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ON WEIGHTED INEQUALITIES WITH GEOMETRIC MEAN OPERATOR GENERATED BY THE HARDY-TYPE INTEGRAL TRANSFORM

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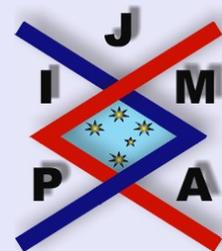
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Abstract

Contents

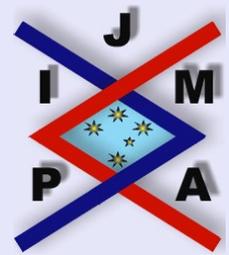


Home Page

Go Back

Close

Quit



Abstract

The generalized geometric mean operator

$$G_K f(x) = \exp \frac{1}{K(x)} \int_0^x k(x, y) \log f(y) dy,$$

with $K(x) := \int_0^x k(x, y) dy$ is considered. A characterization of the weights $u(x)$ and $v(x)$ so that the inequality

$$\left(\int_0^\infty (G_K f(x))^q u(x) dx \right)^{1/q} \leq C \left(\int_0^\infty f(x)^p v(x) dx \right)^{1/p}, \quad f \geq 0,$$

holds is given for all $0 < p, q < \infty$ both for all G_K where $k(x, y)$ satisfies the Oinarov condition and for Riemann-Liouville operators. The corresponding stable bounds of $C = \|G_K\|_{L_v^p \rightarrow L_u^q}$ are pointed out.

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On Weighted Inequalities with Geometric Mean Operator Generated by the Hardy-type Integral Transform

Maria Nassyrova,
Lars-Erik Persson
Vladimir Stepanov

Title Page

Contents



Go Back

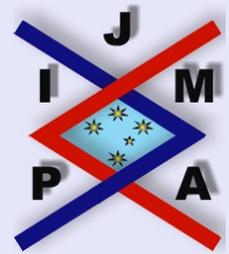
Close

Quit

Page 2 of 37

Contents

1	Introduction	4
2	General Schemes	7
3	Hardy-type Operators. Case $0 < p \leq q < \infty$	10
4	Hardy-type Operators. Case $0 < q < p < \infty$	17
5	Riemann-Liouville Operators	26
References		



On Weighted Inequalities with Geometric Mean Operator Generated by the Hardy-type Integral Transform

Maria Nassyrova,
Lars-Erik Persson
Vladimir Stepanov

Title Page

Contents



Go Back

Close

Quit

Page 3 of 37

1. Introduction

Let $\mathbb{R}_+ := [0, \infty)$ and let $k(x, y) \geq 0$ be a locally integrable kernel defined on $\mathbb{R}_+ \times \mathbb{R}_+$ and such that

$$\int_{\mathbb{R}_+} k(x, y) dy = 1$$

for almost all $x \in \mathbb{R}_+$.

Denote

$$Kf(x) := \int_0^\infty k(x, y)f(y)dy, \quad f(y) \geq 0.$$

If $Kf(x) < \infty$, then there exists a limit

$$(1.1) \quad G_K f(x) := \lim_{\alpha \downarrow 0} [Kf^\alpha(x)]^{1/\alpha}$$

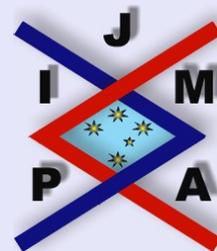
and

$$(1.2) \quad G_K f(x) = \exp \int_0^\infty k(x, y) \log f(y) dy.$$

For $0 < p < \infty$ and a weight function $v(x) \geq 0$ we put

$$\|f\|_{L_v^p} := \left(\int_0^\infty |f(x)|^p v(x) dx \right)^{1/p}$$

and make use of the abbreviation $\|f\|_{L^p}$ when $v(x) \equiv 1$.



**On Weighted Inequalities with
Geometric Mean Operator
Generated by the Hardy-type
Integral Transform**

Maria Nassyrova,
Lars-Erik Persson
Vladimir Stepanov

Title Page

Contents



Go Back

Close

Quit

Page 4 of 37

Suppose $u(x) \geq 0$ and $v(x) \geq 0$ are weight functions and $0 < p, q < \infty$. This paper deals with $L_v^p - L_u^q$ inequalities of the form

$$(1.3) \quad \left(\int_0^\infty (G_K f)^q u \right)^{1/q} \leq C \left(\int_0^\infty f^p v \right)^{1/p}, \quad f \geq 0,$$

where a constant C is independent on f and we always assume that C is the least possible, that is $C = \|G_K\|_{L_v^p \rightarrow L_u^q}$, where

$$(1.4) \quad \|G_K\|_{L_v^p \rightarrow L_u^q} := \sup_{f \geq 0} \frac{\|G_K f\|_{L_u^q}}{\|f\|_{L_v^p}}.$$

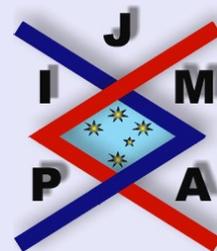
For the classical case $k(x, y) = \frac{1}{x} \chi_{[0, x]}(y)$ with the Hardy averaging operator

$$(1.5) \quad Hf(x) := \frac{1}{x} \int_0^x f(y) dy$$

the inequality (1.3) was characterized in [6] (see also [7]).

In Section 2 of this paper we present a general scheme on how inequalities of the type (1.3) can be characterized via the limiting procedure in similar characterization (in suitable forms) of some corresponding Hardy-type inequalities. In Section 3 we characterize the weights $u(x)$ and $v(x)$ so that (1.3) with $0 < p \leq q < \infty$ holds when

$$(1.6) \quad G_K f(x) = \exp \frac{1}{K(x)} \int_0^x k(x, y) \log f(y) dy$$



On Weighted Inequalities with Geometric Mean Operator Generated by the Hardy-type Integral Transform

Maria Nassyrova,
Lars-Erik Persson
Vladimir Stepanov

Title Page

Contents



Go Back

Close

Quit

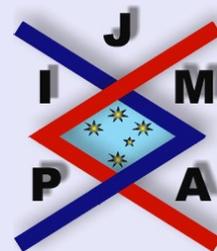
Page 5 of 37

with

$$K(x) := \int_0^x k(x, y)dy < \infty, \quad x > 0,$$

and $K(0) = 0$, $K(\infty) = \infty$, and when $k(x, y)$ satisfies the Oinarov condition (see Theorem 3.1). The corresponding characterization for the case $0 < q < p < \infty$ can be found in Section 4 (see Theorem 4.1). A fairly precise result in the case $k(x, y) = (x - y)^\gamma$, $\gamma > 0$, i.e. when G_K is generated by the Riemann-Liouville convolutional operator, can be found in Section 5 (see Theorem 5.1). In particular, this investigation shows that inequalities of the type (1.3) can be proved also when the Oinarov condition is violated (because for $0 < \gamma < 1$ this kernel does not satisfy this condition). In order to prove these results we need new characterizations of weighted inequalities with the Riemann-Liouville operator which are of independent interest (see Theorem 5.2 and 5.3).

Finally note that the operator (1.6) was studied in a similar connection, by E.R. Love [3], where a sufficient condition was proved for the inequality (1.3) to be valid in the case $p = q = 1$ and special weights.



