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ON WEIGHTED INEQUALITIES WITH GEOMETRIC MEAN OPERATOR GENERATED BY THE HARDY-TYPE INTEGRAL TRANSFORM

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ABSTRACT. The generalized geometric mean operator

$$G_K f(x) = \exp \frac{1}{K(x)} \int_0^x k(x,y) \log f(y) dy,$$

with $K(x) := \int_0^x k(x,y)dy$ is considered. A characterization of the weights $u(x)$ and $v(x)$ so that the inequality

$$\left(\int_0^\infty (G_K f(x))^q \, u(x) \, dx \right)^{1/q} \le C \left(\int_0^\infty f(x)^p v(x) dx \right)^{1/p}, \quad f \ge 0,$$

holds is given for all $0 < p, q < \infty$ both for all G_K where $k(x,y)$ satisfies the Oinarov condition and for Riemann-Liouville operators. The corresponding stable bounds of $C = \|G_K\|_{L_v^p \to L_u^q}$ are pointed out.

Key words and phrases: Integral inequalities, Weights, Geometric mean operator, Kernels, Riemann-Liouville operators.

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1. INTRODUCTION

Let $\mathbb{R}_+ := [0,\infty)$ and let $k(x,y) \geq 0$ be a locally integrable kernel defined on $\mathbb{R}_+ \times \mathbb{R}_+$ and such that

$$\int_{\mathbb{R}_+} k(x,y)dy = 1$$

for almost all $x \in \mathbb{R}_+$.

Denote

$$Kf(x) := \int_0^\infty k(x,y)f(y)dy, \quad f(y) \geq 0.$$

If $Kf(x) < \infty$, then there exists a limit

$$G_K f(x) := \lim_{\alpha \downarrow 0} [Kf^\alpha(x)]^{1/\alpha} \tag{1.1}$$

and

$$G_K f(x) = \exp \int_0^\infty k(x,y) \log f(y)dy. \tag{1.2}$$

For $0 < p < \infty$ and a weight function $v(x) \geq 0$ we put

$$\|f\|_{L^p_v} := \left(\int_0^\infty |f(x)|^p v(x)dx \right)^{1/p}$$

and make use of the abbreviation $\|f\|_{L^p}$ when $v(x) \equiv 1$.

Suppose $u(x) \geq 0$ and $v(x) \geq 0$ are weight functions and $0 < p, q < \infty$. This paper deals with $L^p_v - L^q_u$ inequalities of the form

$$\left(\int_0^\infty (G_K f)^q u \right)^{1/q} \leq C \left(\int_0^\infty f^p v \right)^{1/p}, \quad f \geq 0, \tag{1.3}$$

where a constant C is independent on f and we always assume that C is the least possible, that is $C = \|G_K\|_{L^p_v \to L^q_u}$, where

$$\|G_K\|_{L^p_v \to L^q_u} := \sup_{f \geq 0} \frac{\|G_K f\|_{L^q_u}}{\|f\|_{L^p_v}}. \tag{1.4}$$

For the classical case $k(x,y) = \frac{1}{x}\chi_{[0,x]}(y)$ with the Hardy averaging operator

$$Hf(x) := \frac{1}{x} \int_0^x f(y)dy \tag{1.5}$$

the inequality (1.3) was characterized in [6] (see also [7]).

In Section 2 of this paper we present a general scheme on how inequalities of the type (1.3) can be characterized via the limiting procedure in similar characterization (in suitable forms) of some corresponding Hardy-type inequalities. In Section 3 we characterize the weights $u(x)$ and $v(x)$ so that (1.3) with $0 < p \leq q < \infty$ holds when

$$G_{\mathbf{K}} f(x) = \exp \frac{1}{K(x)} \int_0^x k(x,y) \log f(y)dy \tag{1.6}$$

with

$$K(x) := \int_0^x k(x,y)dy < \infty, \quad x > 0,$$

and $K(0) = 0$, $K(\infty) = \infty$, and when $k(x,y)$ satisfies the Oinarov condition (see Theorem 3.1). The corresponding characterization for the case $0 < q < p < \infty$ can be found in Section 4 (see Theorem 4.1). A fairly precise result in the case $k(x,y) = (x-y)^\gamma$, $\gamma > 0$, i.e. when

G_K is generated by the Riemann-Liouville convolutional operator, can be found in Section 5 (see Theorem 5.1). In particular, this investigation shows that inequalities of the type (1.3) can be proved also when the Oinarov condition is violated (because for $0 < \gamma < 1$ this kernel does not satisfy this condition). In order to prove these results we need new characterizations of weighted inequalities with the Riemann-Liouville operator which are of independent interest (see Theorem 5.3 and 5.5).

Finally note that the operator (1.6) was studied in a similar connection, by E.R. Love [3], where a sufficient condition was proved for the inequality (1.3) to be valid in the case $p = q = 1$ and special weights.

2. GENERAL SCHEMES

We begin with some general remarks. By using the elementary property

$$G_K(f^s) = (G_K f)^s, \quad -\infty < s < \infty, \tag{2.1}$$

we see that

$$\|G_K\|_{L^p_v \to L^q_u} = \|G_K\|_{L^p \to L^q_w} = \|G_K\|^{s/p}_{L^s \to L^{sq/p}_w}, \tag{2.2}$$

where

$$w := \left[G_K\left(\frac{1}{v}\right)\right]^{q/p} u. \tag{2.3}$$

Moreover, from (1.1) and (2.2)

$$\|G_K\|_{L^p_v \to L^q_u} = \lim_{\alpha \downarrow 0} \|K\|^{1/\alpha}_{L^{p/\alpha} \to L^{q/\alpha}_w}. \tag{2.4}$$

The last formula generates the "precise" scheme for characterization of (1.3) provided the norm of the associated integral operator K has very accurate two-sided estimates of the norm

$$c_1(p,q)F(w,p,q) \le \|K\|_{L^p \to L^q_w} \le c_2(p,q)F(w,p,q) \tag{2.5}$$

in the sense that there exist the limits

$$c_i(p,q,F) = \lim_{\alpha \downarrow 0} \left[c_i\left(\frac{p}{\alpha}, \frac{q}{\alpha}\right) F\left(w, \frac{p}{\alpha}, \frac{q}{\alpha}\right)\right]^{1/\alpha}, \quad i = 1, 2, \tag{2.6}$$

and

$$c_i(p,q,F) \in (0,\infty), \quad i = 1, 2.$$

A natural consequence of (2.2) and (2.4) is then the two-sided estimate

$$c_1(p,q,F) \le \|G_K\|_{L^p \to L^q_w} \le c_2(p,q,F), \tag{2.7}$$

which characterizes (1.3) with the least possible constants provided the estimates in (2.5) are also the best.

