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THE HYPO-EUCLIDEAN NORM OF AN $n-$TUPLE OF VECTORS IN INNER PRODUCT SPACES AND APPLICATIONS

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ABSTRACT. The concept of hypo-Euclidean norm for an $n-$tuple of vectors in inner product spaces is introduced. Its fundamental properties are established. Upper bounds via the Boas-Bellman [1]-[3] and Bombieri [2] type inequalities are provided. Applications for $n-$tuples of bounded linear operators defined on Hilbert spaces are also given.

Key words and phrases: Inner product spaces, Norms, Bessel's inequality, Boas-Bellman and Bombieri inequalities, Bounded linear operators, Numerical radius.

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1. INTRODUCTION

Let $(E, \|\cdot\|)$ be a normed linear space over the real or complex number field $\mathbb{K}$. On $\mathbb{K}^n$ endowed with the canonical linear structure we consider a norm $\|\cdot\|_n$ and the unit ball

$$\mathbb{B}(\|\cdot\|_n) := \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{K}^n | \, \|\lambda\|_n \le 1\}.$$

As an example of such norms we should mention the usual $p-$*norms*

$$\|\lambda\|_{n,p} := \begin{cases} \max\{|\lambda_1|, \dots, |\lambda_n|\} & \text{if } p = \infty; \\ \left(\sum_{k=1}^n |\lambda_k|^p\right)^{\frac{1}{p}} & \text{if } p \in [1, \infty). \end{cases} \tag{1.1}$$

The *Euclidean norm* is obtained for $p = 2$, i.e.,

$$\|\lambda\|_{n,2} = \left(\sum_{k=1}^n |\lambda_k|^2\right)^{\frac{1}{2}}.$$

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It is well known that on $E^n := E \times \cdots \times E$ endowed with the canonical linear structure we can define the following $p-$*norms*:

$$\|X\|_{n,p} := \begin{cases} \max\{\|x_1\|, \dots, \|x_n\|\} & \text{if } p = \infty; \\ \left(\sum_{k=1}^{n} \|x_k\|^p\right)^{\frac{1}{p}} & \text{if } p \in [1, \infty); \end{cases} \tag{1.2}$$

where $X = (x_1, \dots, x_n) \in E^n$.

For a given norm $\|\cdot\|_n$ on $\mathbb{K}^n$ we define the functional $\|\cdot\|_{h,n} : E^n \to [0, \infty)$ given by

$$\|X\|_{h,n} := \sup_{(\lambda_1,\dots,\lambda_n) \in B(\|\cdot\|_n)} \left\| \sum_{j=1}^{n} \lambda_j x_j \right\|, \tag{1.3}$$

where $X = (x_1, \dots, x_n) \in E^n$.

It is easy to see that:

(i) $\|X\|_{h,n} \geq 0$ for any $X \in E^n$;
(ii) $\|X + Y\|_{h,n} \leq \|X\|_{h,n} + \|Y\|_{h,n}$ for any $X, Y \in E^n$;
(iii) $\|\alpha X\|_{h,n} = |\alpha| \|X\|_{h,n}$ for each $\alpha \in \mathbb{K}$ and $X \in E^n$;

and therefore $\|\cdot\|_{h,n}$ is a *semi-norm* on E^n. This will be called the *hypo-semi-norm* generated by the norm $\|\cdot\|_n$ on X^n.

We observe that $\|X\|_{h,n} = 0$ if and only if $\sum_{j=1}^{n} \lambda_j x_j = 0$ for any $(\lambda_1, \dots, \lambda_n) \in B(\|\cdot\|_n)$. If there exists $\lambda_1^0, \dots, \lambda_n^0 \neq 0$ such that $(\lambda_1^0, 0, \dots, 0), (0, \lambda_2^0, \dots, 0), \dots, (0, 0, \dots, \lambda_n^0) \in B(\|\cdot\|_n)$ then the semi-norm generated by $\|\cdot\|_n$ is a *norm* on E^n.

If by $\mathbb{B}_{n,p}$ with $p \in [1, \infty]$ we denote the balls generated by the $p-$norms $\|\cdot\|_{n,p}$ on $\mathbb{K}^n$, then we can obtain the following *hypo-p-norms* on X^n :

$$\|X\|_{h,n,p} := \sup_{(\lambda_1,\dots,\lambda_n) \in \mathbb{B}_{n,p}} \left\| \sum_{j=1}^{n} \lambda_j x_j \right\|, \tag{1.4}$$

with $p \in [1, \infty]$.

For $p = 2$, we have the Euclidean ball in $\mathbb{K}^n$, which we denote by $\mathbb{B}_n$,

$$\mathbb{B}_n = \left\{ \lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{K}^n \left| \sum_{i=1}^{n} |\lambda_i|^2 \leq 1 \right. \right\}$$

that generates the *hypo-Euclidean norm* on E^n, i.e.,

$$\|X\|_{h,e} := \sup_{(\lambda_1,\dots,\lambda_n) \in \mathbb{B}_n} \left\| \sum_{j=1}^{n} \lambda_j x_j \right\|. \tag{1.5}$$

Moreover, if $E = H$, H is a Hilbert space over $\mathbb{K}$, then the *hypo-Euclidean norm* on H^n will be denoted simply by

$$\|(x_1, \dots, x_n)\|_e := \sup_{(\lambda_1,\dots,\lambda_n) \in \mathbb{B}_n} \left\| \sum_{j=1}^{n} \lambda_j x_j \right\|, \tag{1.6}$$

and its properties will be extensively studied in the present paper.

Both the notation in (1.6) and the necessity of investigating its main properties are motivated by the recent work of G. Popescu [9] who introduced a similar norm on the Cartesian product of Banach algebra $B(H)$ of all bounded linear operators on H and used it to investigate various properties of $n-$tuple of operators in Multivariable Operator Theory. The study is also motivated by the fact that the hypo-Euclidean norm is closely related to the quadratic form

$\sum_{j=1}^{n} |\langle x, x_j \rangle|^2$ (see the representation Theorem 2.2) that plays a key role in many problems arising in the Theory of Fourier expansions in Hilbert spaces.

The paper is structured as follows: in Section 2 we establish the equivalence of the hypo-Euclidean norm with the usual Euclidean norm on H^n, provide a representation result and obtain some lower bounds for it. In Section 3, on utilising the classical results of Boas-Bellman and Bombieri as well as some recent similar results obtained by the author, we give various upper bounds for the hypo-Euclidean norm. These are complemented in Section 4 with other inequalities between $p-$norms and the hypo-Euclidean norm. Section 5 is devoted to the presentation of some conditional reverse inequalities between the hypo-Euclidean norm and the norm of the sum of the vectors involved. In Section 6, the natural connection between the hypo-Euclidean norm and the operator norm $\|(\cdot, \dots, \cdot)\|_e$ introduced by Popescu in [9] is investigated. A representation result is obtained and some applications for operator inequalities are pointed out. Finally, in Section 7, a new norm for operators is introduced and some natural inequalities are obtained.

2. FUNDAMENTAL PROPERTIES

Let $(H; \langle \cdot, \cdot \rangle)$ be a Hilbert space over $\mathbb{K}$ and $n \in \mathbb{N}$, $n \geq 1$. In the Cartesian product $H^n := H \times \cdots \times H$, for the $n-$tuples of vectors $X = (x_1, \dots, x_n)$, $Y = (y_1, \dots, y_n) \in H^n$, we can define the inner product $\langle \cdot, \cdot \rangle$ by

$$\langle X, Y \rangle := \sum_{j=1}^{n} \langle x_j, y_j \rangle, \qquad X, Y \in H^n, \tag{2.1}$$

which generates the Euclidean norm $\|\cdot\|_2$ on H^n, i.e.,

$$\|X\|_2 := \left(\sum_{j=1}^{n} \|x_j\|^2 \right)^{\frac{1}{2}}, \qquad X \in H^n. \tag{2.2}$$

The following result connects the usual Euclidean norm $\|\cdot\|$ with the hypo-Euclidean norm $\|\cdot\|_e$.

Theorem 2.1. *For any $X \in H^n$ we have the inequalities*

$$\|X\|_2 \geq \|X\|_e \geq \frac{1}{\sqrt{n}} \|X\|_2 \,, \tag{2.3}$$

i.e., $\|\cdot\|_2$ and $\|\cdot\|_e$ are equivalent norms on H^n.

Proof. By the Cauchy-Bunyakovsky-Schwarz inequality we have

$$\left\| \sum_{j=1}^{n} \lambda_j x_j \right\| \leq \left(\sum_{j=1}^{n} |\lambda_j|^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} \|x_j\|^2 \right)^{\frac{1}{2}} \tag{2.4}$$

for any $(\lambda_1, \dots, \lambda_n) \in \mathbb{K}^n$. Taking the supremum over $(\lambda_1, \dots, \lambda_n) \in \mathbb{B}_n$ in (2.4) we obtain the first inequality in (2.3).

