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SOME NEW INEQUALITIES SIMILAR TO HILBERT-PACHPATTE TYPE INEQUALITIES

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ABSTRACT. In this paper, some new inequalities similar to Hilbert-Pachpatte type inequalities are given.

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1. INTRODUCTION

In [1, Chap. 9], the well-known Hardy-Hilbert inequality is given as follows.

Theorem 1.1. *Let* $p > 1,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1, a_m, b_n \geq 0,\ 0 < \sum_{n=1}^{\infty} a_n^p < \infty,\ 0 < \sum_{n=1}^{\infty} b_n^q < \infty$. *Then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} \leq \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty} a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}} \tag{1.1}$$

where $\frac{\pi}{\sin(\pi/p)}$ *is best possible.*

The integral analogue of the Hardy-Hilbert inequality can be stated as follows

Theorem 1.2. *Let* $p > 1,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1,\ f(x), g(y) \geq 0,\ 0 < \int_0^{\infty} f^p(x)dx < \infty,\ 0 < \int_0^{\infty} g^q(y)dy < \infty$. *Then*

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}dxdy \leq \frac{\pi}{sin(\pi/p)}\left(\int_0^{\infty} f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^{\infty} g^q(y)dy\right)^{\frac{1}{q}}, \tag{1.2}$$

where $\frac{\pi}{\sin(\pi/p)}$ *is best possible.*

In [1, Chap. 9] the following extension of Hardy-Hilbert's double-series theorem is given.

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Theorem 1.3. *Let* $p > 1,\ q > 1,\ \frac{1}{p} + \frac{1}{q} \geq 1,\ 0 < \lambda = 2 - \frac{1}{p} - \frac{1}{q} = \frac{1}{p} + \frac{1}{q} \leq 1$. *Then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} \leq K\left(\sum_{m=1}^{\infty} a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}},$$

where $K = K(p,q)$ *depends on* p *and* q *only.*

The following integral analogue of Theorem 1.3 is also given in [1, Chap. 9].

Theorem 1.4. *Under the same conditions as in Theorem 1.1 we have*

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{(x+y)^{\lambda}}dxdy \leq K\left(\int_0^{\infty} f^p dx\right)^{\frac{1}{p}}\left(\int_0^{\infty} g^q dy\right)^{\frac{1}{q}},$$

where $K = K(p,q)$ *depends on* p *and* q *only.*

The inequalities in Theorems 1.1 and 1.2 were studied by Yang and Kuang (see [2, 3]). In [4, 5], some new inequalities similar to the inequalities given in Theorems 1.1, 1.2, 1.3 and 1.4 were established.

In this paper, we establish some new inequalities similar to the Hilbert-Pachpatte inequality.

2. MAIN RESULTS

In what follows we denote by $\mathbb{R}$ the set of real numbers. Let $\mathbb{N} = \{1, 2, \dots\}$,$\mathbb{N}_0 = \{0, 1, 2, \dots\}$. We define the operator ∇ by $\nabla u(t) = u(t) - u(t-1)$ for any function u defined on N. For any function $u(t) : [0,\infty) \to \mathbb{R}$, we denote by u' the derivatives of u.

First we introduce some Lemmas.

Lemma 2.1. *(see* [2]*). Let* $p > 1,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 2 - \min\{p,q\}$, *define the weight function* $\omega_1(q,x)$ *as*

$$\omega_1(q,x) := \int_0^{\infty}\frac{1}{(x+y)^{\lambda}}\left(\frac{x}{y}\right)^{\frac{2-\lambda}{q}} dy,\ x \in [0,\infty).$$

Then

$$\omega_1(q,x) = B\left(\frac{q+\lambda-2}{q}, \frac{p+\lambda-2}{p}\right)x^{1-\lambda}, \tag{2.1}$$

where $B(p,q)$ *is* β*-function.*

Lemma 2.2. *(see* [3]*). Let* $p > 1,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 2 - \min\{p,q\}$, *define the weight function* $\omega_2(q,x)$ *as*

$$\omega_2(q,x) := \int_0^{\infty}\frac{1}{x^{\lambda}+y^{\lambda}}\left(\frac{x}{y}\right)^{\frac{2-\lambda}{q}} dy,\ x \in [0,\infty).$$