This scheme was realized in [6] where K is the Hardy averaging operator (1.5) and $0 < p \le q < \infty$. For this purpose a new non-Muckenhoupt form of (2.5) was used. We generalize this result here for the Riemann-Liouville kernel in Section 4.

Unfortunately, the above scheme does not work for a more general operator or even for the Hardy operator, if $q < p$, because the estimate (2.5) happens to be vulnerable, when the parameters p and q tend to ∞, so that the limits $c_i(p,q,F)$ become either 0 or ∞. However, if the functional F is homogeneous in the sense that

$$\left[F\left(w, \frac{p}{\alpha}, \frac{q}{\alpha}\right)\right]^{1/\alpha} = F(w,p,q), \quad \alpha > 0, \tag{2.8}$$

then an alternative "stable" scheme for (2.7) works, provided the lower bound

$$c_3(p,q)F(w,p,q) \le \|G_K\|_{L^p\to L^q_w} \tag{2.9}$$

can be established. The right hand side of (2.7) follows from the upper bound from (2.5), (2.2) and Jensen's inequality

$$G_K f(x) \le Kf(x), \tag{2.10}$$

because (2.8) implies

$$\left[F\left(w,s,\frac{sq}{p}\right)\right]^{s/p} = F(w,p,q), \quad s>1. \tag{2.11}$$

To realize the "stable" scheme for the Hardy averaging operator, when $0<q<p<\infty$ and $p>1$ an alternative (non-Mazya-Rozin) functional F was found in [6] in the form

$$F(w,p,q) = \left(\int_0^\infty \left(\frac{1}{x}\int_0^x w\right)^{q/(p-q)} w(x)dx\right)^{1/q-1/p} \tag{2.12}$$

which obviously satisfies (2.11).

The idea to use (2.2) and (2.10) for the upper bound estimate of the norm $\|G_K\|_{L^p_v\to L^q_u}$ originated from the paper by Pick and Opic [8], where the authors obtained two-sided estimates of $\|G_H\|_{L^p_v\to L^q_u}$, $0<q<p<\infty$. However, they realized (2.5) in Muckenhoupt or Mazya-Rozin form with unstable factors and, therefore, the estimate (2.7) was rather uncertain.

Throughout the paper, expressions of the form $0\cdot\infty$, ∞/∞, $0/0$ are taken to be equal to zero; and the inequality $A\lesssim B$ means $A\le cB$ with a constant $c>0$ independent of weight functions. Moreover, the relationship $A\approx B$ is interpreted as $A\lesssim B\lesssim A$ or $A=cB$. Everywhere the constant C in explored inequalities are considered to be the least possible one. In the case $q<p$ the auxiliary parameter r is defined by $1/r=1/q-1/p$.

3. HARDY-TYPE OPERATORS. CASE $0<p\le q<\infty$

In the sequel we let $k(x,y)\ge 0$ be locally integrable in $\mathbb{R}_+\times\mathbb{R}_+$ and satisfies the following (Oinarov) condition: there exists a constant $d\ge 1$ independent on x, y, z such that

$$\frac{1}{d}(k(x,z)+k(z,y)) \le k(x,y) \le d(k(x,z)+k(z,y)), \quad x\ge z\ge y\ge 0. \tag{3.1}$$

We assume that

$$K(x) := \int_0^x k(x,y)dy < \infty, \quad x>0, \tag{3.2}$$

and $K(0)=0$, $K(\infty)=\infty$.

Without loss of generality we may and shall also assume that $k(x,y)$ is nondecreasing in x and nonincreasing in y.

We consider the following Hardy-type operator

$$\mathbf{K}f(x) := \frac{1}{K(x)}\int_0^x k(x,y)f(y)dy, \quad f(y)\ge 0,\ x\ge 0, \tag{3.3}$$

and the corresponding geometric mean operator $G_{\mathbf{K}}$:

$$G_{\mathbf{K}}f(x) = \exp\frac{1}{K(x)}\int_0^x k(x,y)\log f(y)dy.$$

Our main goal in this section is to find necessary and sufficient conditions on the weights $u(x)$ and $v(x)$ so that the inequality

$$\left(\int_0^\infty (G_{\mathbf{K}} f(x))^q \, u(x) \, dx\right)^{1/q} \leq C \left(\int_0^\infty f(x)^p v(x) dx\right)^{1/p}, \quad f \geq 0, \tag{3.4}$$

holds for $0 < p \leq q < \infty$, and to point out the corresponding stable estimates of $C = \|G_{\mathbf{K}}\|_{L_v^p \to L_u^q}$. In view of our discussion in the previous section we first need to have the two-sided estimates of $\|\mathbf{K}\|_{L^p \to L_w^q}$ in suitable form. First we observe that it follows from ([1, Theorem 2.1]) by changing variables $x \to \frac{1}{x}$, $y \to \frac{1}{y}$ and by using the duality principle (see [11, Section 2.3]) that for $1 < p \leq q < \infty$ we have

$$c_4(d)\mathbb{A} \leq \|\mathbf{K}\|_{L^p \to L_w^q} \leq c_5(p,q,d)\mathbb{A}, \tag{3.5}$$

where d is defined by (3.1),

$$\mathbb{A} = \max(\mathbb{A}_0, \mathbb{A}_1) \tag{3.6}$$

with

$$\mathbb{A}_0 := \sup_{t>0} t^{-1/p} \left(\int_0^t w\right)^{1/q} \tag{3.7}$$

and

$$\mathbb{A}_1 := \sup_{t>0} \left(\int_0^t k(t,y)^{p'} dy\right)^{-1/p} \left(\int_0^t \left(\int_0^x k(x,y)^{p'} dy\right)^q \frac{w(x)}{K(x)^q} dx\right)^{1/q}. \tag{3.8}$$

