Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 3, Issue 5, Article 78, 2002

A NOTE ON THE TRACE INEQUALITY FOR PRODUCTS OF HERMITIAN MATRIX POWER

ZHONG PENG YANG AND XIAO XIA FENG

DEPARTMENT OF MATHEMATICS,
PUTIAN UNIVERSITY,
PUTIAN, FUJIAN 351100
PEOPLE'S REPUBLIC OF CHINA
yangzhongpeng@sina.com

DEPARTMENT OF MATHEMATICS,
BEIHUA UNIVERSITY,
JILIN, JILIN 132013,
PEOPLE'S REPUBLIC OF CHINA.
fengxiaoxia0110@163.com

Received 6 June, 2002; accepted 24 July, 2002
Communicated by D. Bainov

ABSTRACT. Da-wei Zhang [J.M.A.A., 237 (1999): 721-725] obtained the inequality $\operatorname{tr}(AB)^{2^k} \leq \operatorname{tr} A^{2^k} B^{2^k}$ for $Hermitian$ matrices A and B, where k is natural number. Here it is proved that these results hold when the power index of the product of *Hermitian* matrices A and B is a nonnegative even number. In the meantime, it is pointed out that the relation between $\operatorname{tr}(AB)^m$ and $\operatorname{tr} A^m B^m$ is complicated when the power index m is a nonnegative odd number, therefore the above inequality cannot be generalized to all nonnegative integers. As an application, we not only improve the results of Xiaojing Yang [J.M.A.A., 250 (2000), 372-374], Xinmin Yang [J.M.A.A., 263 (2001): 327-333] and Fozi M. Dannan [J. Ineq. Pure and Appl. Math., 2(3) (2001), Art. 34], but also give the complete resolution for the question of the trace inequality about the powers of Hermitian and skew Hermitian matrices that is proposed by Zhengming Jiao.

Key words and phrases: Hermitian matrix, Trace, Inequality, Skew Hermitian matrix.

2000 *Mathematics Subject Classification.* 15A42; 15A57.

1. INTRODUCTION

Let $C^{n\times n}$ be the set of all $n \times n$ matrices over the complex number field $\mathbb{C}$. The modulus of all diagonal entries of the matrix $A = (a_{ij}) \in C^{n\times n}$ are arranged in decreasing order as $|\delta_1(A)| \geq |\delta_2(A)| \geq \cdots \geq |\delta_n(A)|$, i.e., $\delta_1(A), \delta_2(A), \cdots, \delta_n(A)$ is an entire arrangement of $a_{11}, a_{22}, \cdots, a_{nn}$; all its singular values satisfy $\sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_n(A)$. In particular, when the eigenvalues of A are real numbers, let its eigenvalues satisfy $\lambda_1(A) \geq \lambda_2(A) \cdots \geq$

ISSN (electronic): 1443-5756

The research for this paper was supported by the Science Foundation of the Education Department of Fujian, (No. JB01206).
082-02

$\lambda_n(A)$; A^H, $\operatorname{tr} A$ denote its conjugate transpose matrix and trace respectively. Further, let $H(n)$, $H_0^+(n)$, $H^+(n)$, $S(n)$ be the subsets of all *Hermitian*, *Hermitian* semi-positive definite, *Hermitian* positive definite and skew *Hermitian* matrices. Finally, let $A^{1/2}$ represent the quadratic root of $A \in H_0^+(n)$, and $\mathbb{R}$, $\mathbb{N}$ denote the sets of all real numbers and nonnegative integers. The complex number $\sqrt{-1} \in C$ satisfies $(\sqrt{-1})^2 = -1$.

Recently the trace inequality of two powered *Hermitian* matrices was given in [1] as follows:

$$\operatorname{tr}(AB)^{2^k} \leq \operatorname{tr} A^{2^k} B^{2^k}, \ \ A, B \in H(n), \ k \in \mathbb{N}. \tag{1.1}$$

Furthermore, the following two results were proved in [2],

$$0 \leq \operatorname{tr}(AB)^{2m} \leq (\operatorname{tr} A)^2 (\operatorname{tr} A^2)^{m-1} (\operatorname{tr} B^2)^m, \ \ m(\geq 1) \in \mathbb{N}, \ A, B \in H_0^+(n); \tag{1.2}$$

and

$$0 \leq \operatorname{tr}(AB)^{2m+1} \leq \operatorname{tr} A \ \operatorname{tr} B \, (\operatorname{tr} A^2)^m (\operatorname{tr} B^2)^m, \ \ m(\geq 1) \in \mathbb{N}, \ A, B \in H_0^+(n). \tag{1.3}$$

Another two results appeared in [3, Theorem 1] and [4, Theorem 1]. When $A, B \in H^+(n)$, the following inequalities hold:

$$\operatorname{tr}(AB)^m \leq (\operatorname{tr} A^{2m})^{1/2} (\operatorname{tr} B^{2m})^{1/2}, \ \ m \in N; \tag{1.4}$$

and

$$\operatorname{tr}(AB)^m \leq (\operatorname{tr} AB)^m, \ \ m \in N. \tag{1.5}$$

The above two results (1.1), (1.2), (1.3), (1.4) and (1.5) are related to the work of Bellman. In 1980, Bellman [5] proved:

$$\operatorname{tr}(AB)^2 \leq \operatorname{tr} A^2 B^2, \ \ A, B \in H_0^+(n), \tag{1.6}$$

and proposed the conjecture whether

$$\operatorname{tr}(AB)^m \leq \operatorname{tr} A^m B^m, \ \ m \in \mathbb{N}, \ \ A, B \in H_0^+(n) \tag{1.7}$$

holds.

