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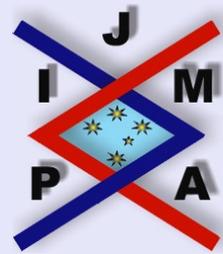
A NOTE ON SUMMABILITY FACTORS

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Abstract

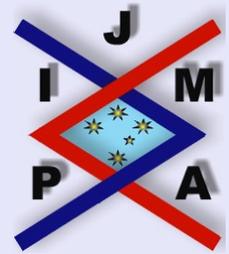
In this note we investigate the relation between two theorems proved by Bor [2, 3] on $|\bar{N}, p_n|_k$ summability of an infinite series.

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1. Introduction

Let $\sum a_n$ be a given infinite series with $\{s_n\}$ as the sequence of its n -th partial sums. Let $\{p_n\}$ be a sequence of positive constants such that $P_n = p_0 + p_1 + p_2 + \dots + p_n \rightarrow \infty$ as $n \rightarrow \infty$.

Let

$$t_n = \frac{1}{P_n} \sum_{\nu=1}^n p_\nu s_\nu.$$

The series $\sum a_n$ is said to be summable $|\bar{N}, p_n|$ if $\sum_1^\infty |t_n - t_{n-1}| < \infty$. It is said to be summable $|\bar{N}, p_n|_k$, $k \geq 1$ [1] if

$$(1.1) \quad \sum_1^\infty \left(\frac{P_n}{p_n}\right)^{k-1} |t_n - t_{n-1}|^k < \infty,$$

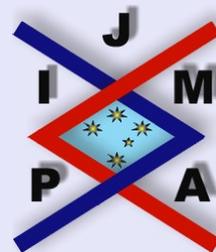
and bounded $[\bar{N}, p_n]_k$, $k \geq 1$ if

$$(1.2) \quad \sum_1^n p_\nu |s_\nu|^k = O(P_n), \quad n \rightarrow \infty.$$

Concerning $|\bar{N}, p_n|$ summability factors of $\sum a_n$, T. Singh [6] proved the following theorem:

Theorem A. *If the sequences $\{p_n\}$ and $\{\lambda_n\}$ satisfy the conditions*

$$(1.3) \quad \sum_1^\infty p_n |\lambda_n| < \infty,$$



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$$(1.4) \quad P_n |\Delta \lambda_n| \leq C p_n |\lambda_n|,$$

C is a constant, and if $\sum a_n$ is bounded $[\bar{N}, p_n]_1$, then $\sum a_n P_n \lambda_n$ is summable $[\bar{N}, p_n]$.

Earlier in 1968 N. Singh [5] had obtained the following theorem.

Theorem B. If $\sum a_n$ is bounded $[\bar{N}, p_n]_1$ and $\{\lambda_n\}$ is a sequence satisfying the following conditions

$$(1.5) \quad \sum_1^{\infty} \frac{p_n |\lambda_n|}{P_n} < \infty,$$

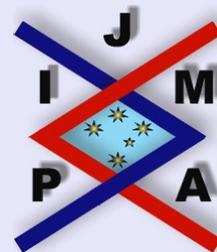
$$(1.6) \quad \frac{P_n}{p_n} \Delta \lambda_n = O(|\lambda_n|),$$

then $\sum a_n \lambda_n$ is summable $[\bar{N}, p_n]$.

In order to extend these theorems to the summability $|\bar{N}, p_n|_k$, $k \geq 1$, Bor [2, 3] proved the following theorems.

Theorem C. Under the conditions (1.2), (1.3) and (1.4), the series $\sum a_n P_n \lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

Theorem D. If $\sum a_n$ is bounded $[\bar{N}, p_n]_k$, $k \geq 1$ and $\{\lambda_n\}$, is a sequence satisfying the conditions (1.4) and (1.5), then $\sum a_n \lambda_n$ is summable $|\bar{N}, p_n|_k$.



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2. Results

In this note we propose to examine the relation between Theorem C and Theorem D.