It is easy to see that $\mathbb{A}_0$ satisfies (2.8) or (2.11), but not $\mathbb{A}_1$. It is also known, that neither $\mathbb{A}_0 < \infty$ nor $\mathbb{A}_1 < \infty$ alone is sufficient for $\|\mathbf{K}\|_{L^p \to L_w^q} < \infty$ in general (see counterexamples in [4], [10] and [5]). For this reason we need to require some additional conditions of a kernel $k(x,y)$ for the property

$$\mathbb{A}_0 \approx \|\mathbf{K}\|_{L^p \to L_w^q}, \tag{3.9}$$

and we will soon discuss this question in detail (see Proposition 3.3). Now we are ready to state our main theorem of this section:

Theorem 3.1. *Let $0 < p \leq q < \infty$ and the kernel $k(x,y) \geq 0$ be such that (3.9) holds. Then (3.4) holds if and only if $\mathbb{A}_0 < \infty$. Moreover,*

$$\mathbb{A}_0 \leq \|G_{\mathbf{K}}\|_{L_v^p \to L_u^q} \lesssim \mathbb{A}_0. \tag{3.10}$$

Proof. The sufficiency including the upper estimate in (3.10) follows from (2.2), (2.10) and (3.9). Moreover, we note that, by (2.2),

$$\left(\int_0^\infty (G_{\mathbf{K}} f)^q \, w\right)^{1/q} \leq \|G_{\mathbf{K}}\|_{L_v^p \to L_u^q} \left(\int_0^\infty f^p\right)^{1/p}. \tag{3.11}$$

By applying this estimate with $f_t(x) = t^{-1/p}\chi_{[0,t]}(x)$ for fixed $t > 0$ (obviously, $\|f_t\|_p = 1$) we see that

$$t^{-1/p} \left(\int_0^t w\right)^{1/q} \leq \|G_{\mathbf{K}}\|_{L_v^p \to L_u^q}.$$

By taking the supremum over t we have also proved the necessity including the lower estimate in (3.10) and the proof is complete. □

Remark 3.2. For the case when $k(x,y) \equiv 1$ we have that $G_{\mathbf{K}}$ coincide with the usual geometric mean operator

$$Gf(x) = \exp \frac{1}{x} \int_0^x \log f(y) dy$$

and thus we see that Theorem 3.1 may be regarded as a genuine generalization of Theorem 2 in [6] (see also[7]).

As mentioned before we shall now discuss the question of for which kernels $k(x,y)$ does the condition (3.9) hold. We need the following notation:

$$K_p(x) := \int_0^x k(x,y)^{p'} dy,$$

$$\alpha_0 := \sup_{t>0} K_p(t)^{1/p'} \left[\sup_{0<x<t} x^{1/p'} k(t,x) \right]^{1/q}, \tag{3.12}$$

$$\alpha_1 := \sup_{t>0} t^{1/p'} \left(\int_t^\infty k(x,t)^q K(x)^{-q} d\left(x^{q/p}\right) \right)^{1/q}, \tag{3.13}$$

and

$$\alpha_2 := \sup_{t>0} t^{1/p'} \left(\int_t^\infty x^{q/p} k(x,t)^q d\left(-K(x)^{-q}\right) \right)^{1/q}. \tag{3.14}$$

Proposition 3.3. *Let $1 < p \leq q < \infty$. If a kernel $k(x,y) \geq 0$ is such that for all weights w (3.9) is true, then $\alpha_0 + \alpha_1 < \infty$. Conversely, if $\alpha_0 + \alpha_2 < \infty$, then (3.9) holds. In particular, if $\alpha_2 \lesssim \alpha_1$, then (3.9) is equivalent with the conditions $\alpha_0 < \infty$ and $\alpha_1 < \infty$.*

Proof. It is known ([11], (see also [2, Theorem 2.13])) that

$$\|\mathbf{K}\|_{L^p \to L^q_w} = \max(A_0, A_1), \tag{3.15}$$

where

$$A_0 := \sup_{t>0} \left(\int_t^\infty k(x,t)^q \frac{w(x)}{K(x)^q} dx \right)^{1/q} t^{1/p'},$$
$$A_1 := \sup_{t>0} \left(\int_t^\infty \frac{w(x)}{K(x)^q} dx \right)^{1/q} K_p(t)^{1/p'}.$$

It follows from a more general result ([5, Theorem 4]) that

$$A_0 \approx \|\mathbf{K}\|_{L^p \to L^q_w} \tag{3.16}$$

for all weights w, if and only if $\alpha_0 < \infty$. Moreover, if (3.9) is true for all weights w, then (3.15) implies

$$A_0 \lesssim \mathbb{A}_0 \tag{3.17}$$

and for $w(x) = x^{q/p-1}$ it brings $\alpha_1 < \infty$. We observe that the inequality

$$\mathbb{A}_0 \lesssim p A_0 \tag{3.18}$$

is always true. Indeed, by applying Minkowski's integral inequality we find that

$$\begin{aligned}\left(\int_0^t w\right)^{1/q} &= \left(\int_0^t \left(\int_0^x k(x,y)dy\right)^q \frac{w(x)}{K(x)^q}dx\right)^{1/q}\\ &\le \int_0^t \left(\int_y^\infty k(x,y)^q \frac{w(x)}{K(x)^q}dx\right)^{1/q} dy\\ &\le A_0 \int_0^t y^{-1/p'}dy\\ &= A_0 p t^{1/p}\end{aligned}$$

and (3.18) follows. Thus, (3.9) implies (3.16). Consequently, $\alpha_0 < \infty$ and, thus, $\alpha_0 + \alpha_1 < \infty$.