**On Weighted Inequalities with
Geometric Mean Operator
Generated by the Hardy-type
Integral Transform**

Maria Nassyrova,
Lars-Erik Persson
Vladimir Stepanov

Title Page

Contents



Go Back

Close

Quit

Page 6 of 37

2. General Schemes

We begin with some general remarks. By using the elementary property

$$(2.1) \quad G_K(f^s) = (G_K f)^s, \quad -\infty < s < \infty,$$

we see that

$$(2.2) \quad \|G_K\|_{L_v^p \rightarrow L_u^q} = \|G_K\|_{L^p \rightarrow L_w^q} = \|G_K\|_{L^s \rightarrow L_w^{sq/p}}^{s/p},$$

where

$$(2.3) \quad w := \left[G_K \left(\frac{1}{v} \right) \right]^{q/p} u.$$

Moreover, from (1.1) and (2.2)

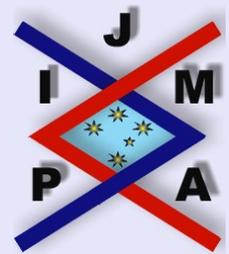
$$(2.4) \quad \|G_K\|_{L_v^p \rightarrow L_u^q} = \lim_{\alpha \downarrow 0} \|K\|_{L^{p/\alpha} \rightarrow L_w^{q/\alpha}}^{1/\alpha}.$$

The last formula generates the “precise” scheme for characterization of (1.3) provided the norm of the associated integral operator K has very accurate two-sided estimates of the norm

$$(2.5) \quad c_1(p, q)F(w, p, q) \leq \|K\|_{L^p \rightarrow L_w^q} \leq c_2(p, q)F(w, p, q)$$

in the sense that there exist the limits

$$(2.6) \quad c_i(p, q, F) = \lim_{\alpha \downarrow 0} \left[c_i \left(\frac{p}{\alpha}, \frac{q}{\alpha} \right) F \left(w, \frac{p}{\alpha}, \frac{q}{\alpha} \right) \right]^{1/\alpha}, \quad i = 1, 2,$$



**On Weighted Inequalities with
Geometric Mean Operator
Generated by the Hardy-type
Integral Transform**

Maria Nassyrova,
Lars-Erik Persson
Vladimir Stepanov

Title Page

Contents



Go Back

Close

Quit

Page 7 of 37

and

$$c_i(p, q, F) \in (0, \infty), \quad i = 1, 2.$$

A natural consequence of (2.2) and (2.4) is then the two-sided estimate

$$(2.7) \quad c_1(p, q, F) \leq \|G_K\|_{L^p \rightarrow L_w^q} \leq c_2(p, q, F),$$

which characterizes (1.3) with the least possible constants provided the estimates in (2.5) are also the best.

This scheme was realized in [6] where K is the Hardy averaging operator (1.5) and $0 < p \leq q < \infty$. For this purpose a new non-Muckenhoupt form of (2.5) was used. We generalize this result here for the Riemann-Liouville kernel in Section 4.

Unfortunately, the above scheme does not work for a more general operator or even for the Hardy operator, if $q < p$, because the estimate (2.5) happens to be vulnerable, when the parameters p and q tend to ∞ , so that the limits $c_i(p, q, F)$ become either 0 or ∞ . However, if the functional F is homogeneous in the sense that

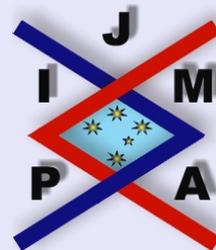
$$(2.8) \quad \left[F\left(w, \frac{p}{\alpha}, \frac{q}{\alpha}\right) \right]^{1/\alpha} = F(w, p, q), \quad \alpha > 0,$$

then an alternative “stable” scheme for (2.7) works, provided the lower bound

$$(2.9) \quad c_3(p, q)F(w, p, q) \leq \|G_K\|_{L^p \rightarrow L_w^q}$$

can be established. The right hand side of (2.7) follows from the upper bound from (2.5), (2.2) and Jensen’s inequality

$$(2.10) \quad G_K f(x) \leq K f(x),$$



On Weighted Inequalities with Geometric Mean Operator Generated by the Hardy-type Integral Transform

Maria Nassyrova,
Lars-Erik Persson
Vladimir Stepanov

Title Page

Contents



Go Back

Close

Quit

Page 8 of 37

because (2.8) implies

$$(2.11) \quad \left[F \left(w, s, \frac{sq}{p} \right) \right]^{s/p} = F(w, p, q), \quad s > 1.$$

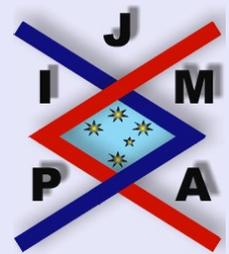
To realize the “stable” scheme for the Hardy averaging operator, when $0 < q < p < \infty$ and $p > 1$ an alternative (non-Mazya-Rozin) functional F was found in [6] in the form

$$(2.12) \quad F(w, p, q) = \left(\int_0^\infty \left(\frac{1}{x} \int_0^x w \right)^{q/(p-q)} w(x) dx \right)^{1/q-1/p}$$

which obviously satisfies (2.11).

The idea to use (2.2) and (2.10) for the upper bound estimate of the norm $\|G_K\|_{L_v^p \rightarrow L_u^q}$ originated from the paper by Pick and Opic [8], where the authors obtained two-sided estimates of $\|G_H\|_{L_v^p \rightarrow L_u^q}$, $0 < q < p < \infty$. However, they realized (2.5) in Muckenhoupt or Mazya-Rozin form with unstable factors and, therefore, the estimate (2.7) was rather uncertain.

Throughout the paper, expressions of the form $0 \cdot \infty$, ∞/∞ , $0/0$ are taken to be equal to zero; and the inequality $A \lesssim B$ means $A \leq cB$ with a constant $c > 0$ independent of weight functions. Moreover, the relationship $A \approx B$ is interpreted as $A \lesssim B \lesssim A$ or $A = cB$. Everywhere the constant C in explored inequalities are considered to be the least possible one. In the case $q < p$ the auxiliary parameter r is defined by $1/r = 1/q - 1/p$.



On Weighted Inequalities with Geometric Mean Operator Generated by the Hardy-type Integral Transform

Maria Nassyrova,
Lars-Erik Persson
Vladimir Stepanov

Title Page

Contents



Go Back

Close

Quit

Page 9 of 37

3. Hardy-type Operators. Case $0 < p \leq q < \infty$

In the sequel we let $k(x, y) \geq 0$ be locally integrable in $\mathbb{R}_+ \times \mathbb{R}_+$ and satisfies the following (Oinarov) condition: there exists a constant $d \geq 1$ independent on x, y, z such that

$$(3.1) \quad \frac{1}{d}(k(x, z)+k(z, y)) \leq k(x, y) \leq d(k(x, z)+k(z, y)), \quad x \geq z \geq y \geq 0.$$

We assume that

$$(3.2) \quad K(x) := \int_0^x k(x, y)dy < \infty, \quad x > 0,$$

and $K(0) = 0, K(\infty) = \infty$.

Without loss of generality we may and shall also assume that $k(x, y)$ is non-decreasing in x and nonincreasing in y .

We consider the following Hardy-type operator

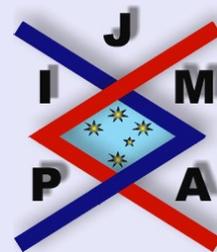
$$(3.3) \quad \mathbf{K}f(x) := \frac{1}{K(x)} \int_0^x k(x, y)f(y)dy, \quad f(y) \geq 0, x \geq 0,$$

and the corresponding geometric mean operator $G_{\mathbf{K}}$:

$$G_{\mathbf{K}}f(x) = \exp \frac{1}{K(x)} \int_0^x k(x, y) \log f(y)dy.$$

Our main goal in this section is to find necessary and sufficient conditions on the weights $u(x)$ and $v(x)$ so that the inequality

$$(3.4) \quad \left(\int_0^\infty (G_{\mathbf{K}}f(x))^q u(x) dx \right)^{1/q} \leq C \left(\int_0^\infty f(x)^p v(x) dx \right)^{1/p}, \quad f \geq 0,$$



On Weighted Inequalities with
Geometric Mean Operator
Generated by the Hardy-type
Integral Transform

