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ON WEIGHTED OSTROWSKI TYPE INEQUALITIES FOR OPERATORS AND VECTOR-VALUED FUNCTIONS

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ABSTRACT. Some weighted Ostrowski type integral inequalities for operators and vector-valued functions in Banach spaces are given. Applications for linear operators in Banach spaces and differential equations are also provided.

Key words and phrases: Ostrowski's Inequality, Vector-Valued Functions, Bochner Integral.

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1. INTRODUCTION

In [12], Pečarić and Savić obtained the following Ostrowski type inequality for weighted integrals (see also [7, Theorem 3]):

Theorem 1.1. *Let* $w : [a,b] \to [0,\infty)$ *be a weight function on* $[a,b]$. *Suppose that* $f : [a,b] \to \mathbb{R}$ *satisfies*

$$|f(t) - f(s)| \leq N |t-s|^{\alpha}, \text{ for all } t,s \in [a,b], \tag{1.1}$$

where $N > 0$ *and* $0 < \alpha \leq 1$ *are some constants. Then for any* $x \in [a,b]$

$$\left| f(x) - \frac{\int_a^b w(t) f(t)\, dt}{\int_a^b w(t)\, dt} \right| \leq N \cdot \frac{\int_a^b |t-x|^{\alpha} w(t)\, dt}{\int_a^b w(t)\, dt}. \tag{1.2}$$

Further, if for some constants c *and* λ

$$0 < c \leq w(t) \leq \lambda c, \text{ for all } t \in [a,b],$$

then for any $x \in [a,b]$, *we have*

$$\left| f(x) - \frac{\int_a^b w(t) f(t)\, dt}{\int_a^b w(t)\, dt} \right| \leq N \cdot \frac{\lambda L(x) J(x)}{L(x) - J(x) + \lambda J(x)}, \tag{1.3}$$

where

$$L(x) := \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right]^{\alpha}$$

and

$$J(x) := \frac{(x-a)^{1+\alpha} + (b-x)^{1+\alpha}}{(1+\alpha)(b-a)}.$$

The inequality (1.2) was rediscovered in [4] where further applications for different weights and in Numerical Analysis were given.

For other results in connection to weighted Ostrowski inequalities, see [3], [8] and [10].

In the present paper we extend the weighted Ostrowski's inequality for vector-valued functions and Bochner integrals and apply the obtained results to operatorial inequalities and linear differential equations in Banach spaces. Some numerical experiments are also conducted.

2. WEIGHTED INEQUALITIES

Let X be a Banach space and $-\infty < a < b < \infty$. We denote by $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators acting on X. The norms of vectors or operators acting on X will be denoted by $\|\cdot\|$.

A function $f : [a,b] \to X$ is called *measurable* if there exists a sequence of simple functions $f_n : [a,b] \to X$ which converges punctually almost everywhere on $[a,b]$ at f. We recall also that a measurable function $f : [a,b] \to X$ is *Bochner integrable* if and only if its norm function (i.e. the function $t \mapsto \|f(t)\| : [a,b] \to \mathbb{R}_+$) is Lebesgue integrable on $[a,b]$.

The following theorem holds.

Theorem 2.1. *Assume that* $B : [a,b] \to \mathcal{L}(X)$ *is Hölder continuous on* $[a,b]$, *i.e.,*

$$\|B(t) - B(s)\| \leq H |t-s|^{\alpha} \text{ for all } t,s \in [a,b], \tag{2.1}$$

where $H > 0$ *and* $\alpha \in (0,1]$.

If $f:[a,b]\to X$ is Bochner integrable on $[a,b]$, then we have the inequality:

$$\left\|B(t)\int_a^b f(s)\,ds-\int_a^b B(s)f(s)\,ds\right\| \tag{2.2}$$
$$\le H\int_a^b |t-s|^{\alpha}\|f(s)\|\,ds$$
$$\le H\times\begin{cases}\dfrac{(b-t)^{\alpha+1}+(t-a)^{\alpha+1}}{\alpha+1}\,|||f|||_{[a,b],\infty} & \textit{if}\quad f\in L_\infty([a,b];X);\\[2ex] \left[\dfrac{(b-t)^{q\alpha+1}+(t-a)^{q\alpha+1}}{q\alpha+1}\right]^{\frac{1}{q}}|||f|||_{[a,b],p} & \textit{if}\quad p>1,\ \frac{1}{p}+\frac{1}{q}=1\\ & \textit{and}\quad f\in L_p([a,b];X);\\[2ex] \left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]^{\alpha}|||f|||_{[a,b],1}\end{cases}$$

for any $t\in[a,b]$, where

$$|||f|||_{[a,b],\infty}:=ess\sup_{t\in[a,b]}\|f(t)\|$$

and

$$|||f|||_{[a,b],p}:=\left(\int_a^b\|f(t)\|^p\,dt\right)^{\frac{1}{p}},\quad p\ge 1.$$