Now, suppose that $\alpha_0 + \alpha_2 < \infty$. Then (3.16) holds and it is sufficient to prove (3.17). To this end we note that

$$\begin{aligned}&\int_t^\infty k(x,t)^q \frac{w(x)}{K(x)^q}dx\\ &\quad\approx k(t,t)^q \int_t^\infty \frac{w(x)}{K(x)^q}dx + \int_t^\infty \frac{w(x)}{K(x)^q}\left(\int_t^x dk(s,t)^q\right)dx\\ &\quad= k(t,t)^q \int_t^\infty w(x)\left(\int_x^\infty d\left(\frac{-1}{K(y)^q}\right)\right)dx + \int_t^\infty dk(s,t)^q \left(\int_s^\infty \frac{w(x)dx}{K(x)^q}\right)\\ &\quad= k(t,t)^q \int_t^\infty d\left(\frac{-1}{K(y)^q}\right)\left(\int_t^y w(x)dx\right) + \int_t^\infty dk(s,t)^q \int_s^\infty w(x)dx \int_x^\infty d\left(\frac{-1}{K(y)^q}\right)\\ &\quad\le \mathbb{A}_0^q \left(\int_t^\infty y^{q/p} d\left(\frac{-1}{K(y)^q}\right)\right) k(t,t)^q + \int_t^\infty dk(s,t)^q \int_s^\infty d\left(\frac{-1}{K(y)^q}\right)\int_s^y w(x)dx\\ &\quad\le \mathbb{A}_0^q \int_t^\infty y^{q/p} k(y,t)^q d\left(\frac{-1}{K(y)^q}\right)\\ &\quad\le \mathbb{A}_0^q \alpha_2^q t^{-q/p'}.\end{aligned}$$

and (3.17) follows. Now (3.5), (3.6), (3.16) and (3.17) imply (3.9). The proof is complete. □

4. HARDY-TYPE OPERATORS. CASE $0 < q < p < \infty$

Put

$$\mathbb{B}_0 := \left(\int_0^\infty \left(\frac{1}{t}\int_0^t w\right)^{r/q} dt\right)^{1/r}. \tag{4.1}$$

A crucial condition for this case corresponding to the condition (3.9) for the case $0 < p \le q < \infty$ is the following:

$$\mathbb{B}_0 \approx \|\mathbf{K}\|_{L^p \to L^q_w}. \tag{4.2}$$

We now state our main result in this Section.

Theorem 4.1. *Let $0 < q < p < \infty$ and the kernel $k(x,y) \ge 0$ be such that (4.2) holds. Also we assume that*

$$\frac{x}{K(x)}\int_0^1 k(x,xt)\log\frac{1}{t}dt \le \alpha_3 < \infty, \quad x > 0. \tag{4.3}$$

Then (3.4) holds if and only if $\mathbb{B}_0 < \infty$. Moreover,

$$\|G_{\mathbf{K}}\|_{L^p_v \to L^q_u} \approx \mathbb{B}_0. \tag{4.4}$$

Proof. The sufficiency including the upper estimate in (4.4) follows by using (2.2), (2.10) and (4.2). Next we show that

$$D \lesssim \|G_{\mathbf{K}}\|_{L^p \to L^q_w}, \tag{4.5}$$

where

$$D := \left(\int_0^\infty \left(\int_x^\infty \frac{w(y)}{y^q} dy \right)^{r/q} x^{r/q'} dx \right)^{1/r}.$$

For this purpose we consider

$$f_0(x) = x^{r/(pq')} \left(\int_x^\infty \frac{w(y)}{y^q} dy \right)^{r/(pq)}.$$

Then

$$D^{r/p} = \|f_0\|_p .$$

We have

$$\begin{aligned}
&\|G_{\mathbf{K}}\|_{L^p \to L^q_w} \|f_0\|_p \\
&\quad = \|G_{\mathbf{K}}\|_{L^p \to L^q_w} D^{r/p} \\
&\quad \geq \left(\int_0^\infty (G_{\mathbf{K}} f_0)^q w \right)^{1/q} \\
&\quad = \left[\int_0^\infty \left(\exp \frac{1}{K(x)} \int_0^x k(x,y) \right.\right. \\
&\qquad \left.\left. \times \log \left\{ y^{r/(pq')} \left(\int_y^\infty \frac{w(z)}{z^q} dz \right)^{r/(pq)} \right\} dy \right)^q w(x) dx \right]^{1/q} \\
&\quad \geq \left[\int_0^\infty \left(\exp \frac{1}{K(x)} \int_0^x k(x,y) \log y dy \right)^{rq/(pq')} \left(\int_x^\infty \frac{w(z)}{z^q} dz \right)^{r/p} w(x) dx \right]^{1/q} \\
&\quad =: J. \qquad (4.6)
\end{aligned}$$

Moreover,

$$\begin{aligned}
\frac{1}{K(x)} \int_0^x k(x,y) \log y dy &= \log x + \frac{1}{K(x)} \int_0^x k(x,y) \log \frac{y}{x} dy \\
&= \log x - \frac{x}{K(x)} \int_0^1 k(x,xt) \log \frac{1}{t} dt \\
&\geq \log x - \alpha_3,
\end{aligned}$$

where α_3 is a constant from (4.3). Then

$$J^q \geq \exp \left\{ \frac{-\alpha_3 rq}{(pq')} \right\} \int_0^\infty \left(\int_x^\infty \frac{w(z)}{z^q} dz \right)^{r/p} w(x) x^{r/(pq')} dx \approx D^r, \tag{4.7}$$

the last estimate is obtained by partial integration. Now (4.5) follows by just combining (4.6) and (4.7).

It remains to show that $D \geq \mathbb{B}_0$. We have

$$\begin{aligned}\int_0^t w(x)dx &= q\int_0^t \left(\int_0^s z^{q-1}dz\right)\frac{w(s)}{s^q}ds\\ &= q\int_0^t \left(\int_z^t \frac{w(s)}{s^q}ds\right) z^{q-1}dz\\ &= q\int_0^t \left\{\left(\int_z^t \frac{w(s)}{s^q}ds\right) z^{q-1+q/(2p)}\right\} z^{-q/(2p)}dz\end{aligned}$$

(by Hölder's inequality with conjugate exponents $\frac{r}{q}$ and $\frac{p}{q}$)

$$\leq q\left(\int_0^t \left(\int_z^t \frac{w(s)}{s^q}ds\right)^{r/q} z^{r/q'+r/(2p)}dz\right)^{q/r}\left(\int_0^t \frac{dz}{\sqrt{z}}\right)^{q/p}.$$