If by σ we denote the rotation-invariant normalised positive Borel measure on the unit sphere $\partial\mathbb{B}_n$ $\left(\partial\mathbb{B}_n = (\lambda_1, \dots, \lambda_n) \in \mathbb{K}^n \left| \sum_{i=1}^{n} |\lambda_i|^2 = 1 \right.\right)$ whose existence and properties have been pointed out in [10], then we can state that

$$\int_{\partial\mathbb{B}_n} |\lambda_k|^2 \, d\sigma(\lambda) = \frac{1}{n} \quad \text{and} \tag{2.5}$$
$$\int_{\partial\mathbb{B}_n} \lambda_k \overline{\lambda_j} d\sigma(\lambda) = 0 \quad \text{if} \quad k \neq j, \; k,j = 1,\dots,n.$$

Utilising these properties, we have

$$\begin{aligned}
\|X\|_e^2 &= \sup_{(\lambda_1,\dots,\lambda_n)\in\mathbb{B}_n} \left\| \sum_{k=1}^n \lambda_k x_k \right\|^2 = \sup_{(\lambda_1,\dots,\lambda_n)\in\mathbb{B}_n} \left[\sum_{k,j=1}^n \lambda_k \overline{\lambda_j} \langle x_k, x_j \rangle \right] \\
&\geq \int_{\partial\mathbb{B}_n} \left[\sum_{k,j=1}^n \lambda_k \overline{\lambda_j} \langle x_k, x_j \rangle \right] d\sigma(\lambda) = \sum_{k,j=1}^n \int_{\partial\mathbb{B}_n} \left[\lambda_k \overline{\lambda_j} \langle x_k, x_j \rangle \right] d\sigma(\lambda) \\
&= \frac{1}{n} \sum_{k=1}^n \|x_k\|^2 = \frac{1}{n} \|X\|_2^2,
\end{aligned}$$

from where we deduce the second inequality in (2.3). □

The following representation result for the hypo-Euclidean norm plays a key role in obtaining various bounds for this norm:

Theorem 2.2. *For any $X \in H^n$ with $X = (x_1, \dots, x_n)$, we have*

$$\|X\|_e = \sup_{\|x\|=1} \left(\sum_{j=1}^n |\langle x, x_j \rangle|^2 \right)^{\frac{1}{2}}. \tag{2.6}$$

Proof. We use the following well known representation result for scalars:

$$\sum_{j=1}^n |z_j|^2 = \sup_{(\lambda_1,\dots,\lambda_n)\in\mathbb{B}_n} \left| \sum_{j=1}^n \lambda_j z_j \right|^2, \tag{2.7}$$

where $(z_1, \dots, z_n) \in \mathbb{K}^n$.

Utilising this property, we thus have

$$\left(\sum_{j=1}^n |\langle x, x_j \rangle|^2 \right)^{\frac{1}{2}} = \sup_{(\lambda_1,\dots,\lambda_n)\in\mathbb{B}_n} \left| \left\langle x, \sum_{j=1}^n \lambda_j x_j \right\rangle \right| \tag{2.8}$$

for any $x \in H$.

Now, taking the supremum over $\|x\| = 1$ in (2.8) we get

$$\begin{aligned}
\sup_{\|x\|=1} \left(\sum_{j=1}^n |\langle x, x_j \rangle|^2 \right)^{\frac{1}{2}} &= \sup_{\|x\|=1} \left[\sup_{(\lambda_1,\dots,\lambda_n)\in\mathbb{B}_n} \left| \left\langle x, \sum_{j=1}^n \lambda_j x_j \right\rangle \right| \right] \\
&= \sup_{(\lambda_1,\dots,\lambda_n)\in\mathbb{B}_n} \left[\sup_{\|x\|=1} \left| \left\langle x, \sum_{j=1}^n \lambda_j x_j \right\rangle \right| \right] \\
&= \sup_{(\lambda_1,\dots,\lambda_n)\in\mathbb{B}_n} \left\| \sum_{j=1}^n \lambda_j x_j \right\|,
\end{aligned}$$

since, in any Hilbert space we have that $\sup_{\|u\|=1} |\langle u, v \rangle| = \|v\|$ for each $v \in H$. □

Corollary 2.3. *If $X = (x_1, \ldots, x_n)$ is an $n-$tuple of orthonormal vectors, i.e., we recall that $\|x_k\| = 1$ and $\langle x_k, x_j\rangle = 0$ for $k, j \in \{1, \ldots, n\}$ with $k \neq j$, then $\|X\|_e \leq 1$.*

The proof is obvious by Bessel's inequality.

The next proposition contains two lower bounds for the hypo-Euclidean norm that are sometimes better than the one in (2.3), as will be shown by some examples later.

Proposition 2.4. *For any $X = (x_1, \ldots, x_n) \in H^n \setminus \{0\}$ we have*

$$\|X\|_e \geq \begin{cases} \frac{1}{\|X\|_2} \left\| \sum_{j=1}^n \|x_j\| \, x_j \right\|, \\ \\ \frac{1}{\sqrt{n}} \left\| \sum_{j=1}^n x_j \right\|. \end{cases} \tag{2.9}$$

Proof. By the definition of the hypo-Euclidean norm we have that, if $(\lambda_1^0, \ldots, \lambda_n^0) \in \mathbb{B}_n$, then obviously

$$\|X\|_e \geq \left\| \sum_{j=1}^n \lambda_j^0 x_j \right\|.$$

The choice

$$\lambda_j^0 := \frac{\|x_j\|}{\|X\|_2}, \qquad j \in \{1, \ldots, n\},$$

which satisfies the condition $(\lambda_1^0, \ldots, \lambda_n^0) \in \mathbb{B}_n$ will produce the first inequality while the selection

$$\lambda_j^0 = \frac{1}{\sqrt{n}}, \qquad j \in \{1, \ldots, n\},$$

will give the second inequality in (2.9). □

Remark 2.5. For $n = 2$, the hypo-Euclidean norm on H^2

$$\|(x, y)\|_e = \sup_{(\lambda,\mu)\in\mathbb{B}_2} \|\lambda x + \mu y\| = \sup_{\|z\|=1} \left[|\langle z, x\rangle|^2 + |\langle z, y\rangle|^2 \right]^{\frac{1}{2}}$$

is bounded below by

$$B_1(x, y) := \frac{1}{\sqrt{2}} \left(\|x\|^2 + \|y\|^2 \right)^{\frac{1}{2}},$$
$$B_2(x, y) := \frac{\| \|x\| \, x + \|y\| \, y \|}{\left(\|x\|^2 + \|y\|^2 \right)^{\frac{1}{2}}}$$

and

$$B_3(x, y) := \frac{1}{\sqrt{2}} \|x + y\|.$$

If $H = \mathbb{C}$ endowed with the canonical inner product $\langle x, y\rangle := x\bar{y}$ where $x, y \in \mathbb{C}$, then

$$B_1(x, y) = \frac{1}{\sqrt{2}} \left(|x|^2 + |y|^2 \right)^{\frac{1}{2}},$$
$$B_2(x, y) = \frac{||x| \, x + |y| \, y|}{\left(|x|^2 + |y|^2 \right)^{\frac{1}{2}}}$$

and

$$B_3(x, y) = \frac{1}{\sqrt{2}} |x + y|, \qquad x, y \in \mathbb{C}.$$

The plots of the differences $D_1(x,y) := B_1(x,y) - B_2(x,y)$ and $D_2(x,y) := B_1(x,y) - B_3(x,y)$ which are depicted in Figure 2.1 and Figure 2.2, respectively, show that the bound B_1 is not always better than B_2 or B_3. However, since the plot of $D_3(x,y) := B_2(x,y) - B_3(x,y)$ (see Figure 2.3) appears to indicate that, at least in the case of $\mathbb{C}^2$, it may be possible that the bound B_2 is always better than B_3, hence we can ask in general which bound from (2.6) is better for a given $n \geq 2$? This is an open problem that will be left to the interested reader for further investigation.