Proof. Firstly, we prove that the $X-$valued function $s\mapsto B(s)f(s)$ is Bochner integrable on $[a,b]$. Indeed, let (f_n) be a sequence of $X-$valued, simple functions which converge almost everywhere on $[a,b]$ at the function f. The maps $s\mapsto B(s)f_n(s)$ are measurable (because they are continuous with the exception of a finite number of points s in $[a,b]$). Then

$$\|B(s)f_n(s)-B(s)f(s)\|\le\|B(s)\|\,\|f_n(s)-f(s)\|\to 0\ \text{ a.e. on }\ [a,b]$$

when $n\to\infty$ so that the function $s\mapsto B(s)f(s):[a,b]\to X$ is measurable. Now, using the estimate

$$\|B(s)f(s)\|\le\sup_{\xi\in[a,b]}\|B(\xi)\|\cdot\|f(s)\|,\quad\text{for all } s\in[a,b],$$

it is easy to see that the function $s\mapsto B(s)f(s)$ is Bochner integrable on $[a,b]$.

We have successively

$$\begin{aligned}\left\|B(t)\int_a^b f(s)\,ds-\int_a^b B(s)f(s)\,ds\right\| &= \left\|\int_a^b (B(t)-B(s))f(s)\,ds\right\|\\ &\le \int_a^b\|(B(t)-B(s))f(s)\|\,ds\\ &\le \int_a^b\|(B(t)-B(s))\|\,\|f(s)\|\,ds\\ &\le H\int_a^b|t-s|^{\alpha}\|f(s)\|\,ds\\ &=: M(t)\end{aligned}$$

for any $t\in[a,b]$, proving the first inequality in (2.2).

Now, observe that

$$\begin{aligned} M(t) &\leq H\,|||f|||_{[a,b],\infty}\int_a^b |t-s|^\alpha\,ds \\ &= H\,|||f|||_{[a,b],\infty}\cdot\frac{(b-t)^{\alpha+1}+(t-a)^{\alpha+1}}{\alpha+1} \end{aligned}$$

and the first part of the second inequality is proved.

Using Hölder's integral inequality, we may state that

$$\begin{aligned} M(t) &\leq H\left(\int_a^b |t-s|^{q\alpha}\,ds\right)^{\frac{1}{q}}\left(\int_a^b \|f(s)\|^p\,ds\right)^{\frac{1}{p}} \\ &= H\left[\frac{(b-t)^{q\alpha+1}+(t-a)^{q\alpha+1}}{q\alpha+1}\right]^{\frac{1}{q}}|||f|||_{[a,b],p}\,, \end{aligned}$$

proving the second part of the second inequality.

Finally, we observe that

$$\begin{aligned} M(t) &\leq H\sup_{s\in[a,b]}|t-s|^\alpha\int_a^b\|f(s)\|\,ds \\ &= H\max\left\{(b-t)^\alpha,(t-a)^\alpha\right\}|||f|||_{[a,b],1} \\ &= H\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]^\alpha|||f|||_{[a,b],1} \end{aligned}$$

and the theorem is proved. □

The following corollary holds.

Corollary 2.2. *Assume that* $B:[a,b]\to\mathcal{L}(X)$ *is Lipschitzian with the constant* $L>0$. *Then we have the inequality*

$$\left\|B(t)\int_a^b f(s)\,ds-\int_a^b B(s)f(s)\,ds\right\| \tag{2.3}$$

$$\leq L\int_a^b|t-s|\,\|f(s)\|\,ds$$

$$\leq L\times\begin{cases}\left[\frac{1}{4}(b-a)^2+\left(t-\frac{a+b}{2}\right)^2\right]|||f|||_{[a,b],\infty} & \text{if } f\in L_\infty([a,b];X);\\ \left[\frac{(b-t)^{q+1}+(t-a)^{q+1}}{q+1}\right]^{\frac{1}{q}}|||f|||_{[a,b],p} & \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ & \text{and } f\in L_p([a,b];X);\\ \left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]|||f|||_{[a,b],1}\end{cases}$$

for any $t\in[a,b]$.

Remark 2.3. If we choose $t = \frac{a+b}{2}$ in (2.2) and (2.3), then we get the following midpoint inequalities:

$$\begin{aligned} &\left\| B\left(\frac{a+b}{2}\right) \int_a^b f(s)\,ds - \int_a^b B(s) f(s)\,ds \right\| \\ &\leq H \int_a^b \left| s - \frac{a+b}{2} \right|^\alpha \|f(s)\|\,ds \\ &\leq H \times \begin{cases} \frac{1}{2^\alpha(\alpha+1)} (b-a)^{\alpha+1} \||f|\|_{[a,b],\infty} & \text{if} \quad f \in L_\infty([a,b];X); \\ \frac{1}{2^\alpha(q\alpha+1)^{\frac{1}{q}}} (b-a)^{\alpha+\frac{1}{q}} \||f|\|_{[a,b],p} & \text{if} \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ & \text{and} \quad f \in L_p([a,b];X); \\ \frac{1}{2^\alpha} (b-a)^\alpha \||f|\|_{[a,b],1} \end{cases} \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} &\left\| B\left(\frac{a+b}{2}\right) \int_a^b f(s)\,ds - \int_a^b B(s) f(s)\,ds \right\| \\ &\leq L \int_a^b \left| s - \frac{a+b}{2} \right| \|f(s)\|\,ds \\ &\leq L \times \begin{cases} \frac{1}{4} (b-a)^2 \||f|\|_{[a,b],\infty} & \text{if} \quad f \in L_\infty([a,b];X); \\ \frac{1}{2(q+1)^{\frac{1}{q}}} (b-a)^{1+\frac{1}{q}} \||f|\|_{[a,b],p} & \text{if} \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ & \text{and} \quad f \in L_p([a,b];X); \\ \frac{1}{2} (b-a) \||f|\|_{[a,b],1} \end{cases} \end{aligned} \tag{2.5}$$

respectively.