This implies

$$\begin{aligned}\mathbb{B}_0^r &\lesssim \int_0^\infty \left(\int_0^t \left(\int_z^\infty \frac{w(s)}{s^q}ds\right)^{r/q} z^{r/q'+r/(2p)}dz\right) t^{r/(2p)-r/q}dt\\ &= \int_0^\infty \left(\int_z^\infty \frac{w(s)}{s^q}ds\right)^{r/q} z^{r/q'+r/(2p)}\left(\int_z^\infty t^{r/(2p)-r/q}dt\right)dz\\ &\approx \int_0^\infty \left(\int_z^\infty \frac{w(s)}{s^q}ds\right)^{r/q} z^{r/q'}dz = D^r.\end{aligned}$$

Thus the lower bound in (4.4) and also the necessity is proved; so the proof is complete. ☐

Next we shall analyze and discuss for which kernels $k\,(x,y)$ the crucial conditions (4.2) holds. First we note that it follows for the case $1 < q < p < \infty$ from a well-known result ([11], see also [2, Theorem 2.19]) that

$$\|\mathbf{K}\|_{L^p\to L^q_w} \approx B_0 + B_1, \tag{4.8}$$

where

$$B_0 := \left(\int_0^\infty \left(\int_t^\infty k(x,t)^q \frac{w(y)}{K(x)^q}dy\right)^{r/q} t^{r/q'}dt\right)^{1/r}$$

and

$$B_1 := \left(\int_0^\infty \left(\int_t^\infty \frac{w(x)}{K(x)^q}dx\right)^{r/p} \frac{K_p(t)^{r/p'}w(t)}{K(t)^q}dt\right)^{1/r}.$$

Moreover, it is established in ([5, Theorem 13]) that

$$B_1 \leq \beta_0 B_0$$

for all weights w for which $B_0 < \infty$, if and only if,

$$\beta_0 := \sup_{t>0} K_p(t)^{1/p'}\left(\int_0^t k\,(t,x)^r\, d\left(x^{r/p'}\right)\right)^{-1/r} < \infty. \tag{4.9}$$

Therefore

$$\|\mathbf{K}\|_{L^p\to L^q_w} \approx B_0 \tag{4.10}$$

under the condition (4.9).

Partly guided by our investigation in the previous section we shall now continue by comparing the constant B_0 and $\mathbb{B}_0$.

Proposition 4.2. *Let* $1 < q < p < \infty$. *Then*

(a) $\mathbb{B}_0 \lesssim B_0$,

(b) $B_0 \le \beta_1 \mathbb{B}_0$, *if*

$$\beta_1 := \left(\int_0^\infty \left(\int_0^x k(x,t)^r t^{r/q'} dt \right)^{p/r} x^{p/q} K(x)^{-p(1+1/q)} d\left(K(x)\right) \right)^{1/p} < \infty. \tag{4.11}$$

Proof. **(a)** We have

$$\mathbb{B}_0^r := \int_0^\infty \left(\int_0^t \left(\int_0^x k(x,y)dy \right)^q \frac{w(x)}{K(x)^q} dx \right)^{r/q} t^{-r/q} dt$$

(applying Minkowski's integral inequality)

$$\le \int_0^\infty \left(\int_0^t \left(\int_y^t k(x,y)^q \frac{w(x)}{K(x)^q} dx \right)^{1/q} dy \right)^r t^{-r/q} dt$$

$$\le \int_0^\infty \left(\int_0^t \left(\int_y^\infty k(x,y)^q \frac{w(x)}{K(x)^q} dx \right)^{1/q} y^\alpha y^{-\alpha} dy \right)^r t^{-r/q} dt$$

(applying Hölder's inequality, $\alpha \in \left(\frac{1}{q'}, \frac{1}{r'}\right)$)

$$\le \int_0^\infty \left(\int_0^t \left(\int_y^\infty k(x,y)^q \frac{w(x)}{K(x)^q} dx \right)^{r/q} y^{\alpha r} dy \right) \left(\int_0^t y^{-\alpha r'} dy \right)^{r-1} t^{-r/q} dt$$

$$= (1-\alpha r')^{1-r} \int_0^\infty \left(\int_y^\infty k(x,y)^q \frac{w(x)}{K(x)^q} dx \right)^{r/q} \left(\int_y^\infty t^{r-1-\alpha r - r/q} dt \right) y^{\alpha r} dy$$

$$= \frac{(1-\alpha r')^{1-r}}{r\left(\alpha - 1/q'\right)} B_0^r.$$

(b) Indeed, following the proof of Proposition 3.3, we find

$$B_0^r \le \int_0^\infty \left(\int_t^\infty k(x,t)^q \left(\int_t^x w \right) d\left(-K(x)^{-q}\right) \right)^{r/q} t^{r/q'} dt$$

(by Minkowski's integral inequality)

$$\le \left(\int_0^\infty \left(\int_0^x k(x,t)^r \left(\int_t^x w \right)^{r/q} t^{r/q'} dt \right)^{q/r} d\left(-K(x)^{-q}\right) \right)^{r/q}$$

$$\le \left(\int_0^\infty \left(\frac{1}{x} \int_0^x w \right) \left(\int_0^x k(x,t)^r t^{r/q'} dt \right)^{q/r} x d\left(-K(x)^{-q}\right) \right)^{r/q}$$

$$= \left(q \int_0^\infty \left(\frac{1}{x} \int_0^x w \right) \left(\int_0^x k(x,t)^r t^{r/q'} dt \right)^{q/r} x K(x)^{-(q+1)} dK(x) \right)^{r/q}$$

(by Hölder's inequality applied with conjugate exponents $\frac{r}{q}$ and $\frac{p}{q}$)

$$\le q^{r/q} \beta_1^r \int_0^\infty \left(\frac{1}{x} \int_0^x w \right)^{r/q} dx = q^{r/q} \beta_1^r \mathbb{B}_0^r$$

and the proof follows. □

Remark 4.3. It is easy to see that the conditions (4.3), (4.9) and (4.11) are satisfied with $k(x,y) \equiv 1$, i.e. when $G_{\mathbf{K}} = G$ (the standard geometric mean operator) and we conclude that Theorem 4.1 may be seen as a generalization of Theorem 4 in [6] (see also [7]).