Since then, many authors have proved that the conjecture (1.7) is correct. In [6], it was pointed out that the inequality (1.7) was also proposed by Lieb and Thiring in 1976, and a similar inequality was proposed also in [7]. R.A. Brualdi [8] commented further work of the inequality (1.7) that was constructed by Lieb and Thiring in [6] and [7].

Whether or not the inequality (1.7) was a conjecture at that time, the condition in [1] was different from that in [2] – [7], which dropped the demand of "semi-positive definite property" for matrices in [1], and examined the trace inequality on the general *Hermitian* matrix powers. Of course, it increases inevitably the discussed difficulty. In 1992 Zhengming Jiao [9] generalised inequality (1.6) for $A \in H(n)$, $B \in S(n)$ and $A, B \in S(n)$, and also presented two questions as follows:

$$\operatorname{tr}(AB)^m \geq \operatorname{tr} A^m B^m, \ \ A \in H(n), \ B \in S(n), \ m \in \mathbb{N}? \tag{1.8}$$

and

$$\operatorname{tr}(AB)^m \leq \operatorname{tr} A^m B^m, \ \ A, B \in S(n), \ m \in \mathbb{N}? \tag{1.9}$$

We will prove that the inequality (1.1) holds when the power index is a nonnegative even number. Thereby the results in [2, 3] and [4] can be obtained and improved. Moreover, a simpler proof for the inequality (1.7) may be presented. As an application of the obtained result, we answer completely two questions mentioned in [9] in the form of (1.8) and (1.9).

2. SOME LEMMAS

Lemma 2.1. *Let* $A, B \in C^{n\times n}$*, then*

$$\sum_{i=1}^{t} |\delta_i((AB)^m)| \le \sum_{i=1}^{t} \lambda_i\left((A^H ABB^H)^m\right) \le \sum_{i=1}^{t} \lambda_i\left((A^H A)^m (BB^H)^m\right), \quad 1 \le t \le n,\ m \in \mathbb{N}. \tag{2.1}$$

Proof. From [10, Theorem 8.9], it follows that

$$\sum_{i=1}^{t} |\delta_i(F)| \le \sum_{i=1}^{t} \sigma_i(F),\ \ 1 \le t \le n,\ \ F \in C^{n\times n} \tag{2.2}$$

and by [11, Theorem 1],

$$\sum_{i=1}^{t} \sigma_i\left(\prod_{j=1}^{p} G_j\right) \le \sum_{i=1}^{t}\prod_{j=1}^{p} \sigma_i(G_j),\ \ 1 \le t \le n,\ \ G_1, G_2, \cdots, G_p \in C^{n\times n}. \tag{2.3}$$

Moreover via [7, Theorem 4], it is derived that

$$\sum_{i=1}^{t} \lambda_i^m(FG) \le \sum_{i=1}^{t} \lambda_i\left(F^m G^m\right),\ \ 1 \le t \le n,\ \ m \in \mathbb{N},\ \ F, G \in H_0^+(n). \tag{2.4}$$

therefore through $A^H A, BB^H \in H_0^+(n)$, it is known that all the eigenvalues of $A^H ABB^H \in C^{n\times n}$ are real. Meanwhile

$$\lambda_i\left(\left(A^H ABB^H\right)^m\right) = \lambda_i^m\left(A^H ABB^H\right) = \left(\lambda_i\left(A^H ABB^H\right)\right)^m,\ \ i = 1, 2, \dots, n;$$

and from (2.2), (2.3), and (2.4), it holds that

$$\begin{aligned}
\sum_{i=1}^{t} |\delta_i\left((AB)^{2m}\right)| &\le \sum_{i=1}^{t} \sigma_i\left((AB)^{2m}\right) \\
&\le \sum_{i=1}^{t} \left(\sigma_i(AB)\right)^{2m} \\
&= \sum_{i=1}^{t} \left(\sigma_i^2(AB)\right)^m \\
&= \sum_{i=1}^{t} \lambda_i\left(\left(A^H ABB^H\right)^m\right) \\
&\le \sum_{i=1}^{t} \lambda_i\left(\left(A^H A\right)^m \left(BB^H\right)^m\right)
\end{aligned}$$

that is, (2.1) holds. □

It is well known that eigenvalues of the product AB for $A, B \in H(n)$ are not real numbers, but we can obtain the following lemma.

Lemma 2.2. *Let* $A = (a_{ij}),\ B = (b_{ij}) \in H(n)(S(n))$*, then* $\operatorname{tr} AB \in \mathbb{R}$.

Proof. When $A, B \in H(n)$, according to [10, pp. 219] it is known that $\operatorname{tr} AB \in \mathbb{R}$.

By the simple fact,

$$F \in S(n) \quad \text{if and only if} \quad \sqrt{-1}F \in H(n), \tag{2.5}$$

it follows that $\sqrt{-1}A, \sqrt{-1}B \in H(n)$ holds when $A, B \in S(n)$. Thus from the proved result, at this time, it is easy to know

$$\operatorname{tr} AB = \operatorname{tr}\left(-\left(\sqrt{-1}A\right)\left(\sqrt{-1}B\right)\right) = -\operatorname{tr}\left(\sqrt{-1}A\right)\left(\sqrt{-1}B\right) \in \mathbb{R}.$$

□

By [10, Theorem 6.5.3], the following Lemma holds.