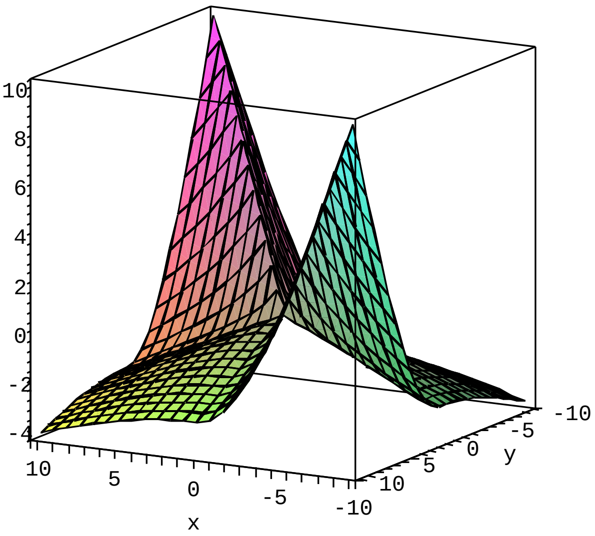

Figure 2.1: The behaviour of $D_1(x,y)$

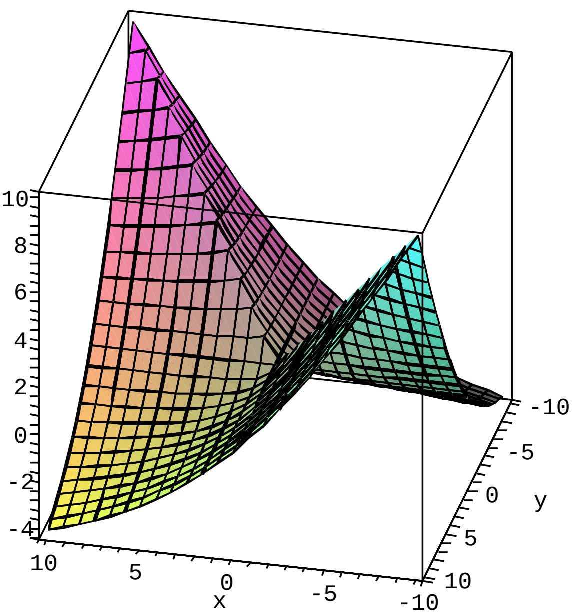

Figure 2.2: The behaviour of $D_2(x,y)$

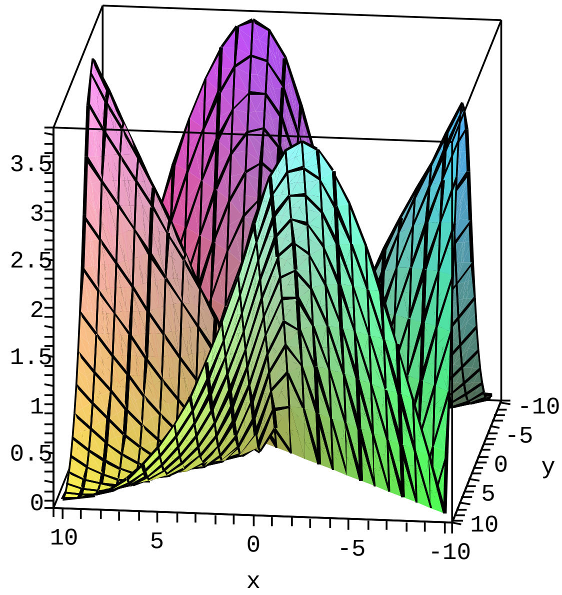

Figure 2.3: The behaviour of $D_3(x,y)$

3. Upper Bounds via the Boas-Bellman and Bombieri Type Inequalities

In 1941, R.P. Boas [3] and in 1944, independently, R. Bellman [1] proved the following generalisation of Bessel's inequality that can be stated for any family of vectors $\{y_1,\dots,y_n\}$(see also [8, p. 392] or [5, p. 125]):

$$\sum_{j=1}^{n} |\langle x, y_j\rangle|^2 \le \|x\|^2 \left[\max_{1\le j\le n} \|y_j\|^2 + \left(\sum_{1\le j\ne k\le n} |\langle y_k, y_j\rangle|^2\right)^{\frac{1}{2}}\right] \tag{3.1}$$

for any $x, y_1 \dots, y_n$ vectors in the real or complex inner product space $(H; \langle\cdot,\cdot\rangle)$. This result is known in the literature as the *Boas-Bellman inequality*.

The following result provides various upper bounds for the hypo-Euclidean norm:

Theorem 3.1. *For any $X = (x_1,\dots,x_n) \in H^n$, we have*

$$\|X\|_e^2 \le \begin{cases} \max\limits_{1\le j\le n} \|x_j\|^2 + \left(\sum\limits_{1\le j\ne k\le n} |\langle x_k, x_j\rangle|^2\right)^{\frac{1}{2}}, \\ \max\limits_{1\le j\le n} \|x_j\|^2 + (n-1) \max\limits_{1\le j\ne k\le n} |\langle x_k, x_j\rangle|; \end{cases} \tag{3.2}$$

$$\|X\|_e^2 \le \left[\max_{1\le j\le n} \|x_j\|^2 \sum_{j=1}^{n} \|x_j\|^2 + \max_{1\le j\ne k\le n} \{\|x_j\|\,\|x_k\|\} \sum_{1\le j\ne k\le n} |\langle x_j, x_k\rangle|\right]^{\frac{1}{2}} \tag{3.3}$$

and

$$\|X\|_e^4 \le \begin{cases} \max\limits_{1\le j\le n} \|x_j\|^2 \sum\limits_{j=1}^{n} \|x_j\|^2 + (n-1)\|X\|_e^2 \max\limits_{1\le j\ne k\le n} |\langle x_j, x_k\rangle|, \\ \|X\|_e^2 \max\limits_{1\le j\le n} \|x_j\|^2 + \max\limits_{1\le j\ne k\le n} \{\|x_j\|\,\|x_k\|\} \sum\limits_{1\le j\ne k\le n} |\langle x_j, x_k\rangle|. \end{cases} \tag{3.4}$$

Proof. Taking the supremum over $\|x\| = 1$ in (3.1) and utilising the representation (2.6), we deduce the first inequality in (3.2).

In [4], we proved amongst others the following inequalities

$$\left|\sum_{j=1}^{n} c_j \langle x, y_j\rangle\right|^2 \le \|x\|^2 \times \begin{cases} \max\limits_{1\le j\le n} |c_j|^2 \sum\limits_{j=1}^{n} \|y_j\|^2, \\ \sum\limits_{j=1}^{n} |c_j|^2 \max\limits_{1\le j\le n} \|y_j\|^2, \end{cases} + \|x\|^2 \times \begin{cases} \max\limits_{1\le j\ne k\le n} \{|c_j c_k|\} \sum\limits_{1\le j\ne k\le n} |\langle y_j, y_k\rangle|, \\ (n-1) \sum\limits_{j=1}^{n} |c_j|^2 \max\limits_{1\le j\ne k\le n} |\langle y_j, y_k\rangle|, \end{cases} \tag{3.5}$$

for any $y_1, \dots, y_n, x \in H$ and $c_1, \dots, c_n \in \mathbb{K}$, where (3.5) should be seen as all possible configurations.

The choice $c_j = \overline{\langle x, y_j\rangle}$, $j \in \{1, \dots, n\}$ will produce the following four inequalities:

$$\left[\sum_{j=1}^{n} |\langle x, y_j\rangle|^2\right]^2 \le \|x\|^2 \times \begin{cases} \max\limits_{1\le j\le n} |\langle x, y_j\rangle|^2 \sum\limits_{j=1}^{n} \|y_j\|^2, \\ \sum\limits_{j=1}^{n} |\langle x, y_j\rangle|^2 \max\limits_{1\le j\le n} \|y_j\|^2, \end{cases} + \|x\|^2 \times \begin{cases} \max\limits_{1\le j\ne k\le n} \{|\langle x, y_j\rangle| \, |\langle x, y_k\rangle|\} \sum\limits_{1\le j\ne k\le n} |\langle y_j, y_k\rangle|, \\ (n-1) \sum\limits_{j=1}^{n} |\langle x, y_j\rangle|^2 \max\limits_{1\le j\ne k\le n} |\langle y_j, y_k\rangle|. \end{cases} \tag{3.6}$$

Taking the supremum over $\|x\| = 1$ and utilising the representation (2.6) we easily deduce the rest of the four inequalities. □

A different generalisation of Bessel's inequality for non-orthogonal vectors is the *Bombieri inequality* (see [2] or [8, p. 397] and [5, p. 134]):

$$\sum_{j=1}^{n} |\langle x, y_j\rangle|^2 \le \|x\|^2 \max_{1\le j\le n} \left\{\sum_{k=1}^{n} |\langle y_j, y_k\rangle|\right\}, \tag{3.7}$$

for any $x \in H$, where $y_1, \dots, y_n$ are vectors in the real or complex inner product space $(H; \langle\cdot,\cdot\rangle)$.

Note that, the Bombieri inequality was not stated in the general case of inner product spaces in [2]. However, the inequality presented there easily leads to (3.7) which, apparently, was firstly mentioned as is in [8, p. 394].

On utilising the Bombieri inequality (3.7) and the representation Theorem 2.2, we can state the following simple upper bound for the hypo-Euclidean norm $\|\cdot\|_e$.

Theorem 3.2. *For any $X = (x_1, \dots, x_n) \in H^n$, we have*

$$\|X\|_e^2 \le \max_{1\le j\le n} \left\{\sum_{k=1}^{n} |\langle x_j, x_k\rangle|\right\}. \tag{3.8}$$

In [6] (see also [5, p. 138]), we have established the following norm inequalities:

$$\left\|\sum_{j=1}^{n} \alpha_j z_j\right\|^2 \le n^{\frac{1}{p}+\frac{1}{t}-1} \sum_{k=1}^{n} |\alpha_k|^2 \left[\sum_{k=1}^{n} \left(\sum_{j=1}^{n} |\langle z_j, z_k\rangle|^q\right)^{\frac{u}{q}}\right]^{\frac{1}{u}}, \tag{3.9}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{t}+\frac{1}{u}=1$ and $1<p\le 2$, $1<t\le 2$ and $\alpha_j\in\mathbb{C}$, $z_j\in H$, $j\in\{1,\dots,n\}$.