We finish this Section by showing that the kernel, investigated in [4], satisfies the condition (4.3).

Example 4.1. Let the kernel $k(x,y)$ be given by

$$k(x,y) = \varphi\left(\frac{y}{x}\right), \tag{4.12}$$

where $\varphi(t) \geq 0$ is decreasing function on $(0,1)$ satisfying

$$\varphi(ts) \leq d\left(\varphi(t) + \varphi(s)\right), \quad 0 < t, s < 1. \tag{4.13}$$

Then (3.1) is obviously valid.

If $\int_0^1 \varphi(t)dt < \infty$, then the kernel $k(x,y)$ of the form (4.12) satisfies (4.3). Indeed,

$$\begin{aligned}
\int_0^1 k(x,xt)\log\frac{1}{t}dt &= \int_0^1 \varphi(t)\log\frac{1}{t}dt \\
&= \sum_{k=0}^{\infty}\int_{2^{-k-1}}^{2^{-k}} \varphi(t)\log\frac{1}{t}dt \\
&\leq \sum_{k=0}^{\infty}\varphi(2^{-k-1})\int_{2^{-k-1}}^{2^{-k}} \log\frac{1}{t}dt \\
&\lesssim \sum_{k=0}^{\infty}\varphi(2^{-k-1})\,(k+1)\,2^{-k-1} \\
&= \sum_{k=1}^{\infty}\varphi(2^{-k})k2^{-k} \\
&\leq 2d\sum_{k=1}^{\infty}\varphi\left(2^{-k/2}\right)k2^{-k} \\
&\leq 2cd\sum_{k=1}^{\infty}\varphi\left(2^{-k/2}\right)2^{-k/2} \\
&= 2^{3/2}cd\sum_{k=1}^{\infty}\varphi\left(2^{-k/2}\right)\int_{2^{(-k-1)/2}}^{2^{-k/2}} dx \\
&\leq 2^{3/2}cd\sum_{k=1}^{\infty}\int_{2^{(-k-1)/2}}^{2^{-k/2}} \varphi(x)\,dx \\
&\leq 2^{3/2}cd\int_0^1 \varphi(t)dt \\
&= 2^{3/2}cd\int_0^1 k(x,xt)dt \\
&= 2^{3/2}cd\frac{1}{x}\int_0^x k(x,z)dz,
\end{aligned}$$

where $c = \sup_{k\geq 0} k2^{-k/2}$.

5. RIEMANN-LIOUVILLE OPERATORS

Let $\gamma > 0$ and consider the following Riemann-Liouville operators:

$$\mathcal{R}_\gamma f(x) := \frac{\gamma}{x^\gamma} \int_0^x (x-y)^{\gamma-1} f(y)dy, \tag{5.1}$$

and corresponding geometric mean operators

$$G_\gamma f(x) = \exp\left[\frac{\gamma}{x^\gamma} \int_0^x (x-y)^{\gamma-1} \log f(y)dy\right].$$

In this section we shall study the question of characterization of the weights $u(x)$ and $v(x)$ so that the inequality

$$\left(\int_0^\infty (G_\gamma f(x))^q \, u(x)\, dx\right)^{1/q} \leq C \left(\int_0^\infty f(x)^p v(x) dx\right)^{1/p}, \quad 0 < p, q < \infty \tag{5.2}$$

holds and also to point out the corresponding stable estimates of $C = \|G_\gamma\|_{L_v^p \to L_u^q}$.

We note that in the case $0 < \gamma < 1$ the kernel $k(x,y) = (x-y)^{\gamma-1}$ in (5.1) does not satisfy the Oinarov condition (3.1) so the results in Theorems 3.1 and 4.1 cannot be applied. However, this kernel has this property for the case $\gamma \geq 1$ so the question above can be solved by simply applying Theorems 3.1 and 4.1. Here we unify both cases in the next theorem and give a separate proof to this operator which gives a better estimate of the upper bound.

Theorem 5.1. **(a)** *Let $0 < p \leq q < \infty$. Then (5.2) holds if and only if $\mathbb{A}_0 < \infty$. Moreover,*

$$\mathbb{A}_0 \leq \|G_\gamma\|_{L_v^p \to L_u^q} \leq \gamma e^{1/p} \mathbb{A}_0, \quad 1 \leq \gamma < \infty. \tag{5.3}$$

and

$$\mathbb{A}_0 \leq \|G_\gamma\|_{L_v^p \to L_u^q} \lesssim \mathbb{A}_0, \quad 0 < \gamma < 1. \tag{5.4}$$

(b) *Let $0 < q < p < \infty$. Then (5.2) holds if and only if $\mathbb{B}_0 < \infty$. Moreover,*

$$\|G_\gamma\|_{L_v^p \to L_u^q} \approx \mathbb{B}_0. \tag{5.5}$$

The factors of equivalence in (5.4) and (5.5) depend on p, q and γ only.

Remark 5.2. By applying Theorem 5.1 with $\gamma = 1$ we obtain Theorems 2 and 4 in [6] (see also [7]) even with the same constants in the upper estimate in (5.3).

Partly guided by the technique used in [6], we postpone the proof of Theorem 5.1 and first prove two auxiliary results of independent interest, namely a characterization of the inequality

$$\left(\int_0^\infty (\mathcal{R}_\gamma f)^q \, w\right)^{1/q} \leq \mathbb{C} \left(\int_0^\infty f^p\right)^{1/p}, \quad f \geq 0, \tag{5.6}$$

in a form suitable for our purpose.

The following two theorems may be seen as a unification and generalization of the results from ([6, Theorems 1 and 3]) and ([9, Theorems 1 and 2]).

Theorem 5.3. **(a)** *Let $\gamma \geq 1$ and $1 < p \leq q \leq \infty$. Then*

$$\mathbb{A}_0 \leq \|\mathcal{R}_\gamma\|_{L^p \to L_w^q} \leq \gamma p' \mathbb{A}_0. \tag{5.7}$$

(b) *Let $0 < \gamma < 1$ and $1/\gamma < p \leq q \leq \infty$. Then*

$$\mathbb{A}_0 \leq \|\mathcal{R}_\gamma\|_{L^p \to L_w^q} \leq \gamma \left[\left(\frac{p-1}{p\gamma - 1}\right)^{1/p'} \left(2^{1/p} + 2^{2/p}\right) + 2^{1-\gamma} p'\right] \mathbb{A}_0. \tag{5.8}$$

Remark 5.4. The lower bound $\mathbb{A}_0 \leq \|\mathcal{R}_\gamma\|_{L^p \to L_w^q}$ holds for all $\gamma > 0$ and $0 < p, q \leq \infty$.