Lemma 2.3. *Let $A, B \in H_0^+(n)$, then*

$$0 \le \operatorname{tr} AB \le \operatorname{tr} A \operatorname{tr} B; \tag{2.6}$$

and

$$0 \le \operatorname{tr} A^m \le (\operatorname{tr} A)^m, \quad m \in \mathbb{N}. \tag{2.7}$$

Lemma 2.4. *Let $A, B \in H(n)$, then $\operatorname{tr}(AB)^m, \ \operatorname{tr} A^m B^m \in \mathbb{R}$ for all $m \in N$.*

Proof. When $m = 0, 1$, obviously the result holds by Lemma 2.2. When $m \ge 2$, via

$$\left((AB)^{m-1}A\right)^H = (A(BAB \cdots BAB)A)^H = (AB)^{m-1}A \in H(n),$$

and Lemma 2.2, it follows that $\operatorname{tr}(AB)^m = \operatorname{tr}\left((AB)^{m-1}A\right)B \in \mathbb{R}$.

For $A^m, B^m \in H(n)$ and from Lemma 2.2, it may be surmised that $\operatorname{tr} A^m B^m \in \mathbb{R}$. □

Lemma 2.5. *Let $A \in H(n)$, $B \in S(n)$, $m \in \mathbb{N}$, then*

$$\begin{aligned} \operatorname{tr}(AB)^m &= \left(-\sqrt{-1}\right)^m \operatorname{tr}\left(A(\sqrt{-1}B)\right)^m, \\ \operatorname{tr} A^m B^m &= \left(-\sqrt{-1}\right)^m \operatorname{tr} A^m \left(\sqrt{-1}B\right)^m; \end{aligned} \tag{2.8}$$

and for $m = 2t (t \in N)$, $\operatorname{tr}(AB)^m$, $\operatorname{tr} A^m B^m$ are all real. Further, when $m = 2t+1 (t \in N)$, $\operatorname{tr}(AB)^m$, $\operatorname{tr} A^m B^m$ are all zeros or pure imaginary numbers.

Proof. Without loss of generality, assume that $m \ge 2$, similarly

$$\begin{aligned} \operatorname{tr}(AB)^m &= \operatorname{tr}\left(A\left(-\sqrt{-1}(\sqrt{-1}B)\right)\right)^m = \left(-\sqrt{-1}\right)^m \operatorname{tr}\left(A\left(\sqrt{-1}B\right)\right)^m, \\ \operatorname{tr} A^m B^m &= \operatorname{tr} A^m \left(-\sqrt{-1}(\sqrt{-1}B)\right)^m = \left(-\sqrt{-1}\right)^m \operatorname{tr} A^m \left(\sqrt{-1}B\right)^m, \end{aligned}$$

so that (2.8) holds.

When $m = 2t \ (t \in \mathbb{N})$, $\left(-\sqrt{-1}\right)^m = (-1)^{3t} \in \mathbb{R}$, thus by (2.8) and Lemma 2.4, one obtains that $\operatorname{tr}(AB)^m$, $\operatorname{tr} A^m B^m$ are all real. When $m = 2t+1 \ (t \in \mathbb{N})$, $\left(-\sqrt{-1}\right)^m = (-1)^{3t+1}\sqrt{-1} \notin \mathbb{R}$, then we have that $\operatorname{tr}(AB)^m$, $\operatorname{tr} A^m B^m$ are all zeros or pure imaginary numbers by (2.8) and Lemma 2.4. □

Similar to the proof of Lemma 2.5, it also follows that:

Lemma 2.6. *Let $A, B \in S(n), \ m \in N$, then*

$$\begin{aligned} \operatorname{tr}(AB)^m &= (-1)^m \operatorname{tr}\left(\left(\sqrt{-1}A\right)\left(\sqrt{-1}B\right)\right)^m, \\ \operatorname{tr} A^m B^m &= (-1)^m \operatorname{tr}\left(\sqrt{-1}A\right)^m \left(\sqrt{-1}B\right)^m \in \mathbb{R}. \end{aligned} \tag{2.9}$$

3. MAIN RESULTS

Theorem 3.1. *Let* $A, B \in C^{n\times n}$*, then*

$$|\operatorname{tr}(AB)^{2m}| \le \operatorname{tr}\left(A^H ABB^H\right)^m \le \operatorname{tr}\left(A^H A\right)^m \left(BB^H\right)^m, \quad m \in \mathbb{N}; \tag{3.1}$$

$$\begin{aligned} |\operatorname{tr}(AB)^{2m}| &\le \operatorname{tr}\left(A^H ABB^H\right)^m \\ &\le \operatorname{tr}\left(A^H A\right)^m \left(BB^H\right)^m \\ &\le \operatorname{tr}\left(\left(A^H A\right)^{1/2}\right)^2 \operatorname{tr}\left(A^H A\right)^{m-1} \operatorname{tr}\left(BB^H\right)^m \\ &\le \left(\operatorname{tr}\left(A^H A\right)^{1/2}\right)^2 \left(\operatorname{tr} A^H A\right)^{m-1} \left(\operatorname{tr} BB^H\right)^m, \quad m(\ge 1) \in \mathbb{N}. \end{aligned} \tag{3.2}$$

Proof. Take $t = n$ in (2.1), then we have that

$$\begin{aligned} \left|\operatorname{tr}(AB)^{2m}\right| &= \left|\sum_{i=1}^{n} \delta_i\left((AB)^{2m}\right)\right| \\ &\le \sum_{i=1}^{n} \left|\delta_i\left((AB)^{2m}\right)\right| \\ &\le \sum_{i=1}^{n} \lambda_i\left(\left(A^H ABB^H\right)^m\right) \\ &= \operatorname{tr}\left(A^H ABB^H\right)^m \\ &\le \sum_{i=1}^{n} \lambda_i\left(\left(A^H A\right)^m \left(BB^H\right)^m\right) = \operatorname{tr}\left(\left(A^H A\right)^m \left(BB^H\right)^m\right), \end{aligned}$$

giving (3.1).