Then

$$\omega_2(q,x) = \frac{1}{\lambda}B\left(\frac{q+\lambda-2}{q\lambda}, \frac{p+\lambda-2}{p\lambda}\right)x^{1-\lambda}. \tag{2.2}$$

Lemma 2.3. *Let* $p > 1,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 2 - \min\{p,q\}$, *define the weight function* $\omega_3(q,m)$ *as*

$$\omega_3(q,m) := \sum_{n=1}^{\infty}\frac{1}{(m+n)^{\lambda}}\left(\frac{m}{n}\right)^{\frac{2-\lambda}{q}},\ m \in \{1, 2, \dots\}.$$

Then

$$\omega_3(q,m) < B\left(\frac{q+\lambda-2}{q}, \frac{p+\lambda-2}{p}\right) m^{1-\lambda}, \tag{2.3}$$

where $B(p,q)$ is β-function.

Proof. By Lemma 2.1,we have

$$\omega_3(q,m) < \int_0^\infty \frac{1}{(m+y)^\lambda}\left(\frac{m}{y}\right)^{\frac{2-\lambda}{q}} dy = B\left(\frac{q+\lambda-2}{q}, \frac{p+\lambda-2}{p}\right) m^{1-\lambda}.$$

The proof is completed. □

Lemma 2.4. *Let $p>1,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1,\ \lambda > 2-\min\{p,q\}$, define the weight function $\omega_4(q,m)$ as*

$$\omega_4(q,m) := \sum_{n=1}^{\infty} \frac{1}{m^\lambda+n^\lambda}\left(\frac{m}{n}\right)^{\frac{2-\lambda}{q}},\ m\in\{1,2,\dots\}.$$

Then

$$\omega_4(q,m) < \frac{1}{\lambda} B\left(\frac{q+\lambda-2}{q\lambda}, \frac{p+\lambda-2}{p\lambda}\right) m^{1-\lambda}. \tag{2.4}$$

Proof. By Lemma 2.2, we have

$$\omega_4(q,m) < \int_0^\infty \frac{1}{m^\lambda+y^\lambda}\left(\frac{m}{y}\right)^{\frac{2-\lambda}{q}} dy = \frac{1}{\lambda} B\left(\frac{q+\lambda-2}{q\lambda}, \frac{p+\lambda-2}{p\lambda}\right) m^{1-\lambda}.$$

The proof is completed. □

Our main result is given in the following theorem.

Theorem 2.5. *Let $p>1,\ \frac{1}{p}+\frac{1}{q}=1$, and $f(x), g(y)$ be real-valued continuous functions defined on $[0,\infty)$, respectively, and let $f(0)=g(0)=0$, and*

$$0 < \int_0^\infty \int_0^x |f'(\tau)|^p \, d\tau dx < \infty,\ 0 < \int_0^\infty \int_0^y |g'(\delta)|^q \, d\delta dy < \infty.$$

Then

$$\int_0^\infty \int_0^\infty \frac{|f(x)|\,|g(y)|}{(qx^{p-1}+py^{q-1})(x+y)} dxdy \le \frac{\pi}{\sin(\pi/p)pq}\left(\int_0^\infty \int_0^x |f'(\tau)|^p \, d\tau dx\right)^{\frac{1}{p}} \left(\int_0^\infty \int_0^y |g'(\delta)|^q \, d\delta dy\right)^{\frac{1}{q}}. \tag{2.5}$$

In particular, when $p=q=2$, we have

$$\int_0^\infty \int_0^\infty \frac{|f(x)|\,|g(y)|}{(x+y)^2} dxdy \le \frac{\pi}{2}\left(\int_0^\infty \int_0^x |f'(\tau)|^2 \, d\tau dx\right)^{\frac{1}{2}} \left(\int_0^\infty \int_0^y |g'(\delta)|^2 \, d\delta dy\right)^{\frac{1}{2}}. \tag{2.6}$$

