0 < \theta < r,\ a.p.t(I),$$

hence

$$(3.22)\qquad |g(s,z+\theta v) - g(s,z)| \leq k_z(s)\theta\left(\|\overline{z}\| + r\right),\ 0 < \theta < r,\ a.p.t(I).$$

Therefore, from (3.22) and the fact that $b(z+\theta v) \to b(z)$ when $(\theta, v) \to (0_+, \overline{z})$, we have that:

$$\begin{aligned}
&\limsup_{(\theta,v)\to(0_+,\overline{z})} \left| \frac{1}{\theta} \int_{b(z)}^{b(z+\theta v)} [g(s, z+\theta v) - g(s,z)] ds \right| \\
&\qquad \leq \limsup_{(\theta,v)\to(0_+,\overline{z})} \frac{1}{\theta} \int_{b(z)}^{b(z+\theta v)} |g(s, z+\theta v) - g(s,z)| \, ds \\
&\qquad \leq \limsup_{(\theta,v)\to(0_+,\overline{z})} \frac{1}{\theta} \int_{b(z)}^{b(z+\theta v)} k_z(s)\theta \left(\|\overline{z}\| + r\right) ds \\
&\qquad = \limsup_{(\theta,v)\to(0_+,\overline{z})} \int_{b(z)}^{b(z+\theta v)} k_z(s) \left(\|\overline{z}\| + r\right) ds \\
&\qquad = 0,
\end{aligned}$$

hence, by using the absolute continuity of the Lebesgue integral and the fact that $b(z+\theta v) \to b(z)$ when $(\theta, v) \to (0_+, \overline{z})$, we deduce the relation (3.21). Furthermore, from the relation (3.21) we have that:

$$\begin{aligned}
(3.23) \qquad &\limsup_{(\theta,v)\to(0_+,\overline{z})} \frac{1}{\theta} \int_{b(z)}^{b(z+\theta v)} g(s, z+\theta v) ds \\
&= \limsup_{(\theta,v)\to(0_+,\overline{z})} \frac{1}{\theta} \int_{b(z)}^{b(z+\theta v)} g(s,z) ds \\
&= \limsup_{(\theta,v)\to(0_+,\overline{z})} \frac{b(z+\theta v) - b(z)}{\theta} \left[\frac{1}{b(z+\theta v) - b(z)} \int_{b(z)}^{b(z+\theta v)} g(s,z) ds \right].
\end{aligned}$$

We consider the functions:

$$\alpha(\theta, v) := \frac{b(z+\theta v) - b(z)}{\theta}, \ \beta(\theta, v) := \frac{1}{b(z+\theta v) - b(z)} \int_{b(z)}^{b(z+\theta v)} g(s,z) ds$$

as well and we note on one hand that, since $b(\cdot)$ is contingent differentiable at the point $z \in Z$ in the direction $\overline{z} \in K_z^{\pm}$, then there exist: $\limsup_{(\theta,v)\to(0_+,\overline{z})} \alpha(\theta, v) = b_K^{+}(z; \overline{z})$ and this limit is finite; on the other hand, by the hypothesis of Theorem 3.3 we have that $g(\cdot, z) \in L_\infty^{loc}(I; \mathbb{R})$, $(\forall)\, z \in Z$, such that, on the basis of Lemma 3.2, we have that:

$$ess \liminf_{s \to b(z)_+} g(s,z) \leq \liminf_{(\theta,v)\to(0_+,\overline{z})} \beta(\theta, v) \leq \limsup_{(\theta,v)\to(0_+,\overline{z})} \beta(\theta, v) \leq ess \limsup_{s \to b(z)_+} g(s,z),$$

hence, we deduce that the function $\beta(\cdot)$ is bounded in a neighborhood of the point $(0, \overline{z})$. Therefore, from (3.14) and from (3.23) we obtain that:

$$\begin{aligned}
\limsup_{(\theta,v)\to(0_+,\overline{z})} \frac{1}{\theta} \int_{b(z)}^{b(z+\theta v)} g(s, z+\theta v) ds &= \limsup_{(\theta,v)\to(0_+,\overline{z})} \left[\alpha(\theta, v)\, \beta(\theta, v)\right] \\
&= \begin{cases} b_K^{+}(z; \overline{z}) \displaystyle\limsup_{(\theta,v)\to(0_+,\overline{z})} \beta(\theta, v), \text{ if } b_K^{+}(z; \overline{z}) \geq 0 \\[2ex] b_K^{\pm}(z; \overline{z}) \displaystyle\liminf_{(\theta,v)\to(0_+,\overline{z})} \beta(\theta, v), \text{ if } b_K^{\pm}(z; \overline{z}) < 0, \end{cases}
\end{aligned}$$

hence, from Lemma 3.2, we get the inequality (3.20).