Remark 2.4. Consider the function $\Psi_\alpha : [a,b] \to \mathbb{R}$, $\Psi_\alpha(t) := \int_a^b |t-s|^\alpha \|f(s)\|\,ds$, $\alpha \in (0,1)$. If f is continuous on $[a,b]$, then Ψ_α is differentiable and

$$\begin{aligned} \frac{d\Psi_\alpha(t)}{dt} &= \frac{d}{dt}\left[\int_a^t (t-s)^\alpha \|f(s)\|\,ds + \int_t^b (s-t)^\alpha \|f(s)\|\,ds\right] \\ &= \alpha\left[\int_a^t \frac{\|f(s)\|}{(t-s)^{1-\alpha}}ds - \int_t^b \frac{\|f(s)\|}{(s-t)^{1-\alpha}}ds\right]. \end{aligned}$$

If $t_0 \in (a,b)$ is such that

$$\int_a^{t_0} \frac{\|f(s)\|}{(t_0-s)^{1-\alpha}}ds = \int_{t_0}^b \frac{\|f(s)\|}{(s-t_0)^{1-\alpha}}ds$$

and $\Psi'(\cdot)$ is negative on (a,t_0) and positive on (t_0,b) , then the best inequality we can get in the first part of (2.2) is the following one

$$\left\| B(t_0) \int_a^b f(s)\,ds - \int_a^b B(s) f(s)\,ds \right\| \leq H \int_a^b |t_0 - s|^\alpha \|f(s)\|\,ds. \tag{2.6}$$

If $\alpha = 1$, then, for

$$\Psi(t) := \int_a^b |t-s| \|f(s)\|\,ds,$$

we have

$$\frac{d\Psi(t)}{dt} = \int_a^t \|f(s)\|\,ds - \int_t^b \|f(s)\|\,ds, \quad t \in (a,b),$$
$$\frac{d^2\Psi(t)}{dt^2} = 2\|f(t)\| \geq 0, \quad t \in (a,b),$$

which shows that Ψ is convex on (a,b).

If $t_m \in (a,b)$ is such that

$$\int_a^{t_m} \|f(s)\|\,ds = \int_{t_m}^b \|f(s)\|\,ds,$$

then the best inequality we can get from the first part of (2.3) is

$$\left\|B(t_m)\int_a^b f(s)\,ds - \int_a^b B(s)\,f(s)\,ds\right\| \leq L\int_a^b sgn(s-t_m)\,s\,\|f(s)\|\,ds. \tag{2.7}$$

Indeed, as

$$\begin{aligned}
&\inf_{t\in[a,b]}\int_a^b |t-s|\,\|f(s)\|\,ds \\
&= \int_a^b |t_m - s|\,\|f(s)\|\,ds \\
&= \int_a^{t_m}(t_m - s)\,\|f(s)\|\,ds + \int_{t_m}^b (s-t_m)\,\|f(s)\|\,ds \\
&= t_m\left(\int_a^{t_m}\|f(s)\|\,ds - \int_{t_m}^b \|f(s)\|\,ds\right) + \int_{t_m}^b s\,\|f(s)\|\,ds - \int_a^{t_m} s\,\|f(s)\|\,ds \\
&= \int_{t_m}^b s\,\|f(s)\|\,ds - \int_a^{t_m} s\,\|f(s)\|\,ds \\
&= \int_a^b sgn(s-t_m)\,s\,\|f(s)\|\,ds,
\end{aligned}$$

then the best inequality we can get from the first part of (2.3) is obtained for $t = t_m \in (a,b)$.

We recall that a function $F : [a,b] \to \mathcal{L}(X)$ is said to be *strongly continuous* if for all $x \in X$, the maps $s \mapsto F(s)\,x : [a,b] \to X$ are continuous on $[a,b]$. In this case the function $s \mapsto \|B(s)\| : [a,b] \to \mathbb{R}_+$ is (Lebesgue) measurable and bounded ([6]). The linear operator $L = \int_a^b F(s)\,ds$ (defined by $Lx := \int_a^b F(s)\,xds$ for all $x \in X$) is bounded, because

$$\|Lx\| \leq \left(\int_a^b \|F(s)\|\,ds\right)\cdot\|x\| \quad \text{for all } x \in X.$$

In a similar manner to Theorem 2.1, we may prove the following result as well.

Theorem 2.5. *Assume that $f : [a,b] \to X$ is Hölder continuous, i.e.,*

$$\|f(t) - f(s)\| \leq K\,|t-s|^{\beta} \quad \textit{for all } t,s \in [a,b], \tag{2.8}$$

where $K > 0$ and $\beta \in (0,1]$.