Proof. From the hypotheses, we have the following identities

$$f(x) = \int_0^x f'(\tau) d\tau, \tag{2.7}$$

and

$$g(y) = \int_0^y g'(\delta) d\delta \tag{2.8}$$

for $x, y \in (0, \infty)$. From (2.7) and (2.8) and using Hölder's integral inequality, respectively, we have

$$|f(x)| \leq x^{\frac{1}{q}} \left(\int_0^x |f'(\tau)|^p \, d\tau \right)^{\frac{1}{p}} \tag{2.9}$$

and

$$|g(y)| \leq y^{\frac{1}{p}} \left(\int_0^y |g'(\delta)|^q \, d\delta \right)^{\frac{1}{q}} \tag{2.10}$$

for $x, y \in (0, \infty)$. From (2.9) and (2.10) and using the elementary inequality

$$z_1 z_2 \leq \frac{z_1^p}{p} + \frac{z_2^q}{q}, \; z_1 \geq 0, \; z_2 \geq 0, \; \frac{1}{p} + \frac{1}{q} = 1, \; p > 1, \tag{2.11}$$

we observe that

$$\begin{aligned} |f(x)| \, |g(y)| &\leq x^{\frac{1}{q}} y^{\frac{1}{p}} \left(\int_0^x |f'(\tau)|^p \, d\tau \right)^{\frac{1}{p}} \left(\int_0^y |g'(\delta)|^q \, d\delta \right)^{\frac{1}{q}} \\ &\leq \left(\frac{x^{p-1}}{p} + \frac{y^{q-1}}{q} \right) \left(\int_0^x |f'(\tau)|^p \, d\tau \right)^{\frac{1}{p}} \left(\int_0^y |g'(\delta)|^q \, d\delta \right)^{\frac{1}{q}} \end{aligned} \tag{2.12}$$

for $x, y \in (0, \infty)$. From (2.12) we observe that

$$\frac{|f(x)| \, |g(y)|}{qx^{p-1} + py^{q-1}} \leq \frac{1}{pq} \left(\int_0^x |f'(\tau)|^p \, d\tau \right)^{\frac{1}{p}} \left(\int_0^y |g'(\delta)|^q \, d\delta \right)^{\frac{1}{q}}. \tag{2.13}$$

Hence

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{|f(x)| \, |g(y)|}{(qx^{p-1} + py^{q-1})(x+y)} dx dy \\ &\qquad \leq \frac{1}{pq} \int_0^\infty \int_0^\infty \frac{\left(\int_0^x |f'(\tau)|^p \, d\tau \right)^{\frac{1}{p}} \left(\int_0^y |g'(\delta)|^q \, d\delta \right)^{\frac{1}{q}}}{x+y} dx dy. \end{aligned} \tag{2.14}$$

By Hölder's integral inequality and (2.1), we have

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{\left(\int_0^x |f'(\tau)|^p \, d\tau \right)^{\frac{1}{p}} \left(\int_0^y |g'(\delta)|^q \, d\delta \right)^{\frac{1}{q}}}{x+y} dx dy \\ &\quad = \int_0^\infty \int_0^\infty \frac{\left(\int_0^x |f'(\tau)|^p \, d\tau \right)^{\frac{1}{p}}}{(x+y)^{\frac{1}{p}}} \left(\frac{x}{y} \right)^{\frac{1}{pq}} \frac{\left(\int_0^y |g'(\delta)|^q \, d\delta \right)^{\frac{1}{q}}}{(x+y)^{\frac{1}{q}}} \left(\frac{y}{x} \right)^{\frac{1}{pq}} dx dy \\ &\quad \leq \left(\int_0^\infty \int_0^\infty \frac{\int_0^x |f'(\tau)|^p \, d\tau}{x+y} \left(\frac{x}{y} \right)^{\frac{1}{q}} dx dy \right)^{\frac{1}{p}} \\ &\qquad \times \left(\int_0^\infty \int_0^\infty \frac{\int_0^y |g'(\delta)|^q \, d\delta}{x+y} \left(\frac{y}{x} \right)^{\frac{1}{p}} dx dy \right)^{\frac{1}{q}} \\ &\quad \leq \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty \int_0^x |f'(\tau)|^p \, d\tau dx \right)^{\frac{1}{p}} \left(\int_0^\infty \int_0^y |g'(\delta)|^q \, d\delta dy \right)^{\frac{1}{q}} \end{aligned} \tag{2.15}$$

by (2.14) and (2.15), we get (2.5). The proof of Theorem 2.5 is complete. □

In a similar way to the proof of Theorem 2.5, we can prove the following theorems.