Maria Nassyrova,
Lars-Erik Persson
Vladimir Stepanov

Title Page

Contents



Go Back

Close

Quit

Page 10 of 37

holds for $0 < p \leq q < \infty$, and to point out the corresponding stable estimates of $C = \|G_{\mathbf{K}}\|_{L_v^p \rightarrow L_w^q}$. In view of our discussion in the previous section we first need to have the two-sided estimates of $\|\mathbf{K}\|_{L^p \rightarrow L_w^q}$ in suitable form. First we observe that it follows from ([1, Theorem 2.1]) by changing variables $x \rightarrow \frac{1}{x}, y \rightarrow \frac{1}{y}$ and by using the duality principle (see [11, Section 2.3]) that for $1 < p \leq q < \infty$ we have

$$(3.5) \quad c_4(d)\mathbb{A} \leq \|\mathbf{K}\|_{L^p \rightarrow L_w^q} \leq c_5(p, q, d)\mathbb{A},$$

where d is defined by (3.1),

$$(3.6) \quad \mathbb{A} = \max(\mathbb{A}_0, \mathbb{A}_1)$$

with

$$(3.7) \quad \mathbb{A}_0 := \sup_{t>0} t^{-1/p} \left(\int_0^t w \right)^{1/q}$$

and

$$(3.8) \quad \mathbb{A}_1 := \sup_{t>0} \left(\int_0^t k(t, y)^{p'} dy \right)^{-1/p} \times \left(\int_0^t \left(\int_0^x k(x, y)^{p'} dy \right)^q \frac{w(x)}{K(x)^q} dx \right)^{1/q}.$$

It is easy to see that \mathbb{A}_0 satisfies (2.8) or (2.11), but not \mathbb{A}_1 . It is also known, that neither $\mathbb{A}_0 < \infty$ nor $\mathbb{A}_1 < \infty$ alone is sufficient for $\|\mathbf{K}\|_{L^p \rightarrow L_w^q} < \infty$ in general



On Weighted Inequalities with Geometric Mean Operator Generated by the Hardy-type Integral Transform

Maria Nassyrova,
Lars-Erik Persson
Vladimir Stepanov

Title Page

Contents



Go Back

Close

Quit

Page 11 of 37

(see counterexamples in [4], [10] and [5]). For this reason we need to require some additional conditions of a kernel $k(x, y)$ for the property

$$(3.9) \quad \mathbb{A}_0 \approx \|\mathbf{K}\|_{L^p \rightarrow L_w^q},$$

and we will soon discuss this question in detail (see Proposition 3.2). Now we are ready to state our main theorem of this section:

Theorem 3.1. *Let $0 < p \leq q < \infty$ and the kernel $k(x, y) \geq 0$ be such that (3.9) holds. Then (3.4) holds if and only if $\mathbb{A}_0 < \infty$. Moreover,*

$$(3.10) \quad \mathbb{A}_0 \leq \|G_{\mathbf{K}}\|_{L_v^p \rightarrow L_u^q} \lesssim \mathbb{A}_0.$$

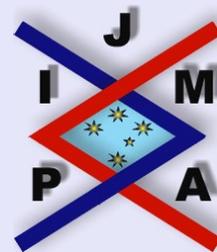
Proof. The sufficiency including the upper estimate in (3.10) follows from (2.2), (2.10) and (3.9). Moreover, we note that, by (2.2),

$$(3.11) \quad \left(\int_0^\infty (G_{\mathbf{K}} f)^q w \right)^{1/q} \leq \|G_{\mathbf{K}}\|_{L_v^p \rightarrow L_u^q} \left(\int_0^\infty f^p \right)^{1/p}.$$

By applying this estimate with $f_t(x) = t^{-1/p} \chi_{[0,t]}(x)$ for fixed $t > 0$ (obviously, $\|f_t\|_p = 1$) we see that

$$t^{-1/p} \left(\int_0^t w \right)^{1/q} \leq \|G_{\mathbf{K}}\|_{L_v^p \rightarrow L_u^q}.$$

By taking the supremum over t we have also proved the necessity including the lower estimate in (3.10) and the proof is complete. \square



On Weighted Inequalities with Geometric Mean Operator Generated by the Hardy-type Integral Transform

Maria Nassyrova,
Lars-Erik Persson
Vladimir Stepanov

Title Page

Contents



Go Back

Close

Quit

Page 12 of 37

Remark 3.1. For the case when $k(x, y) \equiv 1$ we have that G_K coincide with the usual geometric mean operator

$$Gf(x) = \exp \frac{1}{x} \int_0^x \log f(y) dy$$

and thus we see that Theorem 3.1 may be regarded as a genuine generalization of Theorem 2 in [6] (see also [7]).

As mentioned before we shall now discuss the question of for which kernels $k(x, y)$ does the condition (3.9) hold. We need the following notation:

$$K_p(x) := \int_0^x k(x, y)^{p'} dy,$$

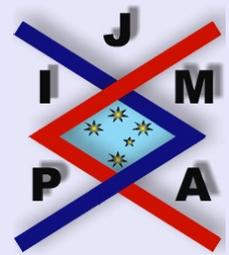
$$(3.12) \quad \alpha_0 := \sup_{t>0} K_p(t)^{1/p'} \left[\sup_{0<x<t} x^{1/p'} k(t, x) \right]^{1/q},$$

$$(3.13) \quad \alpha_1 := \sup_{t>0} t^{1/p'} \left(\int_t^\infty k(x, t)^q K(x)^{-q} d(x^{q/p}) \right)^{1/q},$$

and

$$(3.14) \quad \alpha_2 := \sup_{t>0} t^{1/p'} \left(\int_t^\infty x^{q/p} k(x, t)^q d(-K(x)^{-q}) \right)^{1/q}.$$

Proposition 3.2. Let $1 < p \leq q < \infty$. If a kernel $k(x, y) \geq 0$ is such that for all weights w (3.9) is true, then $\alpha_0 + \alpha_1 < \infty$. Conversely, if $\alpha_0 + \alpha_2 < \infty$, then (3.9) holds. In particular, if $\alpha_2 \lesssim \alpha_1$, then (3.9) is equivalent with the conditions $\alpha_0 < \infty$ and $\alpha_1 < \infty$.



On Weighted Inequalities with Geometric Mean Operator Generated by the Hardy-type Integral Transform

Maria Nassyrova,
Lars-Erik Persson
Vladimir Stepanov

Title Page

Contents



Go Back

Close

Quit

Page 13 of 37

Proof. It is known ([11], (see also [2, Theorem 2.13])) that

$$(3.15) \quad \|\mathbf{K}\|_{L^p \rightarrow L_w^q} = \max(A_0, A_1),$$

where

$$A_0 := \sup_{t>0} \left(\int_t^\infty k(x, t)^q \frac{w(x)}{K(x)^q} dx \right)^{1/q} t^{1/p'},$$

$$A_1 := \sup_{t>0} \left(\int_t^\infty \frac{w(x)}{K(x)^q} dx \right)^{1/q} K_p(t)^{1/p'}.$$

It follows from a more general result ([5, Theorem 4]) that

$$(3.16) \quad A_0 \approx \|\mathbf{K}\|_{L^p \rightarrow L_w^q}$$

for all weights w , if and only if $\alpha_0 < \infty$. Moreover, if (3.9) is true for all weights w , then (3.15) implies

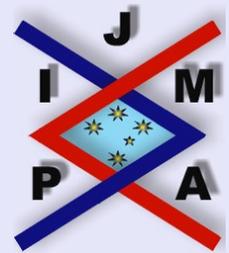
$$(3.17) \quad A_0 \lesssim \mathbb{A}_0$$

and for $w(x) = x^{q/p-1}$ it brings $\alpha_1 < \infty$. We observe that the inequality

$$(3.18) \quad \mathbb{A}_0 \lesssim pA_0$$

is always true. Indeed, by applying Minkowski's integral inequality we find that

$$\left(\int_0^t w \right)^{1/q} = \left(\int_0^t \left(\int_0^x k(x, y) dy \right)^q \frac{w(x)}{K(x)^q} dx \right)^{1/q}$$



**On Weighted Inequalities with
Geometric Mean Operator
Generated by the Hardy-type
Integral Transform**

Maria Nassyrova,
Lars-Erik Persson
Vladimir Stepanov

Title Page

Contents



Go Back

Close

Quit

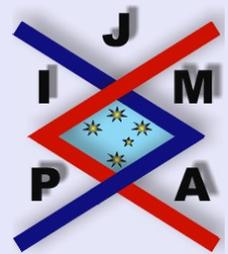
Page 14 of 37

$$\begin{aligned}
&\leq \int_0^t \left(\int_y^\infty k(x, y)^q \frac{w(x)}{K(x)^q} dx \right)^{1/q} dy \\
&\leq A_0 \int_0^t y^{-1/p'} dy \\
&= A_0 p t^{1/p}
\end{aligned}$$

and (3.18) follows. Thus, (3.9) implies (3.16). Consequently, $\alpha_0 < \infty$ and, thus, $\alpha_0 + \alpha_1 < \infty$.