An interesting particular case of (3.9) obtained for $p=q=2$, $t=u=2$ is incorporated in

$$\left\|\sum_{j=1}^{n}\alpha_j z_j\right\|^2 \le \sum_{k=1}^{n}|\alpha_k|^2\left(\sum_{j,k=1}^{n}|\langle z_j,z_k\rangle|^2\right)^{\frac{1}{2}}. \tag{3.10}$$

Other similar inequalities for norms are the following ones [6] (see also [5, pp. 139-140]):

$$\left\|\sum_{j=1}^{n}\alpha_j z_j\right\|^2 \le n^{\frac{1}{p}}\sum_{k=1}^{n}|\alpha_k|^2\max_{1\le j\le n}\left\{\left[\sum_{k=1}^{n}|\langle z_j,z_k\rangle|^q\right]^{\frac{1}{q}}\right\}, \tag{3.11}$$

provided that $1<p\le 2$ and $\frac{1}{p}+\frac{1}{q}=1$, $\alpha_j\in\mathbb{C}$, $z_j\in H$, $j\in\{1,\dots,n\}$. In the particular case $p=q=2$, we have

$$\left\|\sum_{j=1}^{n}\alpha_j z_j\right\|^2 \le \sqrt{n}\sum_{k=1}^{n}|\alpha_k|^2\max_{1\le j\le n}\left[\sum_{k=1}^{n}|\langle z_j,z_k\rangle|^2\right]^{\frac{1}{2}}. \tag{3.12}$$

Also, if $1<m\le 2$, then [6]:

$$\left\|\sum_{j=1}^{n}\alpha_j z_j\right\|^2 \le n^{\frac{1}{m}}\sum_{k=1}^{n}|\alpha_k|^2\left\{\sum_{j=1}^{n}\left[\max_{1\le k\le n}|\langle z_j,z_k\rangle|^l\right]\right\}^{\frac{1}{l}}, \tag{3.13}$$

where $\frac{1}{m}+\frac{1}{l}=1$. For $m=l=2$, we get

$$\left\|\sum_{j=1}^{n}\alpha_j z_j\right\|^2 \le \sqrt{n}\sum_{k=1}^{n}|\alpha_k|^2\left[\sum_{j=1}^{n}\left(\max_{1\le k\le n}|\langle z_j,z_k\rangle|^2\right)\right]^{\frac{1}{2}}. \tag{3.14}$$

Finally, we can also state the inequality [6]:

$$\left\|\sum_{j=1}^{n}\alpha_j z_j\right\|^2 \le n\sum_{k=1}^{n}|\alpha_k|^2\max_{1\le j,k\le n}|\langle z_j,z_k\rangle|. \tag{3.15}$$

Utilising the above norm-inequalities and the definition of the hypo-Euclidean norm, we can state the following result which provides other upper bounds than the ones outlined in Theorem 3.1 and 3.2:

Theorem 3.3. *For any $X=(x_1,\dots,x_n)\in H^n$, we have*

$$\|X\|_e^2 \le \begin{cases} n^{\frac{1}{p}+\frac{1}{t}-1}\left\{\sum_{k=1}^{n}\left(\sum_{j=1}^{n}|\langle x_j,x_k\rangle|^q\right)^{\frac{u}{q}}\right\}^{\frac{1}{u}} & \text{where } \frac{1}{p}+\frac{1}{q}=1, \\ & \frac{1}{t}+\frac{1}{u}=1 \text{ and } 1<p\le 2,\ 1<t\le 2; \\ n^{\frac{1}{p}}\max_{1\le j\le n}\left\{\left[\sum_{j=1}^{n}|\langle x_j,x_k\rangle|^q\right]^{\frac{1}{q}}\right\} & \text{where } \frac{1}{p}+\frac{1}{q}=1 \\ & \text{and } 1<p\le 2; \\ n^{\frac{1}{m}}\left\{\sum_{j=1}^{n}\left[\max_{1\le k\le n}|\langle x_j,x_k\rangle|^l\right]\right\}^{\frac{1}{l}} & \text{where } \frac{1}{m}+\frac{1}{l}=1 \\ & \text{and } 1<m\le 2; \\ n\max_{1\le k\le n}|\langle x_k,z_j\rangle|; \end{cases} \tag{3.16}$$

and, in particular,

$$\|X\|_e^2 \le \begin{cases} \left[\sum_{j,k=1}^{n}|\langle x_j,x_k\rangle|^2\right]^{\frac{1}{2}}; \\ \sqrt{n}\max_{1\le j\le n}\left[\sum_{k=1}^{n}|\langle x_j,x_k\rangle|^2\right]^{\frac{1}{2}}; \\ \sqrt{n}\left[\sum_{j=1}^{n}\left(\max_{1\le k\le n}\left\{|\langle x_j,x_k\rangle|^2\right\}\right)\right]^{\frac{1}{2}}. \end{cases} \tag{3.17}$$

4. Various Inequalities for the Hypo-Euclidean Norm

For an $n-$tuple $X=(x_1,\dots,x_n)$ of vectors in H, we consider the usual $p-$norms:

$$\|X\|_p := \left(\sum_{j=1}^{n}\|x_j\|^p\right)^{\frac{1}{p}},$$

where $p\in[1,\infty)$, and denote with S the sum $\sum_{j=1}^{n}x_j$.

With these notations we can state the following reverse of the inequality $\|X\|_2 \ge \|X\|_e$, that has been pointed out in Theorem 2.1.

Theorem 4.1. *For any $X=(x_1,\dots,x_n)\in H^n$, we have*

$$(0\le)\ \|X\|_2^2-\|X\|_e^2\le \|X\|_1^2-\|S\|^2. \tag{4.1}$$

If

$$\|X\|_{(2)}^2 := \sum_{j,k=1}^{n}\left\|\frac{x_j+x_k}{2}\right\|^2,$$

then also

$$(0\le)\ \|X\|_2^2-\|X\|_e^2\le \|X\|_{(2)}^2-\|S\|^2 \tag{4.2}$$
$$\left(\le n\|X\|_2^2-\|S\|^2\right).$$

Proof. We observe, for any $x \in H$, that

$$\left|\sum_{j=1}^{n}\langle x,x_j\rangle\right|^2 = \sum_{j=1}^{n}\langle x,x_j\rangle\overline{\sum_{k=1}^{n}\langle x,x_k\rangle} = \left|\sum_{j=1}^{n}\langle x,x_j\rangle\sum_{k=1}^{n}\overline{\langle x,x_k\rangle}\right| \tag{4.3}$$
$$= \left|\sum_{k=1}^{n}|\langle x,x_k\rangle|^2 + \sum_{1\le j\neq k\le n}\langle x,x_j\rangle\langle x_k,x\rangle\right|$$
$$\le \sum_{k=1}^{n}|\langle x,x_k\rangle|^2 + \left|\sum_{1\le j\neq k\le n}\langle x,x_j\rangle\langle x_k,x\rangle\right|$$
$$\le \sum_{k=1}^{n}|\langle x,x_k\rangle|^2 + \sum_{1\le j\neq k\le n}|\langle x,x_j\rangle|\,|\langle x_k,x\rangle|.$$

Taking the supremum over $\|x\|=1$, we get

$$\sup_{\|x\|=1}\left|\sum_{j=1}^{n}\langle x,x_j\rangle\right|^2 \le \sup_{\|x\|=1}\sum_{k=1}^{n}|\langle x,x_k\rangle|^2 + \sum_{1\le j\neq k\le n}\sup_{\|x\|=1}|\langle x,x_j\rangle|\cdot\sup_{\|x\|=1}|\langle x_k,x\rangle|. \tag{4.4}$$

However,

$$\sup_{\|x\|=1}\left|\sum_{j=1}^{n}\langle x,x_j\rangle\right|^2 = \sup_{\|x\|=1}\left|\left\langle x,\sum_{j=1}^{n}x_j\right\rangle\right|^2 = \|S\|^2,$$

$$\sup_{\|x\|=1}|\langle x,x_j\rangle| = \|x_j\| \quad \text{and} \quad \sup_{\|x\|=1}|\langle x,x_k\rangle| = \|x_k\|$$

for $j,k\in\{1,\dots,n\}$, and by (4.4) we get

$$\|S\|^2 \le \|X\|_e^2 + \sum_{1\le j\neq k\le n}\|x_j\|\,\|x_j\|$$
$$= \|X\|_e^2 + \sum_{j,k=1}^{n}\|x_j\|\,\|x_k\| - \sum_{k=1}^{n}\|x_k\|^2$$
$$= \|X\|_e^2 + \|X\|_1^2 - \|X\|_2^2,$$

which is clearly equivalent with (4.1).