Theorem 2.6. *Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 2 - \min\{p, q\}$, and $f(x), g(y)$ be real-valued continuous functions defined on $[0, \infty)$,respectively, and let $f(0) = g(0) = 0$, and*

$$0 < \int_0^\infty \int_0^x x^{1-\lambda} \left|f'(\tau)\right|^p d\tau dx < \infty, \quad 0 < \int_0^\infty \int_0^y y^{1-\lambda} \left|g'(\delta)\right|^q d\delta dy < \infty,$$

then

$$\int_0^\infty \int_0^\infty \frac{|f(x)|\,|g(y)|}{(qx^{p-1} + py^{q-1})(x+y)^\lambda} dxdy \leq \frac{B\left(\frac{q+\lambda-2}{q}, \frac{p+\lambda-2}{p}\right)}{pq} \left(\int_0^\infty \int_0^x x^{1-\lambda} \left|f'(\tau)\right|^p d\tau dx\right)^{\frac{1}{p}} \times \left(\int_0^\infty \int_0^y y^{1-\lambda} \left|g'(\delta)\right|^q d\delta dy\right)^{\frac{1}{q}}. \tag{2.16}$$

In particular, when $p = q = 2$,

$$\int_0^\infty \int_0^\infty \frac{|f(x)|\,|g(y)|}{(x+y)^{1+\lambda}} dxdy \leq \frac{B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)}{2} \left(\int_0^\infty \int_0^x x^{1-\lambda} \left|f'(\tau)\right|^2 d\tau dx\right)^{\frac{1}{2}} \left(\int_0^\infty \int_0^y y^{1-\lambda} \left|g'(\delta)\right|^2 d\delta dy\right)^{\frac{1}{2}}. \tag{2.17}$$

Theorem 2.7. *Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 2 - \min\{p, q\}$, and $f(x), g(y)$ be real-valued continuous functions defined on $[0, \infty)$, respectively, and let $f(0) = g(0) = 0$, and*

$$0 < \int_0^\infty \int_0^x x^{1-\lambda} \left|f'(\tau)\right|^p d\tau dx < \infty, \quad 0 < \int_0^\infty \int_0^y y^{1-\lambda} \left|g'(\delta)\right|^q d\delta dy < \infty.$$

Then

$$\int_0^\infty \int_0^\infty \frac{|f(x)|\,|g(y)|}{(qx^{p-1} + py^{q-1})(x^\lambda + y^\lambda)} dxdy \leq \frac{B\left(\frac{q+\lambda-2}{q\lambda}, \frac{p+\lambda-2}{p\lambda}\right)}{\lambda pq} \left(\int_0^\infty \int_0^x x^{1-\lambda} \left|f'(\tau)\right|^p d\tau dx\right)^{\frac{1}{p}} \times \left(\int_0^\infty \int_0^y y^{1-\lambda} \left|g'(\delta)\right|^q d\delta dy\right)^{\frac{1}{q}}. \tag{2.18}$$

In particular, when $p = q = 2$,

$$\int_0^\infty \int_0^\infty \frac{|f(x)|\,|g(y)|}{(x^\lambda + y^\lambda)(x+y)} dxdy \leq \frac{\pi}{2\lambda} \left(\int_0^\infty \int_0^x x^{1-\lambda} \left|f'(\tau)\right|^2 d\tau dx\right)^{\frac{1}{2}} \left(\int_0^\infty \int_0^y y^{1-\lambda} \left|g'(\delta)\right|^2 d\delta dy\right)^{\frac{1}{2}}. \tag{2.19}$$

3. DISCRETE ANALOGUES

Theorem 3.1. *Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and $\{a(m)\}$ and $\{b(n)\}$ be two sequences of real numbers where $m, n \in \mathbb{N}_0$, and $a(0) = b(0) = 0$, and $0 < \sum_{m=1}^{\infty} \sum_{\tau=1}^{m} |\nabla a(\tau)|^p < \infty$, $0 < \sum_{n=1}^{\infty} \sum_{\delta=1}^{n} |\nabla b(\delta)|^q < \infty$, then*