Theorem 5.5. **(a)** *Let* $\gamma \geq 1$*,* $0 < q < p < \infty$ *and* $p > 1$*. Then*

$$\frac{\gamma}{2^{\gamma+1/r+1/q}} (p')^{1/q'} p^{-1/r} r^{-1/r'} q \mathbb{B}_0 \leq \|\mathcal{R}_\gamma\|_{L^p \to L^q_w} \leq \gamma q^{1/q} p' \mathbb{B}_0. \tag{5.9}$$

(b) *Let* $0 < \gamma < 1$*,* $0 < q < p < \infty$ *and* $p\gamma > 1$*. Then*

$$\begin{aligned} &\frac{\gamma \left(2^{q/r} - 1\right)^{1/p} \left(2^{qr/p^2} - 1\right)^{1/r}}{2^{3+1/r}} \mathbb{B}_0 \\ &\leq \|\mathcal{R}_\gamma\|_{L^p \to L^q_w} \\ &\leq \gamma \mathbb{B}_0 \left[\left(\frac{p-1}{p\gamma - 1} \right)^{1/p'} 2^{1+1/q} \left(1 + 2^{1/p'}\right) + \left(\left(\frac{r}{p}\right)^{q/r} 4^q + q\,(p')^q \right)^{1/q} \right]. \end{aligned} \tag{5.10}$$

Proof of Theorem 5.3. For the lower bounds on (5.7) and (5.8) we replace f in (5.6) by the test function $f_t(x) = \chi_{[0,t]}(x)$, $t > 0$. Then for $p, q < \infty$

$$t^{1/p}\mathbb{C} \geq \left(\int_0^\infty (\mathcal{R}_\gamma f_t)^q w \right)^{1/q} \geq \left(\int_0^t w \right)^{1/q}.$$

Hence,

$$\|\mathcal{R}_\gamma\|_{L^p \to L^q_w} \geq \mathbb{A}_0$$

for all $\gamma > 0$, $0 < p, q < \infty$. For $p \leq q = \infty$, $p = q = \infty$ or $q < p = \infty$ the arguments are the same.

Clearly,

$$\mathcal{R}_\gamma f(x) \leq \gamma H f(x), \quad \gamma \geq 1 \tag{5.11}$$

and the upper bound in (5.7) follows from Theorem 1 in [6].

For the upper bound in (5.8) we follow the scheme from the proof of Theorem 1 in [9]. Put

$$J := \left(\int_0^\infty \left(\frac{1}{x^\gamma} \int_0^x (x-y)^{\gamma-1} f(y) dy \right)^q w(x) dx \right)^{1/q}$$

and note that, by Minkowski's inequality,

$$J \leq J_1 + J_2 + J_3, \tag{5.12}$$

where

$$J_1^q := \sum_{k \in \mathbb{Z}} \int_{2^k}^{2^{k+1}} \frac{w(x)dx}{x^{\gamma q}} \left(\int_{2^k}^x (x-y)^{\gamma-1} f(y) dy \right)^q,$$

$$J_2^q := \sum_{k \in \mathbb{Z}} \int_{2^k}^{2^{k+1}} \frac{w(x)dx}{x^{\gamma q}} \left(\int_{2^{k-1}}^{2^k} (x-y)^{\gamma-1} f(y) dy \right)^q,$$

and

$$J_3^q := \sum_{k \in \mathbb{Z}} \int_{2^k}^{2^{k+1}} \frac{w(x)dx}{x^{\gamma q}} \left(\int_0^{2^{k-1}} (x-y)^{\gamma-1} f(y) dy \right)^q.$$

Applying Hölder's inequality we find

$$J_1^q \le \sum_{k\in\mathbb{Z}} \left(\int_{2^k}^{2^{k+1}} f^p\right)^{q/p} \int_{2^k}^{2^{k+1}} \frac{w(x)dx}{x^{\gamma q}} \left(\int_{2^k}^{x} (x-y)^{(\gamma-1)p'}\,dy\right)^{q/p'}$$

$$\le \left(\frac{p-1}{p\gamma-1}\right)^{q/p'} \sum_{k\in\mathbb{Z}} \left(\int_{2^k}^{2^{k+1}} f^p\right)^{q/p} \left(\int_{2^k}^{2^{k+1}} w(x)dx\right) 2^{-kq/p}$$

$$\le 2^{q/p}\left(\frac{p-1}{p\gamma-1}\right)^{q/p'} \sum_{k\in\mathbb{Z}} \left(\int_{2^k}^{2^{k+1}} f^p\right)^{q/p} \left(2^{-(k+1)q/p}\int_{0}^{2^{k+1}} w(x)dx\right)$$

$$\le 2^{q/p}\left(\frac{p-1}{p\gamma-1}\right)^{q/p'} \mathbb{A}_0^q \left(\int_0^\infty f^p\right)^{q/p}, \tag{5.13}$$

where the last step follows by the elementary inequality $\sum a_i^\beta \le (\sum a_i)^\beta$, $\beta \ge 1$ and the definition of $\mathbb{A}_0$.

Similarly, we obtain

$$J_2^q \le 4^{q/p}\left(\frac{p-1}{p\gamma-1}\right)^{q/p'} \mathbb{A}_0^q \left(\int_0^\infty f^p\right)^{q/p}. \tag{5.14}$$

For the upper bound of J_3 we note that

$$J_3^q \le 2^{(1-\gamma)q}\int_0^\infty (Hf)^q\, w$$

so that, by Theorem 1 in [6] ,

$$J_3^q \le 2^{(1-\gamma)q}\mathbb{A}_0^q\,(p')^q \left(\int_0^\infty f^p\right)^{q/p}. \tag{5.15}$$

By combining (5.13), (5.14) and (5.15) we obtain the upper bound of (5.8) and the proof is complete.