Now, suppose that $\alpha_0 + \alpha_2 < \infty$. Then (3.16) holds and it is sufficient to prove (3.17). To this end we note that

$$\begin{aligned}
&\int_t^\infty k(x, t)^q \frac{w(x)}{K(x)^q} dx \\
&\approx k(t, t)^q \int_t^\infty \frac{w(x)}{K(x)^q} dx + \int_t^\infty \frac{w(x)}{K(x)^q} \left(\int_t^x dk(s, t)^q \right) dx \\
&= k(t, t)^q \int_t^\infty w(x) \left(\int_x^\infty d \left(\frac{-1}{K(y)^q} \right) \right) dx \\
&\quad + \int_t^\infty dk(s, t)^q \left(\int_s^\infty \frac{w(x) dx}{K(x)^q} \right) \\
&= k(t, t)^q \int_t^\infty d \left(\frac{-1}{K(y)^q} \right) \left(\int_t^y w(x) dx \right) \\
&\quad + \int_t^\infty dk(s, t)^q \int_s^\infty w(x) dx \int_x^\infty d \left(\frac{-1}{K(y)^q} \right)
\end{aligned}$$



On Weighted Inequalities with Geometric Mean Operator Generated by the Hardy-type Integral Transform

Maria Nassyrova,
Lars-Erik Persson
Vladimir Stepanov

Title Page

Contents



Go Back

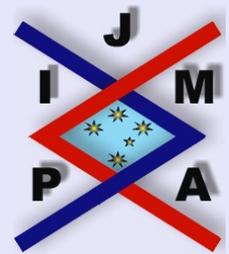
Close

Quit

Page 15 of 37

$$\begin{aligned}
&\leq \mathbb{A}_0^q \left(\int_t^\infty y^{q/p} d \left(\frac{-1}{K(y)^q} \right) \right) k(t, t)^q \\
&\quad + \int_t^\infty dk(s, t)^q \int_s^\infty d \left(\frac{-1}{K(y)^q} \right) \int_s^y w(x) dx \\
&\leq \mathbb{A}_0^q \int_t^\infty y^{q/p} k(y, t)^q d \left(\frac{-1}{K(y)^q} \right) \\
&\leq \mathbb{A}_0^q \alpha_2^q t^{-q/p'}.
\end{aligned}$$

and (3.17) follows. Now (3.5), (3.6), (3.16) and (3.17) imply (3.9). The proof is complete. \square



On Weighted Inequalities with Geometric Mean Operator Generated by the Hardy-type Integral Transform

Maria Nassyrova,
Lars-Erik Persson
Vladimir Stepanov

Title Page

Contents



Go Back

Close

Quit

Page 16 of 37

4. Hardy-type Operators. Case $0 < q < p < \infty$

Put

$$(4.1) \quad \mathbb{B}_0 := \left(\int_0^\infty \left(\frac{1}{t} \int_0^t w \right)^{r/q} dt \right)^{1/r}.$$

A crucial condition for this case corresponding to the condition (3.9) for the case $0 < p \leq q < \infty$ is the following:

$$(4.2) \quad \mathbb{B}_0 \approx \|\mathbf{K}\|_{L^p \rightarrow L_w^q}.$$

We now state our main result in this Section.

Theorem 4.1. *Let $0 < q < p < \infty$ and the kernel $k(x, y) \geq 0$ be such that (4.2) holds. Also we assume that*

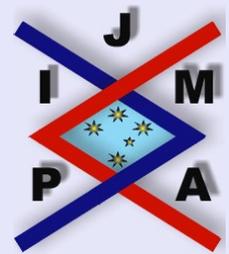
$$(4.3) \quad \frac{x}{K(x)} \int_0^1 k(x, xt) \log \frac{1}{t} dt \leq \alpha_3 < \infty, \quad x > 0.$$

Then (3.4) holds if and only if $\mathbb{B}_0 < \infty$. Moreover,

$$(4.4) \quad \|G_{\mathbf{K}}\|_{L_v^p \rightarrow L_w^q} \approx \mathbb{B}_0.$$

Proof. The sufficiency including the upper estimate in (4.4) follows by using (2.2), (2.10) and (4.2). Next we show that

$$(4.5) \quad D \lesssim \|G_{\mathbf{K}}\|_{L^p \rightarrow L_w^q},$$



On Weighted Inequalities with
Geometric Mean Operator
Generated by the Hardy-type
Integral Transform

Maria Nassyrova,
Lars-Erik Persson
Vladimir Stepanov

Title Page

Contents



Go Back

Close

Quit

Page 17 of 37

where

$$D := \left(\int_0^\infty \left(\int_x^\infty \frac{w(y)}{y^q} dy \right)^{r/q} x^{r/q'} dx \right)^{1/r}.$$

For this purpose we consider

$$f_0(x) = x^{r/(pq')} \left(\int_x^\infty \frac{w(y)}{y^q} dy \right)^{r/(pq)}.$$

Then

$$D^{r/p} = \|f_0\|_p.$$

We have

$$\begin{aligned} & \|G_{\mathbf{K}}\|_{L^p \rightarrow L_w^q} \|f_0\|_p \\ &= \|G_{\mathbf{K}}\|_{L^p \rightarrow L_w^q} D^{r/p} \\ &\geq \left(\int_0^\infty (G_{\mathbf{K}} f_0)^q w \right)^{1/q} \\ &= \left[\int_0^\infty \left(\exp \frac{1}{K(x)} \int_0^x k(x, y) \right. \right. \\ &\quad \left. \left. \times \log \left\{ y^{r/(pq')} \left(\int_y^\infty \frac{w(z)}{z^q} dz \right)^{r/(pq)} \right\} dy \right)^q w(x) dx \right]^{1/q} \\ &\geq \left[\int_0^\infty \left(\exp \frac{1}{K(x)} \int_0^x k(x, y) \log y dy \right)^{rq/(pq')} \right. \end{aligned}$$



**On Weighted Inequalities with
Geometric Mean Operator
Generated by the Hardy-type
Integral Transform**

Maria Nassyrova,
Lars-Erik Persson
Vladimir Stepanov

Title Page

Contents

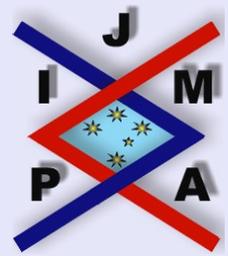


Go Back

Close

Quit

Page 18 of 37



On Weighted Inequalities with Geometric Mean Operator Generated by the Hardy-type Integral Transform

Maria Nassyrova,
Lars-Erik Persson
Vladimir Stepanov

Title Page

Contents



Go Back

Close

Quit

Page 19 of 37

$$\times \left(\int_x^\infty \frac{w(z)}{z^q} dz \right)^{r/p} w(x) dx \Big]^{1/q}$$

$$(4.6) \quad =: J.$$

Moreover,

$$\begin{aligned} \frac{1}{K(x)} \int_0^x k(x, y) \log y dy &= \log x + \frac{1}{K(x)} \int_0^x k(x, y) \log \frac{y}{x} dy \\ &= \log x - \frac{x}{K(x)} \int_0^1 k(x, xt) \log \frac{1}{t} dt \\ &\geq \log x - \alpha_3, \end{aligned}$$

where α_3 is a constant from (4.3). Then

$$(4.7) \quad J^q \geq \exp \left\{ \frac{-\alpha_3 r q}{(p q')} \right\} \int_0^\infty \left(\int_x^\infty \frac{w(z)}{z^q} dz \right)^{r/p} w(x) x^{r/(p q')} dx \approx D^r,$$

the last estimate is obtained by partial integration. Now (4.5) follows by just combining (4.6) and (4.7).