We recall that recently Sarigol and Ozturk [4] constructed an example to demonstrate that the hypotheses of Theorem A are not sufficient for the summability $|\bar{N}, p_n|$ of $\sum a_n P_n \lambda_n$. They proved that Theorem A holds true if we assume the additional condition

$$(2.1) \quad p_{n+1} = O(p_n).$$

From (1.4) we find that

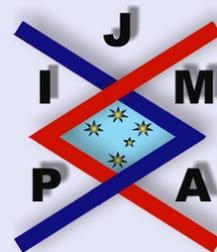
$$\left| \frac{\Delta \lambda_n}{\lambda_n} \right| = \left| 1 - \frac{\lambda_{n+1}}{\lambda_n} \right| \leq \frac{C p_n}{P_n},$$

Hence

$$\begin{aligned} \left| \frac{\lambda_{n+1}}{\lambda_n} \right| &= \left| \frac{\lambda_{n+1}}{\lambda_n} - 1 + 1 \right| \\ &\leq \left| 1 - \frac{\lambda_{n+1}}{\lambda_n} \right| + 1 \\ &\leq \frac{C p_n}{P_n} + 1 \leq C. \end{aligned}$$

Thus $|\lambda_{n+1}| \leq C|\lambda_n|$, and combining this with (2.1) we get

$$(2.2) \quad p_{n+1} |\lambda_{n+1}| \leq C p_n |\lambda_n|.$$



Clearly (2.1) and (1.4) imply (2.2). However (2.2) need not imply (2.1) or (1.4). In view of

$$\Delta(P_n \lambda_n) = P_n \Delta \lambda_n - p_{n+1} \lambda_{n+1}$$

it is clear that if (2.2) holds, then the condition (1.4) is equivalent to the condition

$$(2.3) \quad |\Delta(P_n \lambda_n)| \leq C p_n |\lambda_n|.$$

It can be easily verified that a corrected version of Theorem A and Theorem C and also a slight generalization of the result of Sarigol and Ozturk for $k = 1$ can be stated as

Theorem 2.1. *Under the conditions (1.2), (1.3) (2.2) and (2.3) the series $\sum a_n P_n \lambda_n$ is summable $|\bar{N}, p_n|_k, k \geq 1$*

We now proceed to show that Theorem 2.1 holds good without condition (2.2).

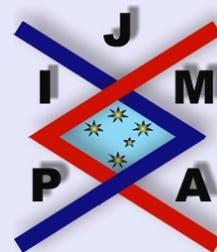
Thus we have:

Theorem 2.2. *Under the conditions (1.2), (1.3) and (2.3) the series $\sum a_n P_n \lambda_n$ is summable $|\bar{N}, p_n|_k, k \geq 1$.*

To prove Theorem 2.2 we first prove the following lemma.

Lemma 2.3. *Under the conditions of Theorem 2.2*

$$(2.4) \quad \sum_1^m p_n |\lambda_n| |s_n|^k = O(1) \text{ as } m \longrightarrow \infty.$$



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3. Proofs

Proof of Lemma 2.3. In view of (1.3) and (2.3)

$$\sum_1^{\infty} |\Delta(\lambda_n P_n)| \leq C \sum_1^{\infty} p_n |\lambda_n| < \infty,$$

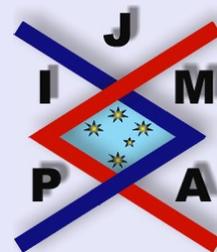
so it follows that $\{P_n \lambda_n\} \in BV$ and hence $P_n |\lambda_n| = O(1)$.

Now

$$\begin{aligned} \sum_1^m p_n \lambda_n ||s_n|^k &= \sum_1^{m-1} \Delta |\lambda_n| \sum_{\nu=1}^n p_\nu |s_\nu|^k + |\lambda_m| \sum_{\nu=1}^m p_\nu |s_\nu|^k \\ &= O(1) \sum_1^{m-1} |\Delta \lambda_n| P_n + O(|\lambda_m| P_m) \\ &= O(1) \left(\sum_1^{m-1} |\Delta(P_n \lambda_n)| + p_{n+1} |\lambda_{n+1}| \right) + O(1) \\ &= O(1) \sum_1^{m-1} p_n |\lambda_n| + O(1) \sum_1^m p_{n+1} |\lambda_{n+1}| + O(1) \\ &= O(1). \end{aligned}$$

□

Proof of Theorem 2.2. Let T_n denote the n^{th} (\bar{N}, p_n) means of the series $\sum a_n P_n \lambda_n$.