If $B : [a, b] \to \mathcal{L}(X)$ *is strongly continuous on* $[a, b]$ *, then we have the inequality:*

$$\left\| \left(\int_a^b B(s)\, ds \right) f(t) - \int_a^b B(s) f(s)\, ds \right\| \tag{2.9}$$

$$\leq K \int_a^b |t-s|^{\beta} \|B(s)\|\, ds$$

$$\leq K \times \begin{cases} \frac{(b-t)^{\beta+1}+(t-a)^{\beta+1}}{\beta+1} \, |||B|||_{[a,b],\infty} & \text{if} \quad \|B(\cdot)\| \in L_{\infty}([a,b];\mathbb{R}_+); \\ \left[\frac{(b-t)^{q\beta+1}+(t-a)^{q\beta+1}}{q\beta+1} \right]^{\frac{1}{q}} |||B|||_{[a,b],p} & \text{if} \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ & \text{and} \quad \|B(\cdot)\| \in L_p([a,b];\mathbb{R}_+); \\ \left[\frac{1}{2}(b-a) + \left| t - \frac{a+b}{2} \right| \right]^{\beta} |||B|||_{[a,b],1} \end{cases}$$

for any $t \in [a, b]$.

The following corollary holds.

Corollary 2.6. *Assume that* f *and* B *are as in Theorem 2.5. If, in addition,* $\int_a^b B(s)\, ds$ *is invertible in* $\mathcal{L}(X)$ *, then we have the inequality:*

$$\left\| f(t) - \left(\int_a^b B(s)\, ds \right)^{-1} \int_a^b B(s) f(s)\, ds \right\| \leq K \left\| \left(\int_a^b B(s)\, ds \right)^{-1} \right\| \int_a^b |t-s|^{\beta} \|B(s)\|\, ds \tag{2.10}$$

for any $t \in [a, b]$.

Remark 2.7. It is obvious that the inequality (2.10) contains as a particular case what is the so called Ostrowski's inequality for weighted integrals (see (1.2)).

3. INEQUALITIES FOR LINEAR OPERATORS

Let $0 \leq a < b < \infty$ and $A \in \mathcal{L}(X)$. We recall that the operatorial norm of A is given by

$$\|A\| = \sup \{ \|Ax\| : \|x\| \leq 1 \}.$$

The *resolvent set* of A (denoted by $\rho(A)$) is the set of all complex scalars λ for which $\lambda I - A$ is an invertible operator. Here I is the identity operator in $\mathcal{L}(X)$. The complementary set of $\rho(A)$ in the complex plane, denoted by $\sigma(A)$, is the *spectrum* of A. It is known that $\sigma(A)$ is a compact set in $\mathbb{C}$. The series $\left(\sum_{n \geq 0} \frac{(tA)^n}{n!} \right)$ converges absolutely and locally uniformly for $t \in \mathbb{R}$. If we denote by e^{tA} its sum, then

$$\left\| e^{tA} \right\| \leq e^{|t| \|A\|}, \quad t \in \mathbb{R}.$$

Proposition 3.1. *Let* X *be a real or complex Banach space,* $A \in \mathcal{L}(X)$ *and* β *be a non-null real number such that* $-\beta \in \rho(A)$*. Then for all* $0 \leq a < b < \infty$ *and each* $s \in [a, b]$*, we have*

$$\left\| \frac{e^{\beta b} - e^{\beta a}}{\beta} \cdot e^{sA} - (\beta I + A)^{-1} \left[e^{b(\beta I + A)} - e^{a(\beta I + A)} \right] \right\| \leq \|A\| e^{b\|A\|} \cdot \left[\frac{1}{4}(b-a)^2 + \left(s - \frac{a+b}{2} \right)^2 \right] \cdot \max \left\{ e^{\beta b}, e^{\beta a} \right\}. \tag{3.1}$$

Proof. We apply the second inequality from Corollary 2.2 in the following particular case.

$$B(\tau) := e^{\tau A}, \quad f(\tau) = e^{\beta\tau}x, \quad \tau \in [a,b], \; x \in X.$$

For all $\xi, \eta \in [a,b]$ there exists an α between ξ and η such that

$$\begin{aligned} \|B(\xi) - B(\eta)\| &= \left\| \sum_{n=1}^{\infty} \frac{(\xi^n - \eta^n)}{n!} A^n \right\| \\ &= \left\| (\xi - \eta) A \sum_{n=0}^{\infty} \frac{(\alpha A)^n}{n!} \right\| \\ &\leq \|A\| \left\| e^{\alpha A} \right\| \cdot |\xi - \eta| \\ &\leq \|A\| e^{b\|A\|} \cdot |\xi - \eta|. \end{aligned}$$