By $A^H A, BB^H \in H_0^+(n)$ and (2.6), (2.7), when $m \ge 1$ one can get

$$\begin{aligned} \operatorname{tr}\left(A^H A\right)^m \left(BB^H\right)^m &\le \operatorname{tr}\left(A^H A\right)\left(A^H A\right)^{m-1} \operatorname{tr}\left(BB^H\right)^m \\ &\le \operatorname{tr}\left(\left(A^H A\right)^{1/2}\right)^2 \operatorname{tr}\left(A^H A\right)^{m-1} \operatorname{tr}\left(BB^H\right)^m \\ &\le \left(\operatorname{tr}\left(A^H A\right)^{1/2}\right)^2 \operatorname{tr}\left(A^H A\right)^{m-1} \left(\operatorname{tr} BB^H\right)^m, \end{aligned}$$

therefore (3.2) is correct, by (3.1). □

Theorem 3.2. *Let* $A, B \in H(n)$*, then*

$$\operatorname{tr}(AB)^{2m} \le |\operatorname{tr}(AB)^{2m}| \le \operatorname{tr}\left(A^2 B^2\right)^m \le \operatorname{tr} A^{2m} B^{2m}, \quad m \in \mathbb{N}. \tag{3.3}$$

Proof. From Lemma 2.4, it is obtained that $\operatorname{tr}(AB)^m$, $\operatorname{tr} A^m B^m$ are all real, then by $\operatorname{tr}(AB)^{2m} \le |\operatorname{tr}(AB)^{2m}|$ and $A^H = A$, $B^H = B$, moreover applying (3.1), (3.3) holds. □

Because 2^k $(k \in \mathbb{N})$ is a nonnegative even number, conclusion (1.1) in [1] can be achieved by (3.3). When $A, B \in H_0^+(n)$, the inequality (1.7) is obtained by replacing A, B of (3.3) with $A^{1/2}, B^{1/2}$. The above procedure indicates we can give a simple proof for the inequality (1.7).

Example 3.1. Let

$$A = \begin{bmatrix} 1 & -2 \\ -2 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \in H(2),$$

and so

$$(AB)^3 = \begin{bmatrix} 16 & -6 \\ 4 & 12 \end{bmatrix}, \quad A^3B^3 = \begin{bmatrix} -29 & 44 \\ -26 & 32 \end{bmatrix},$$
$$(AB)^5 = \begin{bmatrix} -56 & -60 \\ 40 & -96 \end{bmatrix}, \quad A^5B^5 = \begin{bmatrix} -481 & 765 \\ -610 & 954 \end{bmatrix},$$

giving

$$\operatorname{tr}(AB)^3 = 28 > 3 = \operatorname{tr} A^3B^3, \quad \operatorname{tr}(AB)^5 = -152 < 573 = \operatorname{tr} A^5B^5.$$

Example 3.2. Let

$$A = \begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \in H(2),$$

giving

$$(AB)^3 = \begin{bmatrix} -16 & 6 \\ -4 & -12 \end{bmatrix}, \quad A^3B^3 = \begin{bmatrix} 29 & -44 \\ 26 & -32 \end{bmatrix},$$

and so we obtain

$$\operatorname{tr}(AB)^3 = -28 < -3 = \operatorname{tr} A^3B^3.$$

We have generalized the index of the trace inequality (1.1) on Hermitian matrix power ([1, Theorem 1]) from $2^k (k \in N)$ to nonnegative even numbers . Examples 3.1 and 3.2 indicate that it is complex when the power index is a positive odd number. Of course, they also show that the result cannot hold when one gives up the "positive semi-definite" requirement of (1.7). In [12, Theorem 6.3.2], the statement "Marcus (1956) generalized this theorem as following form

$$\operatorname{tr}(AB)^m \le \operatorname{tr} A^mB^m, \quad m \in \mathbb{N}, \quad A, B \in H(n),\text{"}$$

follows the proof of (1.6).

Although we do not have access to the article of Marcus, Examples 3.1 and 3.2 make us unsure that the generalization is not correct.