Following an analogous argument with the previous one we prove that:

$$\liminf_{(\theta,v)\to(0_+,\overline{z})} \frac{1}{\theta}\int_{a(z)}^{a(z+\theta v)} g(s,z+\theta v)ds \geq \overline{A}^{+}(z;\overline{z}). \tag{3.24}$$

From the inequalities in (3.19), (3.20) and (3.24), the inequality in (3.18) implies the following inequality:

$$\overline{D}_K^{+}G(z;\overline{z}) \leq \overline{B}^{+}(z;\overline{z}) - \overline{A}^{+}(z;\overline{z}) + \int_{a(z)}^{b(z)} \overline{D}_K^{+}g(s,\cdot)(z;\overline{z})\,ds. \tag{3.25}$$

Since the left variant (for $\overline{D}_K^{-}G(z;\overline{z})$) of the inequality (3.25) and the similar inequalities which contain the lower contingent derivatives results in the same way, the theorem is proved. □

As to different types of variants of Theorem 3.3 which are going to be obtained taking into account the hypothesis 1) and 2) we are going to deal with only two, which are contained in the following statement:

Corollary 3.4. *Let $I\subset\mathbb{R}$ be an interval, $Y\subset\mathbb{R}^n$, $Z\subset\mathbb{R}\times Y$, and $g(\cdot,\cdot):B:=I\times Z\to\mathbb{R}$ a function that satisfies condition 2) from Theorem 3.3 and such that $g(\cdot,z)$ is regulated ($g(\cdot,z)$ has one-sided limits at each point and at most a countable number of discontinuities, all of the first kind), $(\forall)\, z\in Z$. If, for any $(t,z):=(t,\tau,y)\in B$, $(\overline{t},\overline{z}):=(\overline{t},\overline{\tau},\overline{y})\in K_{(t,z)}^{\pm}B$, we consider:*

$$A^{\pm}\left((t,z);(\overline{t},\overline{z})\right) := \begin{cases} \overline{t}g(t_{\pm},z), \text{ if } \overline{\tau}\geq 0 \\ \overline{t}g(t_{\mp},z), \text{ if } \overline{\tau}<0, \end{cases}$$

$$B^{\pm}\left((\tau,z);(\overline{\tau},\overline{z})\right) := \begin{cases} \overline{\tau}g(t_{\pm},z), \text{ if } \overline{\tau}\geq 0 \\ \overline{\tau}g(t_{\mp},z), \text{ if } \overline{\tau}<0, \end{cases}$$

then, the following are true:

(i) For any $(t,z):=(t,\tau,y)\in B$, $(\overline{t},\overline{z}):=(\overline{t},\overline{\tau},\overline{y})\in K_{(t,z)}^{\pm}B$, the function $G(\cdot,\cdot):B\to\mathbb{R}$, defined by:

$$G(t,z) := \int_t^{\tau} g(s,z)ds \tag{3.26}$$

verifies the inequalities:

$$\begin{aligned} &B^{\pm}\left((t,z);(\overline{t},\overline{z})\right) - A^{\pm}\left((t,z);(\overline{t},\overline{z})\right) + \int_t^{\tau} \underline{D}_K^{\pm}g(s,\cdot)(z;\overline{z})\,ds \\ &\leq \underline{D}_K^{\pm}G\left((t,z);(\overline{t},\overline{z})\right) \leq \overline{D}_K^{\pm}G\left((t,z);(\overline{t},\overline{z})\right) \\ &\leq B^{\pm}\left((t,z);(\overline{t},\overline{z})\right) - A^{\pm}\left((t,z);(\overline{t},\overline{z})\right) + \int_t^{\tau} \overline{D}_K^{\pm}g(s,\cdot)(z;\overline{z})\,ds. \end{aligned}$$

(ii) If the functions $g(t,\cdot)$ are contingent differentiable at $z\in Z$ in the direction $\overline{z}\in K_{(t,z)}^{\pm}Z$, then function $s\to(g(s,\cdot))_K^{\pm}(z,\overline{z})$ is Lebesgue integrable and the function $G(\cdot,\cdot)$, defined by (3.26), is contingent differentiable at $(t,z):=(t,\tau,y)\in B$, in direction $(\overline{t},\overline{z}):=(\overline{t},\overline{\tau},\overline{y})\in K_{(t,z)}^{\pm}B$, and its contingent derivative is given by:

$$(G)_K^{\pm}\left((t,z);(\overline{t},\overline{z})\right) = B^{\pm}\left((t,z);(\overline{t},\overline{z})\right) - A^{\pm}\left((t,z);(\overline{t},\overline{z})\right) + \int_t^{\tau} (g(s,\cdot))_K^{\pm}(z;\overline{z})\,ds.$$

Proof. We consider the function $G(\cdot) : B \to \mathbb{R}$, defined by:

$$G(\zeta) := \int_{a(\zeta)}^{b(\zeta)} \widetilde{g}(s,z)ds,\ \zeta := (t,z) \in B$$

and we apply the Theorem 3.3 to the particular case for which $a(\zeta) = t$, $b(\zeta) = \tau$, and $\widetilde{g}(s,\zeta) = g(s,z)$, if $\zeta := (t,z) \in B$, by taking into account, at the same time, the fact that, in case, on the one hand, $a_K^{\pm}(\zeta,\overline{\zeta}) = \overline{t}$, $b_K^{\pm}(\zeta,\overline{\zeta}) = \overline{\tau}$, $(\forall)\,\zeta \in K_{(t,z)}^{\pm}B$, and, on the other hand, the fact that, as the functions $g(\cdot,z)$, $z \in Z$, are regulated (i.e. $g(\cdot,z)$ has one-sided limits at each point and at most a countable number of discontinuities, all of the first kind), hence locally bounded, then from (3.1) we have that:

$$ess\lim_{s\to t_{\pm}}\inf g(s,z) = ess\lim_{s\to t_{\pm}}\sup g(s,z) = g(t_{\pm},z),\ \ (\forall)\,z \in Z.$$

□

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