The function $\tau \mapsto e^{\tau A}$ is thus Lipschitzian on $[a,b]$ with the constant $L := \|A\| e^{b \cdot \|A\|}$. On the other hand we have

$$\begin{aligned} \int_a^b e^{\tau A}\left(e^{\beta\tau}x\right) d\tau &= \int_a^b e^{\tau A}\left(e^{\beta\tau}Ix\right) d\tau \\ &= \int_a^b e^{\tau(A+\beta I)} x d\tau \\ &= (A + \beta I)^{-1}\left[e^{b(A+\beta I)} - e^{a(A+\beta I)}\right] x, \end{aligned}$$

and

$$|||f|||_{[a,b],\infty} = \sup_{\tau \in [a,b]} \left\| e^{\tau\beta}x \right\| = \max\left\{e^{\beta b}, e^{\beta a}\right\} \cdot \|x\|.$$

Placing all the above results in the second inequality from (2.3) and taking the supremum for all $x \in X$, we will obtain the desired inequality (3.1). □

Remark 3.2. Let $A \in \mathcal{L}(X)$ such that $0 \in \rho(A)$. Taking the limit as $\beta \to 0$ in (3.1), we get the inequality

$$\left\|(b-a)e^{sA} - A^{-1}\left[e^{bA} - e^{aA}\right]\right\| \leq \|A\| e^{b\|A\|} \cdot \left[\frac{1}{4}(b-a)^2 + \left(s - \frac{a+b}{2}\right)^2\right],$$

where a, b and s are as in Proposition 3.1.

Proposition 3.3. *Let $A \in \mathcal{L}(X)$ be an invertible operator, $t \geq 0$ and $0 \leq s \leq t$. Then the following inequality holds:*

$$\left\|\frac{t^2}{2}\sin(sA) - A^{-2}\left[\sin(tA) - tA\cos(tA)\right]\right\| \leq \frac{2s^3 + 2t^3 - 3st^2}{6}\|A\|. \tag{3.2}$$

In particular, if $X = \mathbb{R}$, $A = 1$ and $s = 0$ it follows the scalar inequality

$$|\sin t - t\cos t| \leq \frac{t^3}{3}, \quad \textit{for all } t \geq 0.$$

Proof. We apply the inequality from (2.3) in the following particular case:

$$B(\tau) = \sin(\tau A) := \sum_{n=0}^{\infty} (-1)^n \frac{(\tau A)^{2n+1}}{(2n+1)!}, \quad \tau \geq 0,$$

and

$$f(\tau) = \tau \cdot x, \quad \text{for fixed } x \in X. \tag{3.3}$$

For each $\xi, \eta \in [0, t]$, we have

$$\begin{aligned} \|B(\xi) - B(\eta)\| &= \left\| A\left(\sum_{n=0}^{\infty} (-1)^n \frac{(\xi-\eta)\alpha^{2n}}{(2n)!} A^{2n}\right)\right\| \\ &\leq \|A\| |\xi - \eta| \cdot \|\cos(\alpha A)\| \\ &\leq \|A\| |\xi - \eta| , \end{aligned}$$

where α is a real number between ξ and η, i.e., the function $\tau \longmapsto B(\tau) : \mathbb{R}_+ \to \mathcal{L}(X)$ is $\|A\|$ −Lipschitzian.

Moreover, it is easy to see that

$$\int_0^t B(\tau) f(\tau) d\tau = A^{-2} [\sin(tA) - tA\cos(tA)] x$$

and

$$\int_0^t |s - \tau| |f(\tau)| d\tau = \frac{2s^3 + 2t^3 - 3st^2}{6} \|x\| . \tag{3.4}$$

Applying the first inequality from (2.3) and taking the supremum for $x \in X$ with $\|x\| \leq 1$, we get (3.2). □

4. QUADRATURE FORMULAE

Consider the division of the interval $[a, b]$ given by

$$I_n : a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b \tag{4.1}$$

and $h_i := t_{i+1} - t_i$, $\nu(h) := \max_{i=\overline{0,n-1}} h_i$. For the intermediate points $\xi := (\xi_0, \ldots, \xi_{n-1})$ with $\xi_i \in [t_i, t_{i+1}]$, $i = \overline{0, n-1}$, define the sum

$$S_n^{(1)}(B, f; I_n, \xi) := \sum_{i=0}^{n-1} B(\xi_i) \int_{t_i}^{t_{i+1}} f(s) ds. \tag{4.2}$$

Then we may state the following result in approximating the integral

$$\int_a^b B(s) f(s) ds,$$

based on Theorem 2.1.

Theorem 4.1. *Assume that $B : [a, b] \to \mathcal{L}(X)$ is Hölder continuous on $[a, b]$, i.e., it satisfies the condition (2.1) and $f : [a, b] \to X$ is Bochner integrable on $[a, b]$. Then we have the representation*

$$\int_a^b B(s) f(s) ds = S_n^{(1)}(B, f; I_n, \xi) + R_n^{(1)}(B, f; I_n, \xi) , \tag{4.3}$$

where $S_n^{(1)}(B,f;I_n,\xi)$ is as given by (4.2) and the remainder $R_n^{(1)}(B,f;I_n,\xi)$ satisfies the estimate