Further on, we also observe that, for any $x\in H$ we have the identity:

$$\left|\sum_{j=1}^{n}\langle x,x_j\rangle\right|^2 = \operatorname{Re}\left[\sum_{k,j=1}^{n}\langle x,x_j\rangle\langle x_k,x\rangle\right] \tag{4.5}$$
$$= \sum_{k=1}^{n}|\langle x,x_k\rangle|^2 + \sum_{1\le j\neq k\le n}\operatorname{Re}\left[\langle x,x_j\rangle\langle x_k,x\rangle\right].$$

Utilising the elementary inequality for complex numbers

$$\operatorname{Re}(u\bar{v}) \le \frac{1}{4}|u+v|^2, \qquad u,v\in\mathbb{C}, \tag{4.6}$$

we can state that

$$\sum_{1\le k\neq j\le n}\operatorname{Re}\left[\langle x,x_j\rangle\langle x_k,x\rangle\right]\le\frac{1}{4}\sum_{1\le k\neq j\le n}\left|\langle x,x_j\rangle+\langle x,x_k\rangle\right|^2$$
$$=\sum_{1\le k\neq j\le n}\left|\left\langle x,\frac{x_j+x_k}{2}\right\rangle\right|^2,$$

and by (4.5) we get

$$\left|\sum_{j=1}^{n}\langle x,x_j\rangle\right|^2\le\sum_{k=1}^{n}\left|\langle x,x_k\rangle\right|^2+\sum_{1\le k\neq j\le n}\left|\left\langle x,\frac{x_j+x_k}{2}\right\rangle\right|^2 \tag{4.7}$$

for any $x\in H$.

Taking the supremum over $\|x\|=1$ in (4.7) we deduce

$$\left\|\sum_{j=1}^{n}x_j\right\|^2\le\|X\|_e^2+\sum_{1\le k\neq j\le n}\left\|\frac{x_j+x_k}{2}\right\|^2$$
$$=\|X\|_e^2+\sum_{k,j=1}^{n}\left\|\frac{x_j+x_k}{2}\right\|^2-\sum_{k=1}^{n}\|x_k\|^2$$

which provides the first inequality in (4.2).

By the convexity of $\|\cdot\|^2$ we have

$$\sum_{j,k=1}^{n}\left\|\frac{x_j+x_k}{2}\right\|^2\le\frac{1}{2}\sum_{j,k=1}^{n}\left[\|x_j\|^2+\|x_k\|^2\right]=n\sum_{k=1}^{n}\|x_k\|^2$$

and the last part of (4.2) is obvious. □

Remark 4.2. For $n=2$, $X=(x,y)\in H^2$ we have the upper bounds

$$B_1(x,y):=\|x\|^2+\|y\|^2-\|x+y\|^2$$
$$=2\left(\|x\|\,\|y\|-\operatorname{Re}\langle x,y\rangle\right)$$

and

$$B_2(x,y):=\|x\|^2+\|y\|^2$$

for the difference $\|X\|_2^2-\|X\|_e^2$, $X\in H^2$ as provided by (4.1) and (4.2) respectively. If $H=\mathbb{R}$ then $B_1(x,y)=2(|xy|-xy)$, $B_2(x,y)=x^2+y^2$. If we consider the function $\Delta(x,y)=B_2(x,y)-B_1(x,y)$ then the plot of $\Delta(x,y)$ depicted in Figure 4.1 shows that the bounds provided by (4.1) and (4.2) cannot be compared in general, meaning that sometimes the first is better than the second and vice versa.

From a different view-point we can state the following result:

Theorem 4.3. *For any $X=(x_1,\dots,x_n)\in H^n$, we have*

$$\|S\|^2\le\|X\|_e\left[\|X\|_e+\left(\sum_{k=1}^{n}\|S-x_k\|^2\right)^{\frac{1}{2}}\right] \tag{4.8}$$

and

$$\|S\|^2\le\|X\|_e\left[\|X\|_e+\left\{\max_{1\le k\le n}\|S-x_k\|^2+\left(\sum_{1\le k\neq l\le n}\left|\langle S-x_k,S-x_l\rangle\right|^2\right)^{\frac{1}{2}}\right\}^{\frac{1}{2}}\right], \tag{4.9}$$

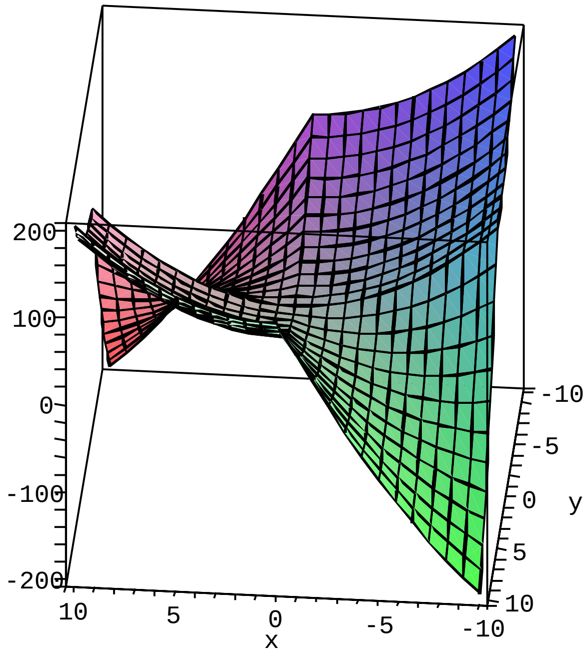

Figure 4.1: The behaviour of $\Delta(x,y)$

respectively.

Proof. Utilising the identity (4.5) above we have

$$\left|\sum_{j=1}^{n}\langle x,x_j\rangle\right|^2=\sum_{k=1}^{n}|\langle x,x_k\rangle|^2+\operatorname{Re}\left\langle x,\sum_{1\le j\ne k\le n}\langle x,x_k\rangle x_j\right\rangle \tag{4.10}$$

for any $x\in H$.

By the Schwarz inequality in the inner product space $(H,\langle\cdot,\cdot\rangle)$, we have that

$$\begin{aligned}\operatorname{Re}\left\langle x,\sum_{1\le j\ne k\le n}\langle x,x_k\rangle x_j\right\rangle &\le \|x\|\left\|\sum_{1\le j\ne k\le n}\langle x,x_k\rangle x_j\right\| \\ &=\|x\|\left\|\sum_{j,k=1}^{n}\langle x,x_k\rangle x_j-\sum_{k=1}^{n}\langle x,x_k\rangle x_k\right\| \\ &=\|x\|\left\|\left\langle x,\sum_{k=1}^{n}x_k\right\rangle\sum_{j=1}^{n}x_j-\sum_{k=1}^{n}\langle x,x_k\rangle x_k\right\| \\ &=\|x\|\left\|\sum_{k=1}^{n}\langle x,x_k\rangle(S-x_k)\right\|.\end{aligned} \tag{4.11}$$

Utilising the Cauchy-Bunyakovsky-Schwarz inequality we have

$$\left\|\sum_{k=1}^{n}\langle x,x_k\rangle(S-x_k)\right\|\le\left(\sum_{k=1}^{n}|\langle x,x_k\rangle|^2\right)^{\frac12}\left(\sum_{k=1}^{n}\|S-x_k\|^2\right)^{\frac12} \tag{4.12}$$

and then by (4.10) – (4.12) we can state the inequality:

$$\left|\sum_{j=1}^{n}\langle x,x_j\rangle\right|^2 \le \left(\sum_{k=1}^{n}|\langle x,x_k\rangle|^2\right)^{\frac{1}{2}}\left[\left(\sum_{k=1}^{n}|\langle x,x_k\rangle|^2\right)^{\frac{1}{2}}+\left(\sum_{k=1}^{n}\|S-x_k\|^2\right)^{\frac{1}{2}}\right] \tag{4.13}$$

for any $x\in H$, $\|x\|=1$. Taking the supremum over $\|x\|=1$ we deduce the desired result (4.8).

Now, following the above argument, we can also state that

$$\left|\left\langle x,\sum_{j=1}^{n}x_j\right\rangle\right|^2 \le \sum_{k=1}^{n}|\langle x,x_k\rangle|^2+\|x\|\left\|\sum_{k=1}^{n}\langle x,x_k\rangle(S-x_k)\right\| \tag{4.14}$$

for any $x\in H$.

Utilising the inequality

$$\left\|\sum_{j=1}^{n}\alpha_j z_j\right\|^2 \le \sum_{j=1}^{n}|\alpha_j|^2\left\{\max_{1\le j\le n}\|z_j\|^2+\left(\sum_{1\le j\ne k\le n}|\langle z_j,z_k\rangle|^2\right)^{\frac{1}{2}}\right\}, \tag{4.15}$$

where $\alpha_j\in\mathbb{C}$, $z_j\in H$, $j\in\{1,\dots,n\}$, that has been obtained in [4], see also [5, p. 128], we can state that

$$\begin{aligned}\left\|\sum_{k=1}^{n}\langle x,x_k\rangle(S-x_k)\right\| \\ \le \left(\sum_{k=1}^{n}|\langle x,x_k\rangle|^2\right)^{\frac{1}{2}}\left\{\max_{1\le k\le n}\|S-x_k\|^2+\left(\sum_{1\le k\ne l\le n}|\langle S-x_k,S-x_l\rangle|^2\right)^{\frac{1}{2}}\right\}^{\frac{1}{2}}\end{aligned} \tag{4.16}$$

for any $x\in H$.