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{|a_m|\,|b_n|}{(qm^{p-1} + pn^{q-1})(m+n)} \leq \frac{\pi}{\sin(\pi/p)pq} \left(\sum_{m=1}^{\infty} \sum_{k=1}^{m} a_k^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} \sum_{r=1}^{n} b_r^q \right)^{\frac{1}{q}}. \tag{3.1}$$

In particular, when $p = q = 2$, we have

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{|a_m|\,|b_n|}{(m+n)^2} \leq \frac{\pi}{2} \left(\sum_{m=1}^{\infty} \sum_{k=1}^{m} a_k^2 \right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} \sum_{r=1}^{n} b_r^2 \right)^{\frac{1}{2}}. \tag{3.2}$$

Proof. From the hypotheses, it is easy to observe that the following identities hold

$$a_m = \sum_{\tau=1}^{m} \nabla a(\tau), \tag{3.3}$$

and

$$b_n = \sum_{\delta=1}^{n} \nabla b(\delta) \tag{3.4}$$

for $m, n \in \mathbb{N}$. From (3.3) and (3.4) and using Hölder's inequality, we have

$$|a_m| \leq m^{\frac{1}{q}} \left(\sum_{\tau=1}^{m} |\nabla a(\tau)|^p \right)^{\frac{1}{p}}, \tag{3.5}$$

and

$$|b_n| \leq n^{\frac{1}{p}} \left(\sum_{\delta=1}^{n} |\nabla b(\delta)|^q \right)^{\frac{1}{q}} \tag{3.6}$$

for $m, n \in \mathbb{N}$. From (3.5) and (3.6) and using the elementary inequality (2.11), we observe that

$$\begin{aligned} |a_m|\,|b_n| &\leq m^{\frac{1}{q}} n^{\frac{1}{p}} \left(\sum_{\tau=1}^{m} |\nabla a(\tau)|^p \right)^{\frac{1}{p}} \left(\sum_{\delta=1}^{n} |\nabla b(\delta)|^q \right)^{\frac{1}{q}} \\ &\leq \left(\frac{m^{p-1}}{p} + \frac{n^{q-1}}{q} \right) \left(\sum_{\tau=1}^{m} |\nabla a(\tau)|^p \right)^{\frac{1}{p}} \left(\sum_{\delta=1}^{n} |\nabla b(\delta)|^q \right)^{\frac{1}{q}} \end{aligned} \tag{3.7}$$

for $m, n \in \mathbb{N}$. From (3.7), we observe that

$$\frac{|a_m|\,|b_n|}{qm^{p-1} + pn^{q-1}} \leq \frac{1}{pq} \left(\sum_{\tau=1}^{m} |\nabla a(\tau)|^p \right)^{\frac{1}{p}} \left(\sum_{\delta=1}^{n} |\nabla b(\delta)|^q \right)^{\frac{1}{q}}. \tag{3.8}$$

Hence

$$\begin{aligned} &\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{|a_m|\,|b_n|}{(qm^{p-1} + pn^{q-1})(m+n)} \\ &\qquad \leq \frac{1}{pq} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\left(\sum_{\tau=1}^{m} |\nabla a(\tau)|^p\right)^{\frac{1}{p}} \left(\sum_{\delta=1}^{n} |\nabla b(\delta)|^q\right)^{\frac{1}{q}}}{m+n}. \end{aligned} \tag{3.9}$$

By the Hölder inequality and (2.3)

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\left(\sum_{\tau=1}^{m}|\nabla a(\tau)|^{p}\right)^{\frac{1}{p}}\left(\sum_{\delta=1}^{n}|\nabla b(\delta)|^{q}\right)^{\frac{1}{q}}}{m+n}$$

$$=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\left(\sum_{\tau=1}^{m}|\nabla a(\tau)|^{p}\right)^{\frac{1}{p}}}{(m+n)^{\frac{1}{p}}}\left(\frac{m}{n}\right)^{\frac{1}{pq}}\frac{\left(\sum_{\delta=1}^{n}|\nabla b(\delta)|^{q}\right)^{\frac{1}{q}}}{(m+n)^{\frac{1}{q}}}\left(\frac{n}{m}\right)^{\frac{1}{pq}}$$