$$\left\|R_n^{(1)}(B,f;I_n,\xi)\right\| \leq H\times\begin{cases}\dfrac{1}{\alpha+1}\left|\left|\left|f\right|\right|\right|_{[a,b],\infty}\displaystyle\sum_{i=0}^{n-1}\left[(t_{i+1}-\xi_i)^{\alpha+1}+(\xi_i-t_i)^{\alpha+1}\right]\\ \dfrac{1}{(q\alpha+1)^{\frac{1}{q}}}\left|\left|\left|f\right|\right|\right|_{[a,b],p}\left\{\displaystyle\sum_{i=0}^{n-1}\left[(t_{i+1}-\xi_i)^{q\alpha+1}+(\xi_i-t_i)^{q\alpha+1}\right]\right\}^{\frac{1}{q}}, \\ \qquad\qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1\\ \left[\dfrac{1}{2}\nu(h)+\displaystyle\max_{i=\overline{0,n-1}}\left|\xi_i-\dfrac{t_{i+1}+t_i}{2}\right|\right]^{\alpha}\left|\left|\left|f\right|\right|\right|_{[a,b],1}\end{cases}$$

$$\leq H\times\begin{cases}\dfrac{1}{\alpha+1}\left|\left|\left|f\right|\right|\right|_{[a,b],\infty}\displaystyle\sum_{i=0}^{n-1}h_i^{\alpha+1}\\ \dfrac{1}{(q\alpha+1)^{\frac{1}{q}}}\left|\left|\left|f\right|\right|\right|_{[a,b],p}\left(\displaystyle\sum_{i=0}^{n-1}h_i^{q\alpha+1}\right)^{\frac{1}{q}}, \qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1\\ \left[\dfrac{1}{2}\nu(h)+\displaystyle\max_{i=\overline{0,n-1}}\left|\xi_i-\dfrac{t_{i+1}+t_i}{2}\right|\right]^{\alpha}\left|\left|\left|f\right|\right|\right|_{[a,b],1}\end{cases}$$

$$\leq H\times\begin{cases}\dfrac{1}{\alpha+1}\left|\left|\left|f\right|\right|\right|_{[a,b],\infty}[\nu(h)]^{\alpha}\\ \dfrac{(b-a)^{\frac{1}{q}}}{(q\alpha+1)^{\frac{1}{q}}}\left|\left|\left|f\right|\right|\right|_{[a,b],p}[\nu(h)]^{\alpha}\\ \left|\left|\left|f\right|\right|\right|_{[a,b],1}[\nu(h)]^{\alpha}.\end{cases}$$

Proof. Applying Theorem 4.1 on $[x_i,x_{i+1}]$ $\left(i=\overline{0,n-1}\right)$, we may write that

$$\left\|\int_{t_i}^{t_{i+1}}B(s)f(s)\,ds-B(\xi_i)\int_{t_i}^{t_{i+1}}f(s)\,ds\right\| \leq H\times\begin{cases}\left[\dfrac{(t_{i+1}-\xi_i)^{\alpha+1}+(\xi_i-t_i)^{\alpha+1}}{\alpha+1}\right]\left|\left|\left|f\right|\right|\right|_{[t_i,t_{i+1}],\infty}\\ \left[\dfrac{(t_{i+1}-\xi_i)^{q\alpha+1}+(\xi_i-t_i)^{q\alpha+1}}{q\alpha+1}\right]^{\frac{1}{q}}\left|\left|\left|f\right|\right|\right|_{[t_i,t_{i+1}],p}\\ \left[\dfrac{1}{2}(t_{i+1}-t_i)+\left|\xi_i-\dfrac{t_{i+1}+t_i}{2}\right|\right]^{\alpha}\left|\left|\left|f\right|\right|\right|_{[t_i,t_{i+1}],1}.\end{cases}$$

Summing over i from 0 to $n-1$ and using the generalised triangle inequality we get

$$\begin{aligned}
&\left\| R_n^{(1)}(B, f; I_n, \xi) \right\| \\
&\leq \sum_{i=0}^{n-1} \left\| \int_{t_i}^{t_{i+1}} B(s) f(s)\, ds - B(\xi_i) \int_{t_i}^{t_{i+1}} f(s)\, ds \right\| \\
&\leq H \times \begin{cases} \dfrac{1}{\alpha+1} \sum_{i=0}^{n-1} \left[(t_{i+1}-\xi_i)^{\alpha+1} + (\xi_i - t_i)^{\alpha+1} \right] |||f|||_{[t_i,t_{i+1}],\infty} \\ \dfrac{1}{(q\alpha+1)^{\frac{1}{q}}} \left[\sum_{i=0}^{n-1} (t_{i+1}-\xi_i)^{q\alpha+1} + (\xi_i - t_i)^{q\alpha+1} \right]^{\frac{1}{q}} |||f|||_{[t_i,t_{i+1}],p} \\ \sum_{i=0}^{n-1} \left[\dfrac{1}{2} h_i + \left| \xi_i - \dfrac{t_{i+1}+t_i}{2} \right| \right]^{\alpha} |||f|||_{[t_i,t_{i+1}],1} . \end{cases}
\end{aligned}$$