Theorem 3.3. *Let* $A, B \in H(n)$, $m \in \mathbb{N}$, *then*

$$\operatorname{tr}(AB)^{2m} \le \operatorname{tr}(A^2B^2)^m \le \operatorname{tr} A^{2m}B^{2m} \le \left(\operatorname{tr} A^{4m}\right)^{1/2} \left(\operatorname{tr} B^{4m}\right)^{1/2}; \tag{3.4}$$

$$\operatorname{tr}(AB)^{2m} \le \operatorname{tr}(A^2B^2)^m \le \left(\operatorname{tr} A^2B^2\right)^m; \tag{3.5}$$

$$\begin{aligned} \operatorname{tr}(AB)^{2m} &\le \operatorname{tr}\left(A^2B^2\right)^m \\ &\le \operatorname{tr} A^{2m}B^{2m} \\ &\le \operatorname{tr} A^2 \ \operatorname{tr} A^{2(m-1)} \operatorname{tr} B^{2m} \\ &\le \left(\operatorname{tr}\left(A^2\right)^{1/2}\right)^2 \left(\operatorname{tr} A^2\right)^{m-1} \left(\operatorname{tr} B^2\right)^m, \quad \textit{when } m \ge 1. \end{aligned} \tag{3.6}$$

Proof. From [10, Problem 7.2.10], it is known that

$$|\operatorname{tr} FG| \le \left(\operatorname{tr} F^2\right)^{1/2} (\operatorname{tr} G)^{1/2}, \quad F, G \in H(n). \tag{3.7}$$

Thus by (3.3), (3.7) and Lemma 2.4, the inequality (3.4) results.

Notice that $AB^2A \in H_0^+(n)$ and $(A^2B^2)^m = A(AB^2A)^{m-1}(AB^2)$, furthermore through (2.7), it follows that

$$\operatorname{tr}\left(A^2B^2\right)^m = \operatorname{tr}\left(AB^2A\right)^m \le \left(\operatorname{tr} AB^2A\right)^m = \left(\operatorname{tr} A^2B^2\right)^m,$$

the inequality (3.5) then follows by (3.3).

Notice that $A^2, (A^2)^{1/2} \in H_0^+(n)$, then via (2.6) and (2.7), it follows that

$$\operatorname{tr} A^{2m} B^{2m} \le \operatorname{tr} A^2 \ \operatorname{tr} A^{2(m-1)} \ \operatorname{tr} B^{2m} \le \left(\operatorname{tr} \left(A^2\right)^{1/2}\right)^2 \left(\operatorname{tr} A^2\right)^{m-1} \left(\operatorname{tr} B^2\right)^m,$$

the inequality (3.6) then results by using (3.5). □

Corollary 3.4. *Let $A, B \in H_0^+(n)$, $m \in \mathbb{N}$, then the inequalities (1.4) and (1.5) hold. Moreover when $m \geq 1$, it follows that*

$$\begin{aligned} 0 &\le \operatorname{tr}(AB)^{2m} \\ &\le \operatorname{tr} \left(A^2 B^2\right)^m \\ &\le \operatorname{tr} A^{2m} B^{2m} \\ &\le \operatorname{tr} A^2 \ \operatorname{tr} A^{2(m-1)} \ \operatorname{tr} B^{2m} \\ &\le (\operatorname{tr} A)^2 \left(\operatorname{tr} A^2\right)^{m-1} \left(\operatorname{tr} B^2\right)^m ; \end{aligned} \tag{3.8}$$

$$\begin{aligned} 0 &\le \operatorname{tr}(AB)^{2m+1} \\ &\le \operatorname{tr} A^{2m+1} B^{2m+1} \\ &\le \operatorname{tr} A \ \operatorname{tr} B \ \operatorname{tr} A^{2m} \ \operatorname{tr} B^{2m} \\ &\le \operatorname{tr} A \ \operatorname{tr} B \left(\operatorname{tr} A^2\right)^m \left(\operatorname{tr} B^2\right)^m . \end{aligned} \tag{3.9}$$

Proof. Using $A^{1/2}$, $B^{1/2}$ instead of A, B in the inequalities (3.4) and (3.5), (1.4) and (1.5) can be obtained. From $A \in H_0^+(n)$ it follows that $(A^2)^{1/2} = A$ and (3.8) is derived by (3.6). Furthermore through (3.6), (3.8), (2.6) and (2.7), it holds that

$$\begin{aligned} 0 &\le \operatorname{tr}(AB)^{2m+1} \\ &= \operatorname{tr} \left(\left(A^{1/2}\right)^2 \left(B^{1/2}\right)^2\right)^{2m+1} \\ &\le \operatorname{tr} \left(A^{1/2}\right)^{2(2m+1)} \left(B^{1/2}\right)^{2(2m+1)} \\ &= \operatorname{tr} A^{2m+1} B^{2m+1} \\ &\le \operatorname{tr} A^{2m+1} \ \operatorname{tr} B^{2m+1} \\ &= \operatorname{tr} AA^{2m} \ \operatorname{tr} BB^{2m} \\ &\le \operatorname{tr} A \ \operatorname{tr} A^{2m} \ \operatorname{tr} B \ \operatorname{tr} B^{2m} \\ &= \operatorname{tr} A \ \operatorname{tr} B \ \operatorname{tr} A^{2m} \ \operatorname{tr} B^{2m} \\ &\le \operatorname{tr} A \ \operatorname{tr} B \left(\operatorname{tr} A^2\right)^m \left(\operatorname{tr} B^2\right)^m , \end{aligned}$$

giving (3.9). □

As an application of the main results, in Corollary 3.4, the basic conclusions (1.2), (1.3), (1.4) and (1.5) of [2, 3] and [4] are summarized. At the same time, in Theorem 3.3, it is shown that the trace inequality (1.2) ([2]) on semi-positive definite *Hermitian* matrix power is extended to general *Hermitian* matrix, according to the form (3.6). By following Example 3.3, it is indicated that the trace inequality (1.3) ([2]) on semi-positive definite Hermitian matrix power cannot be generalised in a similar fashion to Theorem 3.3.