Now, by the use of (4.14) – (4.16) we deduce the desired result (4.9). The details are omitted. □

Remark 4.4. On utilising the inequality:

$$\left\|\sum_{j=1}^{n}\alpha_j z_j\right\|^2 \le \sum_{j=1}^{n}|\alpha_j|^2\left[\max_{1\le k\le n}\|z_k\|^2+(n-1)\max_{1\le k\ne l\le n}|\langle z_k,z_l\rangle|\right], \tag{4.17}$$

where $\alpha_j\in\mathbb{C}$, $z_j\in H$, $j\in\{1,\dots,n\}$, that has been obtained in [4], (see also [5, p. 130]) in place of (4.15) above, we can state the following inequality for the hypo-Euclidean norm as well:

$$\begin{aligned}\|S\|^2 \\ \le \|X\|_e\left[\|X\|_e+\left\{\max_{1\le k\le n}\|S-x_k\|^2+(n-1)\max_{1\le k\ne l\le n}|\langle S-x_k,S-x_l\rangle|^2\right\}^{\frac{1}{2}}\right]\end{aligned} \tag{4.18}$$

for any $X=(x_1,\dots,x_n)\in H^n$.

Other similar results may be stated by making use of the results from [6]. The details are left to the interested reader.

5. REVERSE INEQUALITIES

Before we proceed with establishing some reverse inequalities for the hypo-Euclidean norm, we recall some reverse results of the Cauchy-Bunyakovsky-Schwarz inequality for real or complex numbers as follows:

If $\gamma, \Gamma \in \mathbb{K}$ ($\mathbb{K} = \mathbb{C}, \mathbb{R}$) and $\alpha_j \in \mathbb{K}$, $j \in \{1, \dots, n\}$ with the property that

$$0 \le \operatorname{Re}\left[(\Gamma - \alpha_j)\left(\overline{\alpha_j} - \bar{\gamma}\right)\right] = (\operatorname{Re}\Gamma - \operatorname{Re}\alpha_j)(\operatorname{Re}\alpha_j - \operatorname{Re}\gamma) + (\operatorname{Im}\Gamma - \operatorname{Im}\alpha_j)(\operatorname{Im}\alpha_j - \operatorname{Im}\gamma) \tag{5.1}$$

or, equivalently,

$$\left|\alpha_j - \frac{\gamma + \Gamma}{2}\right| \le \frac{1}{2}\left|\Gamma - \gamma\right| \tag{5.2}$$

for each $j \in \{1, \dots, n\}$, then (see for instance [5, p. 9])

$$n\sum_{j=1}^{n}|\alpha_j|^2 - \left|\sum_{j=1}^{n}\alpha_j\right|^2 \le \frac{1}{4} \cdot n^2\left|\Gamma - \gamma\right|^2. \tag{5.3}$$

In addition, if $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$, then (see for example [5, p. 26]):

$$n\sum_{j=1}^{n}|\alpha_j|^2 \le \frac{1}{4} \cdot \frac{\left\{\operatorname{Re}\left[\left(\bar{\Gamma} + \bar{\gamma}\right)\sum_{j=1}^{n}\alpha_j\right]\right\}^2}{\operatorname{Re}(\Gamma\bar{\gamma})} \le \frac{1}{4} \cdot \frac{|\Gamma + \gamma|^2}{\operatorname{Re}(\Gamma\bar{\gamma})}\left|\sum_{j=1}^{n}\alpha_j\right|^2 \tag{5.4}$$

and

$$n\sum_{j=1}^{n}|\alpha_j|^2 - \left|\sum_{j=1}^{n}\alpha_j\right|^2 \le \frac{1}{4} \cdot \frac{|\Gamma - \gamma|^2}{\operatorname{Re}(\Gamma\bar{\gamma})}\left|\sum_{j=1}^{n}\alpha_j\right|^2. \tag{5.5}$$

Also, if $\Gamma \ne -\gamma$, then (see for instance [5, p. 32]):

$$\left(n\sum_{j=1}^{n}|\alpha_j|^2\right)^{\frac{1}{2}} - \left|\sum_{j=1}^{n}\alpha_j\right| \le \frac{1}{4}n \cdot \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|}. \tag{5.6}$$

Finally, from [7] we can also state that

$$n\sum_{j=1}^{n}|\alpha_j|^2 - \left|\sum_{j=1}^{n}\alpha_j\right|^2 \le n\left[|\Gamma + \gamma| - 2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}\right]\left|\sum_{j=1}^{n}\alpha_j\right|, \tag{5.7}$$

provided $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$.

We notice that a simple sufficient condition for (5.1) to hold is that

$$\operatorname{Re}\Gamma \ge \operatorname{Re}\alpha_j \ge \operatorname{Re}\gamma \quad \text{and} \quad \operatorname{Im}\Gamma \ge \operatorname{Im}\alpha_j \ge \operatorname{Im}\gamma \tag{5.8}$$

for each $j \in \{1, \dots, n\}$.

We can state and prove the following conditional inequalities for the hypo-Euclidean norm $\|\cdot\|_e$:

Theorem 5.1. *Let $\varphi, \phi \in \mathbb{K}$ and $X = (x_1, \dots, x_n) \in H^n$ such that either:*

$$\left| \langle x, x_j \rangle - \frac{\varphi + \phi}{2} \right| \leq \frac{1}{2} |\phi - \varphi| \tag{5.9}$$

or, equivalently,

$$\operatorname{Re}\left[(\phi - \langle x, x_j \rangle) \left(\langle x_j, x \rangle - \bar{\varphi} \right) \right] \geq 0 \tag{5.10}$$

for each $j \in \{1, \dots, n\}$ and for any $x \in H$, $\|x\| = 1$. Then

$$\|X\|_e^2 \leq \frac{1}{n} \|S\|^2 + \frac{1}{4} n |\phi - \varphi|^2 . \tag{5.11}$$

Moreover, if $\operatorname{Re}(\phi\bar{\varphi}) > 0$, then

$$\|X\|_e^2 \leq \frac{1}{4n} \cdot \frac{|\phi + \varphi|^2}{\operatorname{Re}(\phi\overline{\varphi})} \|S\|^2 \tag{5.12}$$

and

$$\|X\|_e^2 \leq \frac{1}{n} \|S\|^2 + \left[|\phi + \varphi| - 2\sqrt{\operatorname{Re}(\phi\bar{\varphi})} \right] \|S\| . \tag{5.13}$$

If $\phi \neq -\varphi$, then

$$\|X\|_e \leq \frac{1}{n} \|S\| + \frac{1}{4} n \cdot \frac{|\phi - \varphi|^2}{|\phi + \varphi|}, \tag{5.14}$$

where $S = \sum_{j=1}^n x_j$.

Proof. We only prove the inequality (5.11).

Let $x \in H$, $\|x\| = 1$. Then, on applying the inequality (5.3) for $\alpha_j = \langle x, x_j \rangle$, $j \in \{1, \dots, n\}$ and $\Gamma = \phi$, $\gamma = \varphi$, we can state that

$$\sum_{j=1}^n |\langle x, x_j \rangle|^2 \leq \frac{1}{n} \left| \left\langle x, \sum_{j=1}^n x_j \right\rangle \right|^2 + \frac{1}{4} n |\phi - \varphi|^2 . \tag{5.15}$$

Now if in (5.15) we take the supremum over $\|x\| = 1$, then we get the desired inequality (5.11).

The other inequalities follow by (5.4), (5.7) and (5.6) respectively. The details are omitted. □

Remark 5.2. Due to the fact that

$$\left| \langle x, x_j \rangle - \frac{\varphi + \phi}{2} \right| \leq \left\| x_j - \frac{\varphi + \phi}{2} \cdot x \right\|$$

for any $j \in \{1, \dots, n\}$ and $x \in H$, $\|x\| = 1$, then a sufficient condition for (5.9) to hold is that

$$\left\| x_j - \frac{\varphi + \phi}{2} \cdot x \right\| \leq \frac{1}{2} |\phi - \varphi|$$

for each $j \in \{1, \dots, n\}$ and $x \in H$, $\|x\| = 1$.

6. APPLICATIONS FOR $n-$TUPLES OF OPERATORS

In [9], the author has introduced the following norm on the Cartesian product $B^{(n)}(H) := B(H) \times \cdots \times B(H)$, where $B(H)$ denotes the Banach algebra of all bounded linear operators defined on the complex Hilbert space H :

$$\|(T_1, \dots, T_n)\|_e := \sup_{(\lambda_1,\dots,\lambda_n)\in\mathbb{B}_n} \|\lambda_1 T_1 + \cdots + \lambda_n T_n\|, \tag{6.1}$$

where $(T_1, \dots, T_n) \in B^{(n)}(H)$ and $\mathbb{B}_n := \left\{(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n \left| \sum_{i=1}^n |\lambda_i|^2 \le 1\right.\right\}$ is the Euclidean closed ball in $\mathbb{C}^n$. It is clear that $\|\cdot\|_e$ is a norm on $B^{(n)}(H)$ and for any $(T_1, \dots, T_n) \in B^{(n)}(H)$ we have

$$\|(T_1, \dots, T_n)\|_e = \|(T_1^*, \dots, T_n^*)\|_e, \tag{6.2}$$

where T_i^* is the adjoint operator of T_i, $i \in \{1, \dots, n\}$.

It has been shown in [9] that the following inequality holds true:

$$\frac{1}{\sqrt{n}} \left\| \sum_{j=1}^n T_j T_j^* \right\|^{\frac{1}{2}} \le \|(T_1, \dots, T_n)\|_e \le \left\| \sum_{j=1}^n T_j T_j^* \right\|^{\frac{1}{2}} \tag{6.3}$$

for any $n-$tuple $(T_1, \dots, T_n) \in B^{(n)}(H)$ and the constants $\frac{1}{\sqrt{n}}$ and 1 are best possible.