Now, observe that

$$\begin{aligned}
&\sum_{i=0}^{n-1} \left[(t_{i+1}-\xi_i)^{\alpha+1} + (\xi_i - t_i)^{\alpha+1} \right] |||f|||_{[t_i,t_{i+1}],\infty} \\
&\qquad \leq |||f|||_{[a,b],\infty} \sum_{i=0}^{n-1} \left[(t_{i+1}-\xi_i)^{\alpha+1} + (\xi_i - t_i)^{\alpha+1} \right] \\
&\qquad \leq |||f|||_{[a,b],\infty} \sum_{i=0}^{n-1} h_i^{\alpha+1} \\
&\qquad \leq |||f|||_{[a,b],\infty} (b-a) [\nu(h)]^{\alpha} .
\end{aligned}$$

Using the discrete Hölder inequality, we may write that

$$\begin{aligned}
&\left[\sum_{i=0}^{n-1} (t_{i+1}-\xi_i)^{q\alpha+1} + (\xi_i - t_i)^{q\alpha+1} \right]^{\frac{1}{q}} |||f|||_{[t_i,t_{i+1}],p} \\
&\leq \left[\sum_{i=0}^{n-1} \left(\left[(t_{i+1}-\xi_i)^{q\alpha+1} + (\xi_i - t_i)^{q\alpha+1} \right]^{\frac{1}{q}} \right)^{q} \right]^{\frac{1}{q}} \times \left[\sum_{i=0}^{n-1} |||f|||_{[t_i,t_{i+1}],p}^{p} \right]^{\frac{1}{p}} \\
&= \left\{ \sum_{i=0}^{n-1} \left[(t_{i+1}-\xi_i)^{q\alpha+1} + (\xi_i - t_i)^{q\alpha+1} \right] \right\}^{\frac{1}{q}} \left(\int_a^b \|f(t)\|^p\, ds \right)^{\frac{1}{p}} \\
&\leq \left(\sum_{i=0}^{n-1} h_i^{q\alpha+1} \right)^{\frac{1}{q}} |||f|||_{[a,b],p}^{p} \\
&\leq (b-a)^{\frac{1}{q}} |||f|||_{[a,b],p} [\nu(h)]^{\alpha} .
\end{aligned}$$

Finally, we have

$$\begin{aligned}
&\sum_{i=0}^{n-1} \left[\frac{1}{2} h_i + \left| \xi_i - \frac{t_{i+1}+t_i}{2} \right| \right]^{\alpha} |||f|||_{[t_i,t_{i+1}],1} \\
&\qquad \leq \left[\frac{1}{2} \max_{i=\overline{0,n-1}} h_i + \max_{i=\overline{0,n-1}} \left| \xi_i - \frac{t_{i+1}+t_i}{2} \right| \right]^{\alpha} |||f|||_{[a,b],1} \\
&\qquad \leq [\nu(h)]^{\alpha} |||f|||_{[a,b],1}
\end{aligned}$$

and the theorem is proved. □

The following corollary holds.

Corollary 4.2. *If B is Lipschitzian with the constant L, then we have the representation (4.3) and the remainder $R_n^{(1)}(B,f;I_n,\xi)$ satisfies the estimates:*

$$\left\|R_n^{(1)}(B,f;I_n,\xi)\right\| \tag{4.4}$$

$$\leq L\times\begin{cases}|||f|||_{[a,b],\infty}\left[\frac{1}{4}\sum_{i=0}^{n-1}h_i^2+\sum_{i=0}^{n-1}\left(\xi_i-\frac{t_{i+1}+t_i}{2}\right)^2\right]\\[2mm] \frac{1}{(q+1)^{\frac{1}{q}}}|||f|||_{[a,b],p}\left\{\sum_{i=0}^{n-1}\left[(t_{i+1}-\xi_i)^{q+1}+(\xi_i-t_i)^{q+1}\right]\right\}^{\frac{1}{q}}, \\ \qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1\\[2mm] \left[\frac{1}{2}\nu(h)+\max_{i=\overline{0,n-1}}\left|\xi_i-\frac{t_{i+1}+t_i}{2}\right|\right]|||f|||_{[a,b],1}\end{cases}$$

$$\leq L\times\begin{cases}\frac{1}{2}|||f|||_{[a,b],\infty}\sum_{i=0}^{n-1}h_i^2\\[2mm] \frac{1}{(q+1)^{\frac{1}{q}}}|||f|||_{[a,b],p}\left(\sum_{i=0}^{n-1}h_i^{q+1}\right)^{\frac{1}{q}}\\[2mm] \left[\frac{1}{2}\nu(h)+\max_{i=\overline{0,n-1}}\left|\xi_i-\frac{t_{i+1}+t_i}{2}\right|\right]|||f|||_{[a,b],1}\end{cases}$$

$$\leq L\times\begin{cases}\frac{1}{2}|||f|||_{[a,b],\infty}(b-a)\,\nu(h)\\[2mm] \frac{(b-a)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}|||f|||_{[a,b],p}\,\nu(h)\\[2mm] |||f|||_{[a,b],1}\,\nu(h).\end{cases}$$

The second possibility we have for approximating the integral $\int_a^b B(s)f(s)\,ds$ is embodied in the following theorem based on Theorem 2.5.