Example 3.3. Let

$$A = \begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \in H(2),$$

then from Examples 3.1 and 3.2, it is shown that

$$(AB)^5 = \begin{bmatrix} 56 & 60 \\ -40 & 96 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix}, \quad B^2 = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

Thus it is easily seen that

$$\begin{aligned} \operatorname{tr}(AB)^5 &= \operatorname{tr}(AB)^{2\times 2+1} \\ &= 152 > 1 \times (-1) \times 13^2 \times 3^2 \\ &= \operatorname{tr} A \operatorname{tr} B \left(\operatorname{tr} A^2\right)^2 \left(\operatorname{tr} B^2\right)^2. \end{aligned}$$

Example 3.4. Let

$$A = \begin{bmatrix} 2 & 3 \\ 3 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

and so from

$$AB = \begin{bmatrix} 6 & 6 \\ 9 & -8 \end{bmatrix}, \quad (AB)^2 = \begin{bmatrix} 90 & -12 \\ -18 & 118 \end{bmatrix},$$

it follows that

$$\operatorname{tr}(AB)^2 = 208 \;>\; 4 = (\operatorname{tr} AB)^2.$$

Example 3.4 indicates that the result (1.5) ([4]) for positive definite matrix cannot be generalized to general *Hermitian* matrix, but the generalized form, similar to (3.5) in Theorem 3.3, may be obtained.

Example 3.5. Let

$$A = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \in H(2),$$

then from $A^2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2I, \;\; B^2 = 5I$, it follows that

$$\left(\operatorname{tr}\left(A^2B^2\right)\right)^2 = 400 > 200 = \operatorname{tr} A^4B^4.$$

Example 3.6. Let

$$A = \begin{bmatrix} 3 & -1 \\ -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 2 \\ 2 & -1 \end{bmatrix} \in H(2),$$

from

$$A^2B^2 = \begin{bmatrix} 274 & 70 \\ -42 & -6 \end{bmatrix}, \quad A^4B^4 = \begin{bmatrix} 87592 & 26152 \\ -19544 & -5816 \end{bmatrix},$$

it is achieved that

$$\left(\operatorname{tr}\left(A^2B^2\right)\right)^2 = 71824 < 81776 = \operatorname{tr} A^4B^4.$$

Examples 3.5 and 3.6 show that the two upper bounds $\operatorname{tr} A^{2m}B^{2m}$ and $(\operatorname{tr} A^2B^2)^m$ for $\operatorname{tr}(AB)^{2m}$ as given by (3.3) and (3.5), are independent of each other, in which $\operatorname{tr}(AB)^{2m}$ is the trace of the product power on the two *Hermitian* matrices A and B. The result of [4] can be derived by replacing the matrices A and B in Examples 3.5 and 3.6 with $A^{1/2}$ and $B^{1/2}$.

For the trace $\operatorname{tr}(AB)^m$ of the product power on positive definite *Hermitian* matrices A and B, the upper bounds $\operatorname{tr} A^mB^m$ and $(\operatorname{tr} AB)^m$ given by (1.7) and (1.5) cannot be compared with each other; but from the upper bound $\left(\operatorname{tr} A^{2m}\right)^{1/2}\left(\operatorname{tr} B^{2m}\right)^{1/2}$ ([3]) determined by (1.4), via (3.4) and Corollary 3.4, it follows that

$$\operatorname{tr} A^mB^m \le \left(\operatorname{tr} A^{2m}\right)^{1/2}\left(\operatorname{tr} B^{2m}\right)^{1/2}.$$

4. TRACE OF THE POWER ON HERMITIAN MATRIX AND SKEW HERMITIAN MATRIX

Theorem 4.1. *Let $A \in H(n)$, $B \in S(n)$, then when $m = 4t$ or $m = 4t + 2$, $t \in \mathbb{N}$, $\operatorname{tr}(AB)^m$ and $\operatorname{tr} A^m B^m$ are all real numbers, and*

$$\operatorname{tr}(AB)^m \le \operatorname{tr}\left(A^2B^2\right)^{m/2} \le \operatorname{tr} A^m B^m, \quad m = 4t, \quad t \in \mathbb{N}; \tag{4.1}$$

$$\operatorname{tr}(AB)^m \ge \operatorname{tr}\left(A^2B^2\right)^{m/2} \ge \operatorname{tr} A^m B^m, \quad m = 4t+2, \quad t \in \mathbb{N}; \tag{4.2}$$

similarly when $m = 4t + 1$ or $m = 4t + 3,\ t \in N$, if $\operatorname{tr}(AB)^m \neq 0$ or $\operatorname{tr} A^m B^m \neq 0$, then $\operatorname{tr}(AB)^m \notin \mathbb{R}$ or $\operatorname{tr} A^m B^m \notin \mathbb{R}$, so $\operatorname{tr}(AB)^m$ and $\operatorname{tr} A^m B^m$ cannot be compared with each other.