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Then

$$\begin{aligned} T_n &= \frac{1}{P_n} \sum_{\nu=0}^n p_\nu \sum_{r=0}^{\nu} a_r P_r \lambda_r \\ &= \frac{1}{P_n} \sum_{\nu=0}^n (P_n - P_{\nu-1}) a_\nu P_\nu \lambda_\nu. \end{aligned}$$

so that for $n \geq 1$

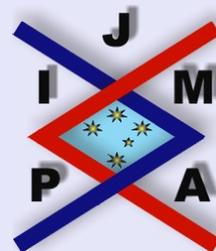
$$\begin{aligned} T_n - T_{n-1} &= \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^n P_{\nu-1} a_\nu P_\nu \lambda_\nu \\ &= \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} \Delta(P_{\nu-1} P_\nu \lambda_\nu) s_\nu + p_n \lambda_n s_n \\ &= L_1 + L_2, \text{ say.} \end{aligned}$$

Thus to prove the theorem it is sufficient to show that

$$\sum_1^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} |L_\nu|^k < \infty, \quad \nu = 1, 2.$$

Now

$$\begin{aligned} |\Delta(P_{\nu-1} P_\nu \lambda_\nu)| &\leq p_\nu P_\nu |\lambda_\nu| + P_\nu |\Delta(P_\nu \lambda_\nu)| \\ &\leq C p_\nu P_\nu |\lambda_\nu| \end{aligned}$$



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in view of (2.3). So

$$\begin{aligned}
 \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{k-1} |L_1|^k &= O(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}^k} \left(\sum_{\nu=1}^{n-1} p_\nu P_\nu |\lambda_\nu| |s_\nu| \right)^k \\
 &= O(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}^k} \left(\sum_{\nu=1}^{n-1} (P_\nu |\lambda_\nu|)^k |s_\nu|^k p_\nu \right) \left(\sum_{\nu=1}^{n-1} p_\nu \right)^{k-1} \\
 &= O(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}^k} \sum_{\nu=1}^{n-1} P_\nu |\lambda_\nu| p_\nu |s_\nu|^k \\
 &= O(1) \sum_{\nu=1}^m p_\nu |\lambda_\nu| |s_\nu|^k = O(1)
 \end{aligned}$$

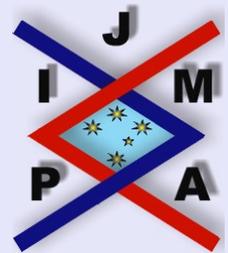
in view of the lemma and $P_n |\lambda_n| = O(1)$.

Also

$$\begin{aligned}
 \sum_1^{m+1} \left(\frac{P_n}{p_n} \right)^{k-1} |L_2|^k &= O(1) \sum_1^{m+1} p_n |\lambda_n|^k |s_n|^k P_n^{k-1} \\
 &= O(1) \sum_1^{m+1} p_n |\lambda_n| |s_n|^k \\
 &= O(1).
 \end{aligned}$$

This proves Theorem 2.2. □

Thus a generalization of a corrected version of Theorem C is Theorem 2.2.



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Writing $\lambda_n = \mu_n P_n$ the conditions (1.5) and (1.4) become

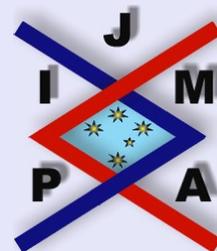
$$(3.1) \quad \sum_1^{\infty} p_n |\mu_n| < \infty,$$

$$(3.2) \quad |\Delta(P_n \mu_n)| \leq C p_n |\mu_n|,$$

consequently Theorem **D** can be stated as:

If $\sum a_n$ is bounded $[\bar{N}, P_n]_k$, $k \geq 1$ and $\{\mu_n\}$ is a sequence satisfying (3.1) and (3.2) then $\sum a_n P_n \mu_n$ is summable $[\bar{N}, p_n]_k$, $k \geq 1$.

Thus Theorem **D** is the same as Theorem 2.2 which is a generalization of the corrected version of Theorem **C**.



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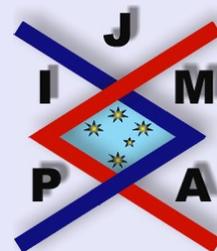
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