In the same paper [9] the author has introduced the *Euclidean operator radius* of an $n-$tuple of operators $(T_1, \dots, T_n)$ by

$$w_e(T_1, \dots, T_n) := \sup_{\|x\|=1} \left(\sum_{j=1}^n |\langle T_j x, x\rangle|^2 \right)^{\frac{1}{2}} \tag{6.4}$$

and proved that $w_e(\cdot)$ is a norm on $B^{(n)}(H)$ and satisfies the double inequality:

$$\frac{1}{2}\|(T_1, \dots, T_n)\|_e \le w_e(T_1, \dots, T_n) \le \|(T_1, \dots, T_n)\|_e \tag{6.5}$$

for each $n-$tuple $(T_1, \dots, T_n) \in B^{(n)}(H)$.

As pointed out in [9], the Euclidean numerical radius also satisfies the double inequality:

$$\frac{1}{2\sqrt{n}} \left\| \sum_{j=1}^n T_j T_j^* \right\|^{\frac{1}{2}} \le w_e(T_1, \dots, T_n) \le \left\| \sum_{j=1}^n T_j T_j^* \right\|^{\frac{1}{2}} \tag{6.6}$$

for any $(T_1, \dots, T_n) \in B^{(n)}(H)$ and the constants $\frac{1}{2\sqrt{n}}$ and 1 are best possible.

We are now able to establish the following natural connections that exists between the hypo-Euclidean norm of vectors in a Cartesian product of Hilbert spaces and the norm $\|\cdot\|_e$ for $n-$tuples of operators in the Banach algebra $B(H)$.

Theorem 6.1. *For any* $(T_1, \dots, T_n) \in B^{(n)}(H)$ *we have*

$$\begin{aligned} \|(T_1, \dots, T_n)\|_e &= \sup_{\|y\|=1} \|(T_1 y, \dots, T_n y)\|_e \\ &= \sup_{\|y\|=1, \|x\|=1} \left(\sum_{j=1}^n |\langle T_j y, x\rangle|^2 \right)^{\frac{1}{2}}. \end{aligned} \tag{6.7}$$

Proof. By the definition of the $\|\cdot\|_e$ −norm on $B^{(n)}(H)$ and the hypo-Euclidean norm on H^n, we have:

$$\|(T_1,\dots,T_n)\|_e = \sup_{(\lambda_1,\dots,\lambda_n)\in\mathbb{B}_n}\left[\sup_{\|y\|=1}\|(\lambda_1T_1+\dots+\lambda_nT_n)\,y\|\right] \tag{6.8}$$
$$= \sup_{\|y\|=1}\left[\sup_{(\lambda_1,\dots,\lambda_n)\in\mathbb{B}_n}\|\lambda_1T_1y+\dots+\lambda_nT_ny\|\right]$$
$$= \sup_{\|y\|=1}\|(T_1y,\dots,T_ny)\|_e\,.$$

Utilising the representation of the hypo-Euclidean norm on H^n from Theorem 2.2, we have

$$\|(T_1y,\dots,T_ny)\|_e = \sup_{\|x\|=1}\left(\sum_{j=1}^n|\langle T_jy,x\rangle|^2\right)^{\frac{1}{2}}. \tag{6.9}$$

Making use of (6.8) and (6.9) we deduce the desired equality (6.7). □

Remark 6.2. Utilising Theorem 2.1, we have

$$\left(\sum_{j=1}^n\|T_jy\|^2\right)^{\frac{1}{2}} \geq \|(T_1y,\dots,T_ny)\|_e \geq \frac{1}{\sqrt{n}}\left(\sum_{j=1}^n\|T_jy\|^2\right)^{\frac{1}{2}} \tag{6.10}$$

for any $y\in H$, $\|y\|=1$.

Since

$$\sum_{j=1}^n\|T_jy\|^2 = \left\langle\sum_{j=1}^n T_j^*T_jy,y\right\rangle,\qquad \|y\|=1$$

hence, on taking the supremum over $\|y\|=1$ in (6.10) and on observing that

$$\sup_{\|y\|=1}\left\langle\sum_{j=1}^n T_j^*T_jy,y\right\rangle = w\left(\sum_{j=1}^n T_j^*T_j\right) = \left\|\sum_{j=1}^n T_j^*T_j\right\| = \left\|\sum_{j=1}^n T_jT_j^*\right\|,$$

we deduce the inequality (6.3) that has been established in [9] by a different argument.

We observe that, due to the representation Theorem 6.1, some inequalities obtained for the hypo-Euclidean norm can be utilised in obtaining various new inequalities for the operator norm $\|\cdot\|_e$ by employing a standard approach consisting in taking the supremum over $\|y\|=1$, as described in the above remark.

The following different lower bound for the Euclidean operator norm $\|\cdot\|_e$ can be stated:

Proposition 6.3. *For any* $(T_1,\dots,T_n)\in B^{(n)}(H)$, *we have*

$$\|(T_1,\dots,T_n)\|_e \geq \frac{1}{\sqrt{n}}\|T_1+\dots+T_n\|\,. \tag{6.11}$$

Proof. Utilising Proposition 2.4 and Theorem 6.1 we have:

$$\|(T_1,\dots,T_n)\|_e = \sup_{\|y\|=1}\|(T_1y,\dots,T_ny)\|_e$$
$$\geq \frac{1}{\sqrt{n}}\sup_{\|y\|=1}\|T_1y+\dots+T_ny\|$$
$$= \frac{1}{\sqrt{n}}\|T_1+\dots+T_n\|$$

which is the desired inequality (6.11). □

We can state the following results concerning various upper bounds for the operator norm $\|(.,\ldots,.)\|_e$:

Theorem 6.4. *For any* $(T_1,\ldots,T_n)\in B^{(n)}(H)$, *we have the inequalities:*

$$\|(T_1,\ldots,T_n)\|_e^2 \le \begin{cases} \max\limits_{1\le j\le n}\left\{\|T_j\|^2\right\} + \left[\sum\limits_{1\le j\ne k\le n} w^2\left(T_k^*T_j\right)\right]^{\frac{1}{2}}; \\ \max\limits_{1\le j\le n}\left\{\|T_j\|^2\right\} + (n-1)\max\limits_{1\le j\ne k\le n}\left\{w\left(T_k^*T_j\right)\right\}; \\ \left[\max\limits_{1\le j\le n}\left\{\|T_j\|^2\right\}\left\|\sum\limits_{j=1}^n T_j^*T_j\right\|^2\right. \\ \qquad\left. + \max\limits_{1\le j\ne k\le n}\left\{\|T_j\|\,\|T_k\|\right\}\sum\limits_{1\le j\ne k\le n} w\left(T_kT_j^*\right)\right]^{\frac{1}{2}}. \end{cases} \tag{6.12}$$

The proof follows by Theorem 3.1 and Theorem 6.1 and the details are omitted.

On utilising the inequalities (3.8) and (3.17) we can state the following result as well:

Theorem 6.5. *For any* $(T_1,\ldots,T_n)\in B^{(n)}(H)$, *we have:*

$$\|(T_1,\ldots,T_n)\|_e^2 \le \begin{cases} \max\limits_{1\le j\le n}\left\{\sum\limits_{k=1}^n w\left(T_k^*T_j\right)\right\}; \\ \left[\sum\limits_{j,k=1}^n w^2\left(T_k^*T_j\right)\right]^{\frac{1}{2}}; \\ n\max\limits_{1\le j\le n}\left[\sum\limits_{k=1}^n w^2\left(T_k^*T_j\right)\right]^{\frac{1}{2}}; \\ n\left[\sum\limits_{j=1}^n \max\limits_{1\le k\le n}\left\{w^2\left(T_k^*T_j\right)\right\}\right]^{\frac{1}{2}}. \end{cases} \tag{6.13}$$

The results from Section 5 can be also naturally used to provide some reverse inequalities that are of interest.

Theorem 6.6. *Let* $(T_1,\ldots,T_n)\in B^{(n)}(H)$ *and* $\varphi,\phi\in\mathbb{K}$ *such that*

$$\left\|T_jy - \frac{\varphi+\phi}{2}\cdot x\right\| \le \frac{1}{2}|\phi-\varphi| \quad \textit{for any} \quad \|x\|=\|y\|=1 \tag{6.14}$$

and for each $j\in\{1,\ldots,n\}$. *Then*

$$\|(T_1,\ldots,T_n)\|_e^2 \le \frac{1}{n}\left\|\sum_{j=1}^n T_j\right\|^2 + \frac{1}{n}|\phi-\varphi|^2. \tag{6.15}$$

In addition, if $\operatorname{Re}(\phi\bar{\varphi})>0$, *then*

$$\|(T_1,\ldots,T_n)\|_e^2 \le \frac{1}{4n}\cdot\frac{|\phi+\varphi|^2}{\operatorname{Re}(\phi\bar{\varphi})}\left\|\sum_{j=1}^n T_j\right\|^2 \tag{6.16}$$

and

$$\|(T_1,\ldots,T_n)\|_e^2 \le \frac{1}{n}\left\|\sum_{j=1}^n T_j\right\|^2 + \left[|\phi+\varphi| - 2\sqrt{\operatorname{Re}(\phi\bar{\varphi})}\right]\left\|\sum_{j=1}^n T_j\right\|. \tag{6.17}$$

If $\phi \neq -\varphi$, then also

$$\|(T_1, \dots, T_n)\|_e \leq \frac{1}{\sqrt{n}} \left\| \sum_{j=1}^{n} T_j \right\|^2 + \frac{1}{4}\sqrt{n} \cdot \frac{|\phi - \varphi|^2}{|\phi + \varphi|}. \tag{6.18}$$

Proof. For any $x, y \in H$ with $\|x\| = \|y\| = 1$ we have

$$\begin{aligned} \left| \langle x, T_j y\rangle - \frac{\varphi + \phi}{2} \right| &= \left| \left\langle x, T_j y - \frac{\varphi + \phi}{2} x \right\rangle \right| \\ &\leq \|x\| \left\| T_j y - \frac{\varphi + \phi}{2} x \right\| \\ &\leq \frac{1}{2} |\phi - \varphi| \end{aligned}$$

for each $j \in \{1, \dots, n\}$.