$$\leq\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sum_{\tau=1}^{m}|\nabla a(\tau)|^{p}}{m+n}\left(\frac{m}{n}\right)^{\frac{1}{q}}\right)^{\frac{1}{p}}\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sum_{\delta=1}^{n}|\nabla b(\delta)|^{q}}{m+n}\left(\frac{n}{m}\right)^{\frac{1}{p}}\right)^{\frac{1}{q}}$$

$$<\frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty}\sum_{\tau=1}^{m}|\nabla a(\tau)|^{p}\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\sum_{\delta=1}^{n}|\nabla b(\delta)|^{q}\right)^{\frac{1}{q}} \tag{3.10}$$

by (3.9) and (3.10), we get (3.1). The proof of Theorem 3.1 is complete. □

In a similar manner to the proof of Theorem 3.1, we can prove the following theorems.

Theorem 3.2. *Let* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\lambda>2-\min\{p,q\}$, *and* $\{a(m)\}$ *and* $\{b(n)\}$ *be two sequences of real numbers where* $m,n\in\mathbb{N}_0$,*and* $a(0)=b(0)=0$, *and*

$$0<\sum_{m=1}^{\infty}\sum_{\tau=1}^{m}m^{1-\lambda}|\nabla a(\tau)|^{p}<\infty,$$

$$0<\sum_{n=1}^{\infty}\sum_{\delta=1}^{n}n^{1-\lambda}|\nabla b(\delta)|^{q}<\infty,$$

then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{|a_m||b_n|}{(qm^{p-1}+pn^{q-1})(m+n)^{\lambda}} \leq\frac{B\left(\frac{q+\lambda-2}{q},\frac{p+\lambda-2}{p}\right)}{pq}\left(\sum_{m=1}^{\infty}\sum_{\tau=1}^{m}m^{1-\lambda}|\nabla a(\tau)|^{p}\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\sum_{\delta=1}^{n}n^{1-\lambda}|\nabla b(\delta)|^{q}\right)^{\frac{1}{q}}. \tag{3.11}$$

In particular, when $p=q=2$, *we have*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{|a_m||b_n|}{(m+n)^{1+\lambda}} \leq\frac{B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)}{2}\left(\sum_{m=1}^{\infty}\sum_{\tau=1}^{m}m^{1-\lambda}|\nabla a(\tau)|^{2}\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}\sum_{\delta=1}^{n}n^{1-\lambda}|\nabla b(\delta)|^{2}\right)^{\frac{1}{2}}. \tag{3.12}$$

Theorem 3.3. *Let* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\lambda>2-\min\{p,q\}$, *and* $\{a(m)\}$ *and* $\{b(n)\}$ *be two sequences of real numbers where* $m,n\in\mathbb{N}_0$, *and* $a(0)=b(0)=0$, *and*

$$0<\sum_{m=1}^{\infty}\sum_{\tau=1}^{m}m^{1-\lambda}|\nabla a(\tau)|^{p}<\infty,$$

$$0<\sum_{n=1}^{\infty}\sum_{\delta=1}^{n}n^{1-\lambda}|\nabla b(\delta)|^{q}<\infty,$$

then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{|a_m|\,|b_n|}{(qm^{p-1}+pn^{q-1})(m^\lambda+n^\lambda)} \le \frac{B\left(\frac{q+\lambda-2}{q\lambda},\frac{p+\lambda-2}{p\lambda}\right)}{\lambda pq}\left(\sum_{m=1}^{\infty}\sum_{\tau=1}^{m}m^{1-\lambda}\left|\nabla a(\tau)\right|^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\sum_{\delta=1}^{n}n^{1-\lambda}\left|\nabla b(\delta)\right|^q\right)^{\frac{1}{q}}. \tag{3.13}$$

In particular, when $p=q=2$*, we have*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{|a_m|\,|b_n|}{(m+n)(m^\lambda+n^\lambda)} \le \frac{\pi}{2\lambda}\left(\sum_{m=1}^{\infty}\sum_{\tau=1}^{m}m^{1-\lambda}\left|\nabla a(\tau)\right|^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}\sum_{\delta=1}^{n}n^{1-\lambda}\left|\nabla b(\delta)\right|^2\right)^{\frac{1}{2}}. \tag{3.14}$$

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