It remains to show that $D \geq \mathbb{B}_0$. We have

$$\begin{aligned} \int_0^t w(x) dx &= q \int_0^t \left(\int_0^s z^{q-1} dz \right) \frac{w(s)}{s^q} ds \\ &= q \int_0^t \left(\int_z^t \frac{w(s)}{s^q} ds \right) z^{q-1} dz \\ &= q \int_0^t \left\{ \left(\int_z^t \frac{w(s)}{s^q} ds \right) z^{q-1+q/(2p)} \right\} z^{-q/(2p)} dz \end{aligned}$$

(by Hölder's inequality with conjugate exponents $\frac{r}{q}$ and $\frac{p}{q}$)

$$\leq q \left(\int_0^t \left(\int_z^t \frac{w(s)}{s^q} ds \right)^{r/q} z^{r/q' + r/(2p)} dz \right)^{q/r} \left(\int_0^t \frac{dz}{\sqrt{z}} \right)^{q/p}.$$

This implies

$$\begin{aligned} \mathbb{B}_0^r &\lesssim \int_0^\infty \left(\int_0^t \left(\int_z^\infty \frac{w(s)}{s^q} ds \right)^{r/q} z^{r/q' + r/(2p)} dz \right) t^{r/(2p) - r/q} dt \\ &= \int_0^\infty \left(\int_z^\infty \frac{w(s)}{s^q} ds \right)^{r/q} z^{r/q' + r/(2p)} \left(\int_z^\infty t^{r/(2p) - r/q} dt \right) dz \\ &\approx \int_0^\infty \left(\int_z^\infty \frac{w(s)}{s^q} ds \right)^{r/q} z^{r/q'} dz = D^r. \end{aligned}$$

Thus the lower bound in (4.4) and also the necessity is proved; so the proof is complete. \square

Next we shall analyze and discuss for which kernels $k(x, y)$ the crucial conditions (4.2) holds. First we note that it follows for the case $1 < q < p < \infty$ from a well-known result ([11], see also [2, Theorem 2.19]) that

$$(4.8) \quad \|\mathbf{K}\|_{L^p \rightarrow L_w^q} \approx B_0 + B_1,$$

where

$$B_0 := \left(\int_0^\infty \left(\int_t^\infty k(x, t)^q \frac{w(y)}{K(x)^q} dy \right)^{r/q} t^{r/q'} dt \right)^{1/r}$$



On Weighted Inequalities with Geometric Mean Operator Generated by the Hardy-type Integral Transform

Maria Nassyrova,
Lars-Erik Persson
Vladimir Stepanov

Title Page

Contents



Go Back

Close

Quit

Page 20 of 37

and

$$B_1 := \left(\int_0^\infty \left(\int_t^\infty \frac{w(x)}{K(x)^q} dx \right)^{r/p} \frac{K_p(t)^{r/p'} w(t)}{K(t)^q} dt \right)^{1/r}.$$

Moreover, it is established in ([5, Theorem 13]) that

$$B_1 \leq \beta_0 B_0$$

for all weights w for which $B_0 < \infty$, if and only if,

$$(4.9) \quad \beta_0 := \sup_{t>0} K_p(t)^{1/p'} \left(\int_0^t k(t,x)^r d(x^{r/p'}) \right)^{-1/r} < \infty.$$

Therefore

$$(4.10) \quad \|\mathbf{K}\|_{L^p \rightarrow L_w^q} \approx B_0$$

under the condition (4.9).

Partly guided by our investigation in the previous section we shall now continue by comparing the constant B_0 and \mathbb{B}_0 .

Proposition 4.2. *Let $1 < q < p < \infty$. Then*

(a) $\mathbb{B}_0 \lesssim B_0$,

(b) $B_0 \leq \beta_1 \mathbb{B}_0$, if

$$(4.11) \quad \beta_1 := \left(\int_0^\infty \left(\int_0^x k(x,t)^r t^{r/q'} dt \right)^{p/r} x^{p/q} \times K(x)^{-p(1+1/q)} d(K(x)) \right)^{1/p} < \infty.$$



On Weighted Inequalities with Geometric Mean Operator Generated by the Hardy-type Integral Transform

Maria Nassyrova,
Lars-Erik Persson
Vladimir Stepanov

Title Page

Contents



Go Back

Close

Quit

Page 21 of 37

Proof. (a) We have

$$\mathbb{B}_0^r := \int_0^\infty \left(\int_0^t \left(\int_0^x k(x,y) dy \right)^q \frac{w(x)}{K(x)^q} dx \right)^{r/q} t^{-r/q} dt$$

(applying Minkowski's integral inequality)

$$\begin{aligned} &\leq \int_0^\infty \left(\int_0^t \left(\int_y^t k(x,y)^q \frac{w(x)}{K(x)^q} dx \right)^{1/q} dy \right)^r t^{-r/q} dt \\ &\leq \int_0^\infty \left(\int_0^t \left(\int_y^\infty k(x,y)^q \frac{w(x)}{K(x)^q} dx \right)^{1/q} y^\alpha y^{-\alpha} dy \right)^r t^{-r/q} dt \end{aligned}$$

(applying Hölder's inequality, $\alpha \in \left(\frac{1}{q'}, \frac{1}{r'}\right)$)

$$\begin{aligned} &\leq \int_0^\infty \left(\int_0^t \left(\int_y^\infty k(x,y)^q \frac{w(x)}{K(x)^q} dx \right)^{r/q} y^{\alpha r} dy \right) \left(\int_0^t y^{-\alpha r'} dy \right)^{r-1} t^{-r/q} dt \\ &= (1 - \alpha r')^{1-r} \int_0^\infty \left(\int_y^\infty k(x,y)^q \frac{w(x)}{K(x)^q} dx \right)^{r/q} \left(\int_y^\infty t^{r-1-\alpha r-r/q} dt \right) y^{\alpha r} dy \\ &= \frac{(1 - \alpha r')^{1-r}}{r(\alpha - 1/q')} B_0^r. \end{aligned}$$

(b) Indeed, following the proof of Proposition 3.2, we find

$$B_0^r \leq \int_0^\infty \left(\int_t^\infty k(x,t)^q \left(\int_t^x w \right) d(-K(x)^{-q}) \right)^{r/q} t^{r/q'} dt$$



On Weighted Inequalities with
Geometric Mean Operator
Generated by the Hardy-type
Integral Transform

Maria Nassyrova,
Lars-Erik Persson
Vladimir Stepanov

Title Page

Contents

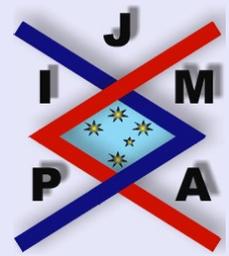


Go Back

Close

Quit

Page 22 of 37



On Weighted Inequalities with Geometric Mean Operator Generated by the Hardy-type Integral Transform

Maria Nassyrova,
Lars-Erik Persson
Vladimir Stepanov

Title Page

Contents



Go Back

Close

Quit

Page 23 of 37

(by Minkowski's integral inequality)

$$\begin{aligned} &\leq \left(\int_0^\infty \left(\int_0^x k(x,t)^r \left(\int_t^x w \right)^{r/q} t^{r/q'} dt \right)^{q/r} d(-K(x)^{-q}) \right)^{r/q} \\ &\leq \left(\int_0^\infty \left(\frac{1}{x} \int_0^x w \right) \left(\int_0^x k(x,t)^r t^{r/q'} dt \right)^{q/r} x d(-K(x)^{-q}) \right)^{r/q} \\ &= \left(q \int_0^\infty \left(\frac{1}{x} \int_0^x w \right) \left(\int_0^x k(x,t)^r t^{r/q'} dt \right)^{q/r} x K(x)^{-(q+1)} dK(x) \right)^{r/q} \end{aligned}$$

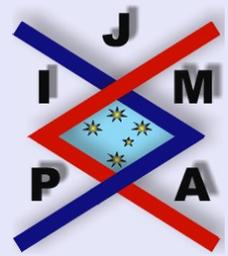
(by Hölder's inequality applied with conjugate exponents $\frac{r}{q}$ and $\frac{p}{q}$)

$$\leq q^{r/q} \beta_1^r \int_0^\infty \left(\frac{1}{x} \int_0^x w \right)^{r/q} dx = q^{r/q} \beta_1^r \mathbb{B}_0^r$$

and the proof follows. □

Remark 4.1. *It is easy to see that the conditions (4.3), (4.9) and (4.11) are satisfied with $k(x,y) \equiv 1$, i.e. when $G_{\mathbf{K}} = G$ (the standard geometric mean operator) and we conclude that Theorem 4.1 may be seen as a generalization of Theorem 4 in [6] (see also [7]).*

We finish this Section by showing that the kernel, investigated in [4], satisfies the condition (4.3).