Proof. By Lemma 2.5, we have that both $\operatorname{tr}(AB)^m$ and $\operatorname{tr} A^m B^m$ are real numbers when $m = 4t$. Furthermore through (3.3), (2.8), it follows that

$$\begin{aligned}
\operatorname{tr}(AB)^m &= \left(-\sqrt{-1}\right)^{4t} \operatorname{tr}\left(A\left(\sqrt{-1}B\right)\right)^{4t} \\
&= \operatorname{tr}\left(A\left(\sqrt{-1}B\right)\right)^{4t} \\
&\le \operatorname{tr}\left(A^2\left(\sqrt{-1}B\right)^2\right)^{2t} \\
&= \sqrt{-1}^{4t} \operatorname{tr}\left(A^2B^2\right)^{2t} \\
&= \operatorname{tr}\left(A^2B^2\right)^{m/2} \\
&\le \operatorname{tr} A^{4t}\left(\sqrt{-1}B\right)^{4t} \\
&= \sqrt{-1}^{4t} \operatorname{tr} A^{4t}B^{4t} \\
&= \operatorname{tr} A^m B^m,
\end{aligned}$$

giving (4.1).

In the same way, when $m = 4t + 2$, $\operatorname{tr}(AB)^m$ and $\operatorname{tr} A^m B^m$ are all real numbers and it holds that

$$\begin{aligned}
\operatorname{tr}(AB)^m &= \left(-\sqrt{-1}\right)^{4t+2} \operatorname{tr}\left(A\left(\sqrt{-1}B\right)\right)^{4t+2} \\
&= -\operatorname{tr}\left(A\left(\sqrt{-1}B\right)\right)^{2(2t+1)} \\
&\ge -\operatorname{tr}\left(A^2\left(\sqrt{-1}B\right)^2\right)^{2t+1} \\
&= -\sqrt{-1}^{4t+2} \operatorname{tr}\left(A^2\left(\sqrt{-1}B\right)^2\right)^{2t+1} \\
&= \operatorname{tr}\left(A^2B^2\right)^{m/2} \\
&\ge -\operatorname{tr} A^{2(2t+1)}\left(\sqrt{-1}B\right)^{2(2t+1)} \\
&= -\sqrt{-1}^{4t+2} \operatorname{tr} A^{4t+2}B^{4t+2} = \operatorname{tr} A^m B^m,
\end{aligned}$$

producing (4.2).

When $m = 4t + 1$ or $m = 4t + 3$, $t \in N$, the result is obtained by Lemma 2.5. □

Example 4.1. Let

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix} \in H(2),\ B = \begin{bmatrix} 2\sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{bmatrix} = \sqrt{-1}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \in S(2).$$

Hence

$$(AB)^3 = \sqrt{-1}\begin{bmatrix} 50 & 11 \\ -22 & -5 \end{bmatrix} \quad \text{and} \quad A^3B^3 = \sqrt{-1}\begin{bmatrix} 104 & 8 \\ -64 & -5 \end{bmatrix}.$$

It is known that both $\operatorname{tr}(AB)^3$ and $\operatorname{tr} A^3B^3$ are pure imaginary numbers by Lemma 2.5, they cannot be compared with each other, but their imaginary parts have the following relation

$$\operatorname{Im}\ \operatorname{tr}(AB)^3 = 45 < 99 = \operatorname{Im}\ \operatorname{tr} A^3B^3.$$

Let

$$C = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} \in H(2),$$

$$D = \sqrt{-1}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \in S(2).$$

$$(CD)^3 = \sqrt{-1}\begin{bmatrix} 15 & -42 \\ 70 & 29 \end{bmatrix},$$

$$C^3D^3 = \sqrt{-1}\begin{bmatrix} 15 & 30 \\ -64 & -35 \end{bmatrix},$$

and hence

$$\operatorname{Im}\ \operatorname{tr}(CD)^3 = 44 > -20 = \operatorname{Im}\ \operatorname{tr} C^3D^3.$$

Example 4.2. Let

$$A = \begin{bmatrix} 1 & -2 \\ -2 & -1 \end{bmatrix} \in H(2), \quad B = \sqrt{-1}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \in S(2).$$

Hence

$$(AB)^5 = \sqrt{-1}\begin{bmatrix} -435 & 543 \\ -905 & -616 \end{bmatrix} \quad \text{and} \quad A^5B^5 = \sqrt{-1}\begin{bmatrix} 525 & 375 \\ -1775 & -670 \end{bmatrix}.$$

Furthermore their imaginary parts satisfy:

$$Im\ \operatorname{tr}(AB)^5 = -1051\ <\ -145 = Im\ \operatorname{tr} A^5B^5.$$

Let $C = -A \in H(2)$, $D = B \in S(2)$, then

$$\operatorname{Im}\ \operatorname{tr}(CD)^5 = -\operatorname{Im}\ \operatorname{tr}(AB)^5 = 1051 > 145 = -\operatorname{Im}\ \operatorname{tr} A^5B^5 = \operatorname{Im}\ \operatorname{tr} C^5D^5.$$

From Example 4.1 and 4.2, it is known that when $m = 4t+1$ or $m = 4t+3$, $t \in N$, in general, the imaginary parts of $\operatorname{tr}(AB)^m$ and $\operatorname{tr} A^mB^m$ as pure imaginary numbers make the positive and reverse direction of the question (1.8) not to hold. It is seen from this that in Theorem 4.1, the question (1.8) that is proposed by [9] is completely resolved.

Theorem 4.2. *Let $A, B \in S(n)$, if $m = 2t$, $t \in N$, then both $\operatorname{tr}(AB)^m$ and $\operatorname{tr} A^mB^m$ are real numbers and*

$$\operatorname{tr}(AB)^m \le |\operatorname{tr}(AB)^m| \le \operatorname{tr}\left(A^2B^2\right)^{m/2} \le \operatorname{tr} A^mB^m, \tag{4.3}$$

holds.