(a) For the lower bound we write

$$\mathbb{C}\left(\int_0^\infty f^p\right)^{1/p} \ge \gamma\left(\int_0^\infty \left(\frac{1}{x^\gamma}\int_0^{x/2} (x-y)^{\gamma-1} f(y)dy\right)^q w(x)dx\right)^{1/q}$$

$$\ge \frac{\gamma}{2^{\gamma-1}}\left(\int_0^\infty \left(\frac{1}{x}\int_0^{x/2} f\right)^q w(x)dx\right)^{1/q}$$

$$\ge \frac{\gamma}{2^{\gamma}}\left(\int_0^\infty \left(\frac{1}{x}\int_0^{x} f\right)^q w(2x)dx\right)^{1/q}.$$

By applying Theorem 3 in [6] we find

$$\|\mathcal{R}_\gamma\|_{L^p\to L^q_w} \ge \frac{\gamma}{2^\gamma} c\,(p,q)\,\mathfrak{B}_0,$$

where

$$\mathfrak{B}_0 := \left(\int_0^\infty \left(\frac{1}{x}\int_0^x w\,(2s)\,ds\right)^{r/q} dx\right)^{1/r} = 2^{-1/r}\mathbb{B}_0$$

and $c\,(p,q) = 2^{-1/q}\,(p')^{1/q'}\,p^{-1/r}r^{-1/r'}q$ and the lower bound in (5.9) is proved. Using (5.11) and again Theorem 3 in [6] we obtain the upper bound of (5.9).

(b) It is shown in Theorem 2 in [9] that

$$\|\mathcal{R}_\gamma\|_{L^p\to L^q_w} \geq \frac{\gamma}{4}\mathbb{E},$$

where

$$\mathbb{E} := \left(\sum_{k\in\mathbb{Z}} 2^{kr/p'}\left(\int_{2^k}^{2^{k+1}} \frac{w(x)dx}{x^q}\right)^{r/q}\right)^{1/r}.$$

We show that

$$\mathbb{E} \geq \frac{\left(2^{q/r}-1\right)^{1/p}\left(2^{qr/p^2}-1\right)^{1/r}}{2^{1+1/r}}\mathbb{B}_0.$$

Indeed,

$$\begin{aligned}
\mathbb{B}_0^r &= \sum_{k\in\mathbb{Z}}\int_{2^{k-1}}^{2^k}\left(\frac{1}{x}\int_0^x w\right)^{r/q}dx \\
&\leq \sum_{k\in\mathbb{Z}} 2^{-(k-1)r/q}\left(\int_0^{2^k} w\right)^{r/q}\int_{2^{k-1}}^{2^k} dx \\
&= \sum_{k\in\mathbb{Z}} 2^{-(k-1)r/p}\left(\sum_{m\leq k}\int_{2^{m-1}}^{2^m} w\right)^{r/q} \\
&= 2^{r/p}\sum_{k\in\mathbb{Z}} 2^{-kr/p}\left(\sum_{m\leq k}\left\{2^{\frac{-mq^2}{rp}}\int_{2^{m-1}}^{2^m} w\right\}2^{\frac{mq^2}{rp}}\right)^{r/q}
\end{aligned}$$

(by applying Hölder's inequality with $\frac{r}{q}$ and $\frac{p}{q}$)

$$\begin{aligned}
&\leq 2^{r/p}\sum_{k\in\mathbb{Z}} 2^{-kr/p}\left(\sum_{m\leq k} 2^{-mq/p}\left(\int_{2^{m-1}}^{2^m} w\right)^{r/q}\right)\left(\sum_{m\leq k} 2^{mq/r}\right)^{r/p} \\
&= \frac{2^{r/p+q/p}}{\left(2^{q/r}-1\right)^{r/p}}\sum_{m\in\mathbb{Z}} 2^{-mq/p}\left(\int_{2^{m-1}}^{2^m} w\right)^{r/q}\sum_{k\geq m} 2^{-kqr/p^2} \\
&= \frac{2^{2r/p}}{\left(2^{q/r}-1\right)^{r/p}\left(2^{qr/p^2}-1\right)}\sum_{m\in\mathbb{Z}} 2^{-mr/p}\left(\int_{2^{m-1}}^{2^m} w\right)^{r/q} \\
&\leq \frac{2^{r(1+1/p)}}{\left(2^{q/r}-1\right)^{r/p}\left(2^{qr/p^2}-1\right)}\mathbb{E}^r.
\end{aligned}$$

Conversely,

$$\begin{aligned}
\mathbb{E}^r &\leq \sum_{k\in\mathbb{Z}} 2^{-(k-1)r/p}\left(\int_{2^{k-1}}^{2^k} w\right)^{r/q} \\
&\leq 2^{r/p}\sum_{k\in\mathbb{Z}} 2^{-kr/p}\left(\int_0^{2^k} w\right)^{r/q}\int_{2^k}^{2^{k+1}} dx
\end{aligned}$$

$$\leq 2^{r/p+r/q} \sum_{k \in \mathbb{Z}} \int_{2^k}^{2^{k+1}} \left(\frac{1}{x} \int_0^x w \right)^{r/q} dx$$

$$= 2^{r/p+r/q} \mathbb{B}_0^r. \tag{5.16}$$

Using the decomposition (5.12) it is shown in Theorem 2 in [9] that

$$J \leq \left(\frac{p-1}{p\gamma - 1} \right)^{1/p'} 2^{1/p'} \left(1 + 2^{1/p'} \right) \mathbb{E} + 2^{1-\gamma} \left(\int_0^\infty (Hf)^q w \right)^{1/q}.$$

Now the upper bound in (5.10) follows from (5.16) and Theorem 3 in [6].

□

We are now ready to complete this section by presenting

Proof of Theorem 5.1. Both sides of (5.3) follow from Theorem 5.3 **(a)** and (2.4). The lower bound in (5.4) follows by using the test functions from the proof of Theorem 3.1 and the upper bound is a consequence of (2.2), (2.10) and (5.8). Similarly, the proof of **(b)** is based upon Theorem 5.5 and we use the same test function for the lower bound as in the proof of Theorem 4.1 with subsequent application of the inequality $\mathbb{B}_0 \leq B_0$. The proof is complete. □

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