Now, on applying Theorem 5.1 for $x_j = T_j y$, we can write from (5.11) the following inequality

$$\|(T_1 y, \dots, T_n y)\|_e \leq \frac{1}{n} \left\| \sum_{j=1}^{n} T_j y \right\| + \frac{1}{4} n |\phi - \varphi|^2$$

for each y with $\|y\| = 1$.

Taking the supremum over $\|y\| = 1$ and utilising Theorem 6.1, we deduce (6.15).

The other inequalities follow by a similar procedure on making use of the inequalities (5.12) – (5.14) and the details are omitted. □

Remark 6.7. The inequality (6.14) is equivalent with

$$\begin{aligned} 0 &\leq \operatorname{Re}\left[(\phi - \langle x, T_j y\rangle)(\langle T_j y, x\rangle - \bar{\varphi})\right] \\ &= (\operatorname{Re}(\phi) - \operatorname{Re}\langle x, T_j y\rangle)(\operatorname{Re}\langle T_j y, x\rangle - \operatorname{Re}(\varphi)) \\ &\quad + (\operatorname{Im}(\phi) - \operatorname{Im}\langle x, T_j y\rangle)(\operatorname{Im}\langle T_j y, x\rangle - \operatorname{Im}(\varphi)) \end{aligned} \tag{6.19}$$

for each $j \in \{1, \dots, n\}$ and $\|x\| = \|y\| = 1$. A sufficient condition for (6.19) to hold is then:

$$\begin{cases} \operatorname{Re}(\varphi) \leq \operatorname{Re}\langle x, T_j y\rangle \leq \operatorname{Re}(\phi) \\ \\ \operatorname{Im}(\varphi) \leq \operatorname{Im}\langle x, T_j y\rangle \leq \operatorname{Im}(\phi) \end{cases} \tag{6.20}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and $j \in \{1, \dots, n\}$.

7. A Norm on $B(H)$

For an operator $A \in B(H)$ we define

$$\delta(A) := \|(A, A^*)\|_e = \sup_{(\lambda,\mu)\in\mathbb{B}_2} \|\lambda A + \mu A^*\|, \tag{7.1}$$

where $\mathbb{B}_2$ is the Euclidean unit ball in $\mathbb{C}^2$.

The properties of this functional are embodied in the following theorem:

Theorem 7.1. *The functional δ is a norm on $B(H)$ and satisfies the double inequality:*

$$\|A\| \leq \delta(A) \leq 2\|A\| \tag{7.2}$$

for any $A \in B(H)$.

Moreover, we have the inequalities

$$\frac{\sqrt{2}}{2}\left\|A^2+(A^*)^2\right\|^{\frac{1}{2}} \le \delta(A) \le \left\|A^2+(A^*)^2\right\|^{\frac{1}{2}}, \tag{7.3}$$

and

$$\frac{\sqrt{2}}{2}\|A+A^*\| \le \delta(A) \le \left[\|A\|^2 + w\left(A^2\right)\right]^{\frac{1}{2}} \tag{7.4}$$

for any $A \in B(H)$, *respectively.*

Proof. First of all, observe, by Theorem 5.1, that we have the representation

$$\delta(A) = \sup_{\|x\|=1,\|y\|=1}\left[|\langle Ay,x\rangle|^2 + |\langle A^*y,x\rangle|^2\right]^{\frac{1}{2}} \tag{7.5}$$

for each $A \in B(H)$.

Obviously $\delta(A) \ge 0$ for each $A \in B(H)$ and of $\delta(A) = 0$ then, by (7.5), $\langle Ay,x\rangle = 0$ for any $x,y \in H$ with $\|x\| = \|y\| = 1$ which implies that $A = 0$. Also, by (7.5), we observe that

$$\begin{aligned}\delta(\alpha A) &= \sup_{\|x\|=1,\|y\|=1}\left[|\langle \alpha Ay,x\rangle|^2 + |\langle \bar{\alpha}A^*y,x\rangle|^2\right]^{\frac{1}{2}} \\ &= |\alpha| \sup_{\|x\|=1,\|y\|=1}\left[|\langle Ay,x\rangle|^2 + |\langle A^*y,x\rangle|^2\right]^{\frac{1}{2}} \\ &= |\alpha|\,\delta(A)\end{aligned}$$

for any $\alpha \in \mathbb{R}$ and $A \in B(H)$.

Now, if $A, B \in B(H)$, then

$$\begin{aligned}\delta(A+B) &= \sup_{(\lambda,\mu)\in\mathbb{B}_2}\|\lambda A + \mu A^* + \lambda B + \mu B^*\| \\ &\le \sup_{(\lambda,\mu)\in\mathbb{B}_2}\|\lambda A + \mu A^*\| + \sup_{(\lambda,\mu)\in\mathbb{B}_2}\|\lambda B + \mu B^*\| \\ &= \delta(A) + \delta(B),\end{aligned}$$

which proves the triangle inequality.

Also, we observe that

$$\delta(A) \ge \sup_{\|x\|=1,\|y\|=1}|\langle Ay,x\rangle| = \|A\|$$

and

$$\begin{aligned}\delta(A) &\le \sup_{\|x\|=1,\|y\|=1}\left[|\langle Ay,x\rangle| + |\langle A^*y,x\rangle|\right] \\ &\le \sup_{\|x\|=1,\|y\|=1}|\langle Ay,x\rangle| + \sup_{\|x\|=1,\|y\|=1}|\langle A^*y,x\rangle| \\ &= 2\|A\|\end{aligned}$$

and the inequality (7.2) is proved.

The inequality (7.3) follows from (6.3) for $n = 2$, $T_1 = A$ and $T_2 = A^*$ while (7.4) follows from Proposition 6.3 and the second inequality in (6.12) for the same choices. □

Remark 7.2. It is easy to see that

$$\frac{\sqrt{2}}{2}\left\|A^2+(A^*)^2\right\|^{\frac{1}{2}} \le \|A\|$$

and

$$\left\|A^2+(A^*)^2\right\|^{\frac{1}{2}},\ \left[\|A\|^2 + w\left(A^2\right)\right]^{\frac{1}{2}} \le 2\|A\|$$

for each $A \in B(H)$. Also, we notice that if A is self-adjoint, then the equality case holds in the second part of (7.3) and in both sides of (7.4). However, it is an open question for the author which of the lower bounds $\|A\|$, $\frac{\sqrt{2}}{2}\|A + A^*\|$ of the norm $\delta(A)$ are better and when. The same question applies for the upper bounds $\left\|A^2 + (A^*)^2\right\|^{\frac{1}{2}}$ and $\left[\|A\|^2 + w(A^2)\right]^{\frac{1}{2}}$, respectively.

REFERENCES

[1] R. BELLMAN, Almost orthogonal series, *Bull. Amer. Math. Soc.*, **50** (1944), 517–519.

[2] E. BOMBIERI, A note on the large sieve, *Acta Arith.*, **18** (1971), 401–404.

[3] R.P. BOAS, A general moment problem, *Amer. J. Math.*, **63** (1941), 361–370.

[4] S.S. DRAGOMIR, On the Boas-Bellman inequality in inner product spaces, *Bull. Austral. Math. Soc.*, **69**(2) (2004), 217–225.

[5] S.S. DRAGOMIR, *Advances in Inequalities of the Schwarz, Grüss and Bessel Type in Inner Product Spaces*, Nova Science Publishers, Inc., 2005.

[6] S.S. DRAGOMIR, On the Bombieri inequality in inner product spaces, *Libertas Math.*, **25** (2005), 13–26.

[7] S.S. DRAGOMIR, Reverses of the Schwarz inequality generalising the Klamkin-McLeneghan result, *Bull. Austral. Math. Soc.*, **73**(1) (2006), 69–78.

[8] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, 1993.

[9] G. POPESCU, Unitary invariants in multivariable operator theory, Preprint, Arχiv.math.04/0410492.

[10] W. RUDIN, *Function Theory in the Unit Ball of* $\mathbb{C}^n$, Springer Verlag, New York, Berlin, 1980.