Proof. By (2.5) and Lemma 2.6, it is known that both $\operatorname{tr}(AB)^m$ and $\operatorname{tr} A^mB^m$ are real numbers and

$$\begin{aligned} \operatorname{tr}(AB)^m &= \operatorname{tr}\left(\left(\sqrt{-1}A\right)\left(\sqrt{-1}B\right)\right)^m, \\ \operatorname{tr} A^mB^m &= \operatorname{tr}\left(\sqrt{-1}A\right)^m\left(\sqrt{-1}B\right)^m. \end{aligned} \tag{4.4}$$

Moreover by (3.3), it follows that

$$\begin{aligned}
\operatorname{tr}(AB)^m &= \operatorname{tr}\left(\left(\sqrt{-1}A\right)\left(\sqrt{-1}B\right)\right)^{2t} \\
&\le \left|\operatorname{tr}\left(\left(\sqrt{-1}A\right)\left(\sqrt{-1}B\right)\right)^{2t}\right| \\
&= |\operatorname{tr}(AB)^m| \\
&\le \operatorname{tr}\left(\left(\sqrt{-1}A\right)^2\left(\sqrt{-1}B\right)^2\right)^t \\
&= \operatorname{tr}\left(\sqrt{-1}^4 A^2B^2\right)^{m/2} \\
&= \operatorname{tr}\left(A^2B^2\right)^{m/2} \\
&\le \operatorname{tr}\left(\sqrt{-1}A\right)^{2t}\left(\sqrt{-1}B\right)^{2t} \\
&= \operatorname{tr}\left(\sqrt{-1}^{4t}A^{2t}B^{2t}\right) = \operatorname{tr} A^mB^m,
\end{aligned}$$

giving (4.3). □

Example 4.3. Let

$$A = \sqrt{-1}\begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}, \quad B = \sqrt{-1}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \in S(2),$$

and

$$(AB)^3 = -\begin{bmatrix} -51 & -11 \\ 22 & 5 \end{bmatrix}, \quad A^3B^3 = -\begin{bmatrix} -104 & -8 \\ 64 & 5 \end{bmatrix},$$

so that

$$\operatorname{tr}(AB)^3 = 45 < 99 = \operatorname{tr} A^3B^3.$$

Let

$$C = \sqrt{-1}\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}, \quad D = \sqrt{-1}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \in S(2),$$

according to

$$(CD)^3 = \begin{bmatrix} 15 & -42 \\ 70 & 29 \end{bmatrix}, \quad C^3D^3 = \begin{bmatrix} 15 & 30 \\ -130 & -35 \end{bmatrix},$$

it is known that

$$\operatorname{tr}(CD)^3 = 44 > -20 = \operatorname{tr} C^3D^3.$$

Via (2.9), we know that both $\operatorname{tr}(AB)^m$ and $\operatorname{tr} A^mB^m$ are real numbers when $m = 2t+1$, $t \in \mathbb{N}$ and $A, B \in S(n)$. Thereby in Example 4.3, it is indicated that the positive and reverse direction of the question (1.9) certainly do not hold, in which the question (1.9) is proposed by [9]; when m is a nonnegative even number, the question as posed by (1.9) is confirmed by Theorem 4.2, thus we completely resolve the question (1.9) of [9].

REFERENCES

[1] DA-WEI ZHANG, Note a matrix trace inequality for products of Hermitian matrices, *J. Math. Anal. Appl.*, **237** (1999), 721–725.

[2] XIAOJING YANG, Note a matrix trace inequality, *J. Math. Anal. Appl.*, **250** (2000), 372–374.

[3] XINMIN YANG, XIAOQI YANG AND KOK LAI TEO, A matrix trace inequality, *J. Math. Anal. Appl.*, **263** (2001), 327–333.

[4] F.M. DANNAN, Matrix and operator inequalities, *Ineq. Pure. and Appl. Math.*, **2**(3) (2001), Art. 34. [ONLINE: http://jipam.vu.edu.au/v2n3/029_01.html

[5] B. BELLMAN, Some inequalities for positive definite matrices, in *General Inequalities* 2, Proceeding 2^{nd} international conference on general inequalities (E. F. Beckenbach, Ed), Birkhausel, Basel, 1980, 89–90.

[6] P. J. BUSHELL AND G. B. TRUSTRUM, Trace inequalities for positive definite matrix power products, *Lin. Alg. Appl.*, **132** (1990), 172–178.

[7] BO-YING WANG AND MING-PENG GONG, Some eigenvalue inequalities for positive semidefinite matrix power products, *Lin. Alg. Appl.*, **184** (1993), 242–249.

[8] R.A. BRUALDI, From the Editior-in-Chief, *Lin. Alg. Appl.*, **220** (1995), 1–6.

[9] ZHENGMING JIAO, On the inequalities for traces of matrix product, *Journal of Shangdong Normal University* (Natural Science Edition), **7**(4) (1992), 19–20.

[10] FUZHEN ZHANG, *Matrix Theory: Basic and Techniques*, Springer-Verlag, New York, 1999.

[11] LING CHEN, Inequalities for singular values and trace, *Lin. Alg. Appl.*, **171** (1992), 109–120.

[12] SONG GUI WANG AND ZHONG ZHEN JIA, *The Inequality in Matrix Theory*, The Educational Publishing House of An'hui, Hefei, 1994.