Example 4.1. Let the kernel $k(x, y)$ be given by

$$(4.12) \quad k(x, y) = \varphi\left(\frac{y}{x}\right),$$

where $\varphi(t) \geq 0$ is decreasing function on $(0, 1)$ satisfying

$$(4.13) \quad \varphi(ts) \leq d(\varphi(t) + \varphi(s)), \quad 0 < t, s < 1.$$

Then (3.1) is obviously valid.

If $\int_0^1 \varphi(t) dt < \infty$, then the kernel $k(x, y)$ of the form (4.12) satisfies (4.3).
Indeed,

$$\begin{aligned} \int_0^1 k(x, xt) \log \frac{1}{t} dt &= \int_0^1 \varphi(t) \log \frac{1}{t} dt \\ &= \sum_{k=0}^{\infty} \int_{2^{-k-1}}^{2^{-k}} \varphi(t) \log \frac{1}{t} dt \\ &\leq \sum_{k=0}^{\infty} \varphi(2^{-k-1}) \int_{2^{-k-1}}^{2^{-k}} \log \frac{1}{t} dt \\ &\lesssim \sum_{k=0}^{\infty} \varphi(2^{-k-1}) (k+1) 2^{-k-1} \\ &= \sum_{k=1}^{\infty} \varphi(2^{-k}) k 2^{-k} \\ &\leq 2d \sum_{k=1}^{\infty} \varphi(2^{-k/2}) k 2^{-k} \end{aligned}$$

**On Weighted Inequalities with
Geometric Mean Operator
Generated by the Hardy-type
Integral Transform**

Maria Nassyrova,
Lars-Erik Persson
Vladimir Stepanov

Title Page

Contents



Go Back

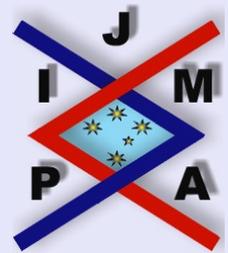
Close

Quit

Page 24 of 37

$$\begin{aligned}
&\leq 2cd \sum_{k=1}^{\infty} \varphi(2^{-k/2}) 2^{-k/2} \\
&= 2^{3/2}cd \sum_{k=1}^{\infty} \varphi(2^{-k/2}) \int_{2^{-(k-1)/2}}^{2^{-k/2}} dx \\
&\leq 2^{3/2}cd \sum_{k=1}^{\infty} \int_{2^{-(k-1)/2}}^{2^{-k/2}} \varphi(x) dx \\
&\leq 2^{3/2}cd \int_0^1 \varphi(t) dt \\
&= 2^{3/2}cd \int_0^1 k(x, xt) dt \\
&= 2^{3/2}cd \frac{1}{x} \int_0^x k(x, z) dz,
\end{aligned}$$

where $c = \sup_{k \geq 0} k 2^{-k/2}$.



**On Weighted Inequalities with
Geometric Mean Operator
Generated by the Hardy-type
Integral Transform**

Maria Nassyrova,
Lars-Erik Persson
Vladimir Stepanov

Title Page

Contents



Go Back

Close

Quit

Page 25 of 37

5. Riemann-Liouville Operators

Let $\gamma > 0$ and consider the following Riemann-Liouville operators:

$$(5.1) \quad \mathcal{R}_\gamma f(x) := \frac{\gamma}{x^\gamma} \int_0^x (x-y)^{\gamma-1} f(y) dy,$$

and corresponding geometric mean operators

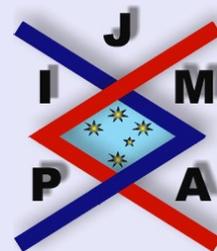
$$G_\gamma f(x) = \exp \left[\frac{\gamma}{x^\gamma} \int_0^x (x-y)^{\gamma-1} \log f(y) dy \right].$$

In this section we shall study the question of characterization of the weights $u(x)$ and $v(x)$ so that the inequality

$$(5.2) \quad \left(\int_0^\infty (G_\gamma f(x))^q u(x) dx \right)^{1/q} \leq C \left(\int_0^\infty f(x)^p v(x) dx \right)^{1/p}, \quad 0 < p, q < \infty$$

holds and also to point out the corresponding stable estimates of $C = \|G_\gamma\|_{L_v^p \rightarrow L_u^q}$.

We note that in the case $0 < \gamma < 1$ the kernel $k(x, y) = (x - y)^{\gamma-1}$ in (5.1) does not satisfy the Oinarov condition (3.1) so the results in Theorems 3.1 and 4.1 cannot be applied. However, this kernel has this property for the case $\gamma \geq 1$ so the question above can be solved by simply applying Theorems 3.1 and 4.1. Here we unify both cases in the next theorem and give a separate proof to this operator which gives a better estimate of the upper bound.



**On Weighted Inequalities with
Geometric Mean Operator
Generated by the Hardy-type
Integral Transform**

Maria Nassyrova,
Lars-Erik Persson
Vladimir Stepanov

Title Page

Contents

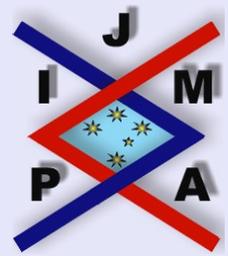


Go Back

Close

Quit

Page 26 of 37



On Weighted Inequalities with
Geometric Mean Operator
Generated by the Hardy-type
Integral Transform

Maria Nassyrova,
Lars-Erik Persson
Vladimir Stepanov

Title Page

Contents



Go Back

Close

Quit

Page 27 of 37

Theorem 5.1. (a) Let $0 < p \leq q < \infty$. Then (5.2) holds if and only if $\mathbb{A}_0 < \infty$. Moreover,

$$(5.3) \quad \mathbb{A}_0 \leq \|G_\gamma\|_{L_v^p \rightarrow L_u^q} \leq \gamma e^{1/p} \mathbb{A}_0, \quad 1 \leq \gamma < \infty.$$

and

$$(5.4) \quad \mathbb{A}_0 \leq \|G_\gamma\|_{L_v^p \rightarrow L_u^q} \lesssim \mathbb{A}_0, \quad 0 < \gamma < 1.$$

(b) Let $0 < q < p < \infty$. Then (5.2) holds if and only if $\mathbb{B}_0 < \infty$. Moreover,

$$(5.5) \quad \|G_\gamma\|_{L_v^p \rightarrow L_u^q} \approx \mathbb{B}_0.$$

The factors of equivalence in (5.4) and (5.5) depend on p, q and γ only.

Remark 5.1. By applying Theorem 5.1 with $\gamma = 1$ we obtain Theorems 2 and 4 in [6] (see also [7]) even with the same constants in the upper estimate in (5.3).

Partly guided by the technique used in [6], we postpone the proof of Theorem 5.1 and first prove two auxiliary results of independent interest, namely a characterization of the inequality

$$(5.6) \quad \left(\int_0^\infty (\mathcal{R}_\gamma f)^q w \right)^{1/q} \leq \mathbb{C} \left(\int_0^\infty f^p \right)^{1/p}, \quad f \geq 0,$$

in a form suitable for our purpose.

The following two theorems may be seen as a unification and generalization of the results from ([6, Theorems 1 and 3]) and ([9, Theorems 1 and 2]).



**On Weighted Inequalities with
Geometric Mean Operator
Generated by the Hardy-type
Integral Transform**

Maria Nassyrova,
Lars-Erik Persson
Vladimir Stepanov

Title Page

Contents



Go Back

Close

Quit

Page 28 of 37

Theorem 5.2. (a) Let $\gamma \geq 1$ and $1 < p \leq q \leq \infty$. Then

$$(5.7) \quad \mathbb{A}_0 \leq \|\mathcal{R}_\gamma\|_{L^p \rightarrow L_w^q} \leq \gamma p' \mathbb{A}_0.$$

(b) Let $0 < \gamma < 1$ and $1/\gamma < p \leq q \leq \infty$. Then

$$(5.8) \quad \mathbb{A}_0 \leq \|\mathcal{R}_\gamma\|_{L^p \rightarrow L_w^q} \leq \gamma \left[\left(\frac{p-1}{p\gamma-1} \right)^{1/p'} (2^{1/p} + 2^{2/p}) + 2^{1-\gamma} p' \right] \mathbb{A}_0.$$

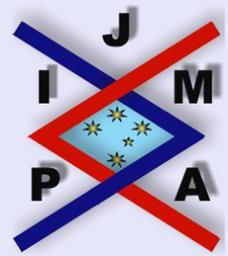
Remark 5.2. The lower bound $\mathbb{A}_0 \leq \|\mathcal{R}_\gamma\|_{L^p \rightarrow L_w^q}$ holds for all $\gamma > 0$ and $0 < p, q \leq \infty$.

Theorem 5.3. (a) Let $\gamma \geq 1$, $0 < q < p < \infty$ and $p > 1$. Then

$$(5.9) \quad \frac{\gamma}{2^{\gamma+1/r+1/q}} (p')^{1/q'} p^{-1/r} r^{-1/r'} q \mathbb{B}_0 \leq \|\mathcal{R}_\gamma\|_{L^p \rightarrow L_w^q} \leq \gamma q^{1/q} p' \mathbb{B}_0.$$

(b) Let $0 < \gamma < 1$, $0 < q < p < \infty$ and $p\gamma > 1$. Then

$$(5.10) \quad \frac{\gamma (2^{q/r} - 1)^{1/p} (2^{qr/p^2} - 1)^{1/r}}{2^{3+1/r}} \mathbb{B}_0 \leq \|\mathcal{R}_\gamma\|_{L^p \rightarrow L_w^q} \leq \gamma \mathbb{B}_0 \left[\left(\frac{p-1}{p\gamma-1} \right)^{1/p'} 2^{1+1/q} (1 + 2^{1/p'}) + \left(\left(\frac{r}{p} \right)^{q/r} 4^q + q (p')^q \right)^{1/q} \right].$$



On Weighted Inequalities with Geometric Mean Operator Generated by the Hardy-type Integral Transform

Maria Nassyrova,
Lars-Erik Persson
Vladimir Stepanov

Title Page

Contents



Go Back

Close

Quit

Page 29 of 37

Proof of Theorem 5.2. For the lower bounds on (5.7) and (5.8) we replace f in (5.6) by the test function $f_t(x) = \chi_{[0,t]}(x)$, $t > 0$. Then for $p, q < \infty$

$$t^{1/p} \mathbb{C} \geq \left(\int_0^\infty (\mathcal{R}_\gamma f_t)^q w \right)^{1/q} \geq \left(\int_0^t w \right)^{1/q}.$$

Hence,

$$\|\mathcal{R}_\gamma\|_{L^p \rightarrow L^q_w} \geq \mathbb{A}_0$$

for all $\gamma > 0$, $0 < p, q < \infty$. For $p \leq q = \infty$, $p = q = \infty$ or $q < p = \infty$ the arguments are the same.

Clearly,

$$(5.11) \quad \mathcal{R}_\gamma f(x) \leq \gamma H f(x), \quad \gamma \geq 1$$

and the upper bound in (5.7) follows from Theorem 1 in [6].

For the upper bound in (5.8) we follow the scheme from the proof of Theorem 1 in [9]. Put

$$J := \left(\int_0^\infty \left(\frac{1}{x^\gamma} \int_0^x (x-y)^{\gamma-1} f(y) dy \right)^q w(x) dx \right)^{1/q}$$

and note that, by Minkowski's inequality,

$$(5.12) \quad J \leq J_1 + J_2 + J_3,$$

where

$$J_1^q := \sum_{k \in \mathbb{Z}} \int_{2^k}^{2^{k+1}} \frac{w(x) dx}{x^{\gamma q}} \left(\int_{2^k}^x (x-y)^{\gamma-1} f(y) dy \right)^q,$$

$$J_2^q := \sum_{k \in \mathbb{Z}} \int_{2^k}^{2^{k+1}} \frac{w(x) dx}{x^{\gamma q}} \left(\int_{2^{k-1}}^{2^k} (x-y)^{\gamma-1} f(y) dy \right)^q,$$

and

$$J_3^q := \sum_{k \in \mathbb{Z}} \int_{2^k}^{2^{k+1}} \frac{w(x) dx}{x^{\gamma q}} \left(\int_0^{2^{k-1}} (x-y)^{\gamma-1} f(y) dy \right)^q.$$

Applying Hölder's inequality we find

$$\begin{aligned} J_1^q &\leq \sum_{k \in \mathbb{Z}} \left(\int_{2^k}^{2^{k+1}} f^p \right)^{q/p} \int_{2^k}^{2^{k+1}} \frac{w(x)}{x^{\gamma q}} dx \\ &\quad \times \left(\int_{2^k}^x (x-y)^{(\gamma-1)p'} dy \right)^{q/p'} \\ &\leq \left(\frac{p-1}{p\gamma-1} \right)^{q/p'} \sum_{k \in \mathbb{Z}} \left(\int_{2^k}^{2^{k+1}} f^p \right)^{q/p} \left(\int_{2^k}^{2^{k+1}} w(x) dx \right)^{2^{-kq/p}} \\ &\leq 2^{q/p} \left(\frac{p-1}{p\gamma-1} \right)^{q/p'} \sum_{k \in \mathbb{Z}} \left(\int_{2^k}^{2^{k+1}} f^p \right)^{q/p} \end{aligned}$$



**On Weighted Inequalities with
Geometric Mean Operator
Generated by the Hardy-type
Integral Transform**

Maria Nassyrova,
Lars-Erik Persson
Vladimir Stepanov

Title Page

Contents

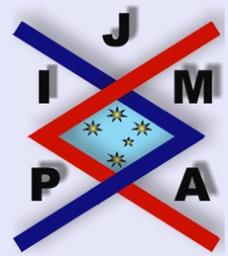


Go Back

Close

Quit

Page 30 of 37



On Weighted Inequalities with Geometric Mean Operator Generated by the Hardy-type Integral Transform

Maria Nassyrova,
Lars-Erik Persson
Vladimir Stepanov

Title Page

Contents



Go Back

Close

Quit

Page 31 of 37

$$(5.13) \quad \times \left(2^{-(k+1)q/p} \int_0^{2^{k+1}} w(x) dx \right) \\ \leq 2^{q/p} \left(\frac{p-1}{p\gamma-1} \right)^{q/p'} \mathbb{A}_0^q \left(\int_0^\infty f^p \right)^{q/p},$$

where the last step follows by the elementary inequality $\sum a_i^\beta \leq (\sum a_i)^\beta$, $\beta \geq 1$ and the definition of \mathbb{A}_0 .

Similarly, we obtain

$$(5.14) \quad J_2^q \leq 4^{q/p} \left(\frac{p-1}{p\gamma-1} \right)^{q/p'} \mathbb{A}_0^q \left(\int_0^\infty f^p \right)^{q/p}.$$

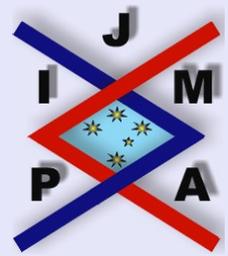
For the upper bound of J_3 we note that

$$J_3^q \leq 2^{(1-\gamma)q} \int_0^\infty (Hf)^q w$$

so that, by Theorem 1 in [6],

$$(5.15) \quad J_3^q \leq 2^{(1-\gamma)q} \mathbb{A}_0^q (p')^q \left(\int_0^\infty f^p \right)^{q/p}.$$

By combining (5.13), (5.14) and (5.15) we obtain the upper bound of (5.8) and the proof is complete.



On Weighted Inequalities with Geometric Mean Operator Generated by the Hardy-type Integral Transform

Maria Nassyrova,
Lars-Erik Persson
Vladimir Stepanov

Title Page

Contents



Go Back

Close

Quit

Page 32 of 37

(a) For the lower bound we write

$$\begin{aligned} \mathbb{C} \left(\int_0^\infty f^p \right)^{1/p} &\geq \gamma \left(\int_0^\infty \left(\frac{1}{x^\gamma} \int_0^{x/2} (x-y)^{\gamma-1} f(y) dy \right)^q w(x) dx \right)^{1/q} \\ &\geq \frac{\gamma}{2^{\gamma-1}} \left(\int_0^\infty \left(\frac{1}{x} \int_0^{x/2} f \right)^q w(x) dx \right)^{1/q} \\ &\geq \frac{\gamma}{2^\gamma} \left(\int_0^\infty \left(\frac{1}{x} \int_0^x f \right)^q w(2x) dx \right)^{1/q}. \end{aligned}$$

By applying Theorem 3 in [6] we find

$$\|\mathcal{R}_\gamma\|_{L^p \rightarrow L_w^q} \geq \frac{\gamma}{2^\gamma} c(p, q) \mathfrak{B}_0,$$

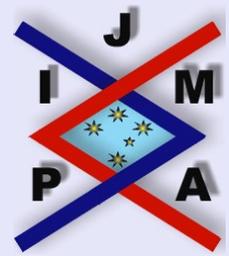
where

$$\mathfrak{B}_0 := \left(\int_0^\infty \left(\frac{1}{x} \int_0^x w(2s) ds \right)^{r/q} dx \right)^{1/r} = 2^{-1/r} \mathbb{B}_0$$

and $c(p, q) = 2^{-1/q} (p')^{1/q'} p^{-1/r} r^{-1/r'}$ and the lower bound in (5.9) is proved. Using (5.11) and again Theorem 3 in [6] we obtain the upper bound of (5.9).

(b) It is shown in Theorem 2 in [9] that

$$\|\mathcal{R}_\gamma\|_{L^p \rightarrow L_w^q} \geq \frac{\gamma}{4} \mathbb{E},$$



On Weighted Inequalities with Geometric Mean Operator Generated by the Hardy-type Integral Transform

Maria Nassyrova,
Lars-Erik Persson
Vladimir Stepanov

Title Page

Contents



Go Back

Close

Quit

Page 33 of 37

where

$$\mathbb{E} := \left(\sum_{k \in \mathbb{Z}} 2^{kr/p'} \left(\int_{2^k}^{2^{k+1}} \frac{w(x) dx}{x^q} \right)^{r/q} \right)^{1/r}.$$

We show that

$$\mathbb{E} \geq \frac{(2^{q/r} - 1)^{1/p} (2^{qr/p^2} - 1)^{1/r}}{2^{1+1/r}} \mathbb{B}_0.$$

Indeed,

$$\begin{aligned} \mathbb{B}_0^r &= \sum_{k \in \mathbb{Z}} \int_{2^{k-1}}^{2^k} \left(\frac{1}{x} \int_0^x w \right)^{r/q} dx \\ &\leq \sum_{k \in \mathbb{Z}} 2^{-(k-1)r/q} \left(\int_0^{2^k} w \right)^{r/q} \int_{2^{k-1}}^{2^k} dx \\ &= \sum_{k \in \mathbb{Z}} 2^{-(k-1)r/p} \left(\sum_{m \leq k} \int_{2^{m-1}}^{2^m} w \right)^{r/q} \\ &= 2^{r/p} \sum_{k \in \mathbb{Z}} 2^{-kr/p} \left(\sum_{m \leq k} \left\{ 2^{-\frac{mq^2}{rp}} \int_{2^{m-1}}^{2^m} w \right\} 2^{\frac{mq^2}{rp}} \right)^{r/q} \end{aligned}$$

(by applying Hölder's inequality with $\frac{r}{q}$ and $\frac{p}{q}$)

$$\leq 2^{r/p} \sum_{k \in \mathbb{Z}} 2^{-kr/p} \left(\sum_{m \leq k} 2^{-mq/p} \left(\int_{2^{m-1}}^{2^m} w \right)^{r/q} \right) \left(\sum_{m \leq k} 2^{mq/r} \right)^{r/p}$$

$$\begin{aligned}
&= \frac{2^{r/p+q/p}}{(2^{q/r} - 1)^{r/p}} \sum_{m \in \mathbb{Z}} 2^{-mq/p} \left(\int_{2^{m-1}}^{2^m} w \right)^{r/q} \sum_{k \geq m} 2^{-kqr/p^2} \\
&= \frac{2^{2r/p}}{(2^{q/r} - 1)^{r/p} (2^{qr/p^2} - 1)} \sum_{m \in \mathbb{Z}} 2^{-mr/p} \left(\int_{2^{m-1}}^{2^m} w \right)^{r/q} \\
&\leq \frac{2^{r(1+1/p)}}{(2^{q/r} - 1)^{r/p} (2^{qr/p^2} - 1)} \mathbb{E}^r.
\end{aligned}$$

Conversely,

$$\begin{aligned}
\mathbb{E}^r &\leq \sum_{k \in \mathbb{Z}} 2^{-(k-1)r/p} \left(\int_{2^{k-1}}^{2^k} w \right)^{r/q} \\
&\leq 2^{r/p} \sum_{k \in \mathbb{Z}} 2^{-kr/p} \left(\int_0^{2^k} w \right)^{r/q} \int_{2^k}^{2^{k+1}} dx \\
&\leq 2^{r/p+r/q} \sum_{k \in \mathbb{Z}} \int_{2^k}^{2^{k+1}} \left(\frac{1}{x} \int_0^x w \right)^{r/q} dx \\
(5.16) \quad &= 2^{r/p+r/q} \mathbb{B}_0^r.
\end{aligned}$$

Using the decomposition (5.12) it is shown in Theorem 2 in [9] that

$$J \leq \left(\frac{p-1}{p\gamma-1} \right)^{1/p'} 2^{1/p'} \left(1 + 2^{1/p'} \right) \mathbb{E} + 2^{1-\gamma} \left(\int_0^\infty (Hf)^q w \right)^{1/q}.$$



On Weighted Inequalities with Geometric Mean Operator Generated by the Hardy-type Integral Transform

Maria Nassyrova,
Lars-Erik Persson
Vladimir Stepanov

Title Page

Contents



Go Back

Close

Quit

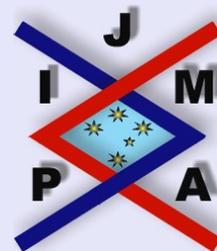
Page 34 of 37

Now the upper bound in (5.10) follows from (5.16) and Theorem 3 in [6].

□

We are now ready to complete this section by presenting

Proof of Theorem 5.1. Both sides of (5.3) follow from Theorem 5.2 (a) and (2.4). The lower bound in (5.4) follows by using the test functions from the proof of Theorem 3.1 and the upper bound is a consequence of (2.2), (2.10) and (5.8). Similarly, the proof of (b) is based upon Theorem 5.3 and we use the same test function for the lower bound as in the proof of Theorem 4.1 with subsequent application of the inequality $\mathbb{B}_0 \leq B_0$. The proof is complete. □



On Weighted Inequalities with
Geometric Mean Operator
Generated by the Hardy-type
Integral Transform

Maria Nassyrova,
Lars-Erik Persson
Vladimir Stepanov

Title Page

Contents



Go Back

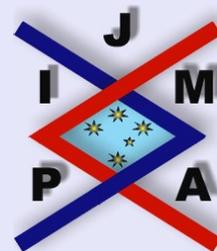
Close

Quit

Page 35 of 37

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On Weighted Inequalities with
Geometric Mean Operator
Generated by the Hardy-type
Integral Transform

Maria Nassyrova,
Lars-Erik Persson
Vladimir Stepanov

Title Page

Contents



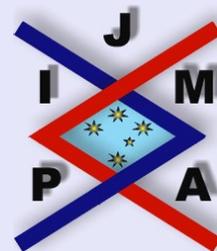
Go Back

Close

Quit

Page 36 of 37

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**On Weighted Inequalities with
Geometric Mean Operator
Generated by the Hardy-type
Integral Transform**

Maria Nassyrova,
Lars-Erik Persson
Vladimir Stepanov

Title Page

Contents



Go Back

Close

Quit

Page 37 of 37