Theorem 4.3. *Assume that $f:[a,b]\to X$ is Hölder continuous, i.e., the condition (2.8) holds. If $B:[a,b]\to\mathcal{L}(X)$ is strongly continuous on $[a,b]$, then we have the representation:*

$$\int_a^b B(s)f(s)\,ds=S_n^{(2)}(B,f;I_n,\xi)+R_n^{(2)}(B,f;I_n,\xi), \tag{4.5}$$

where

$$S_n^{(2)}(B,f;I_n,\xi):=\sum_{i=0}^{n-1}\left(\int_{t_i}^{t_{i+1}}B(s)\,ds\right)f(\xi_i) \tag{4.6}$$

and the remainder $R_n^{(2)}(B,f;I_n,\xi)$ *satisfies the estimate:*

(4.7) $$\left\| R_n^{(2)}(B,f;I_n,\xi)\right\|$$

$$\leq K\times\begin{cases}\dfrac{1}{\beta+1}\left\|\left|B\right|\right\|_{[a,b],\infty}\displaystyle\sum_{i=0}^{n-1}\left[(t_{i+1}-\xi_i)^{\beta+1}+(\xi_i-t_i)^{\beta+1}\right]\\[2ex] \dfrac{1}{(q\beta+1)^{\frac{1}{q}}}\left\|\left|B\right|\right\|_{[a,b],p}\left\{\displaystyle\sum_{i=0}^{n-1}\left[(t_{i+1}-\xi_i)^{q\beta+1}+(\xi_i-t_i)^{q\beta+1}\right]\right\}^{\frac{1}{q}}, \\ \qquad\qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1\\[2ex] \left[\dfrac{1}{2}\nu(h)+\displaystyle\max_{i=\overline{0,n-1}}\left|\xi_i-\dfrac{t_{i+1}+t_i}{2}\right|\right]^{\beta}\left\|\left|B\right|\right\|_{[a,b],1}\end{cases}$$

$$\leq K\times\begin{cases}\dfrac{1}{\beta+1}\left\|\left|B\right|\right\|_{[a,b],\infty}\displaystyle\sum_{i=0}^{n-1}h_i^{\beta+1}\\[2ex] \dfrac{1}{(q\beta+1)^{\frac{1}{q}}}\left\|\left|B\right|\right\|_{[a,b],p}\left\{\displaystyle\sum_{i=0}^{n-1}h_i^{q\beta+1}\right\}^{\frac{1}{q}}, \qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \left[\dfrac{1}{2}\nu(h)+\displaystyle\max_{i=\overline{0,n-1}}\left|\xi_i-\dfrac{t_{i+1}+t_i}{2}\right|\right]^{\beta}\left\|\left|B\right|\right\|_{[a,b],1}\end{cases}$$

$$\leq K\times\begin{cases}\dfrac{1}{\beta+1}\left\|\left|B\right|\right\|_{[a,b],\infty}(b-a)\left[\nu(h)\right]^{\beta}\\[2ex] \dfrac{(b-a)^{\frac{1}{q}}}{(q\beta+1)^{\frac{1}{q}}}\left\|\left|B\right|\right\|_{[a,b],p}\left[\nu(h)\right]^{\beta}, \qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1\\[2ex] \left\|\left|B\right|\right\|_{[a,b],1}\left[\nu(h)\right]^{\beta}\ .\end{cases}$$

If we consider the quadrature

(4.8) $$M_n^{(1)}(B,f;I_n):=\sum_{i=0}^{n-1}B\left(\frac{t_i+t_{i+1}}{2}\right)\int_{t_i}^{t_{i+1}}f(s)\,ds,$$

then we have the representation

(4.9) $$\int_a^b B(s)\,f(s)\,ds=M_n^{(1)}(B,f;I_n)+R_n^{(1)}(B,f;I_n),$$

and the remainder $R_n^{(1)}(B,f;I_n)$ satisfies the estimate:

$$\left\|R_n^{(1)}(B,f;I_n)\right\| \le H\times\begin{cases}\frac{1}{2^\alpha(\alpha+1)}\left|\left|\left|f\right|\right|\right|_{[a,b],\infty}\sum\limits_{i=0}^{n-1}h_i^{\alpha+1}\\ \frac{1}{2^\alpha(q\alpha+1)^{\frac{1}{q}}}\left|\left|\left|f\right|\right|\right|_{[a,b],p}\left[\sum\limits_{i=0}^{n-1}h_i^{q\alpha+1}\right]^{\frac{1}{q}},\ p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \frac{1}{2^\alpha}\left[\nu(h)\right]^\alpha\left|\left|\left|f\right|\right|\right|_{[a,b],1}\end{cases} \tag{4.10}$$

$$\le H\times\begin{cases}\frac{1}{2^\alpha(\alpha+1)}(b-a)\left|\left|\left|f\right|\right|\right|_{[a,b],\infty}\left[\nu(h)\right]^\alpha\\ \frac{(b-a)^{\frac{1}{q}}}{2^\alpha(q\alpha+1)^{\frac{1}{q}}}\left|\left|\left|f\right|\right|\right|_{[a,b],p}\left[\nu(h)\right]^\alpha,\quad p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \frac{1}{2^\alpha}\left|\left|\left|f\right|\right|\right|_{[a,b],1}\left[\nu(h)\right]^\alpha,\end{cases}$$

provided that B and f are as in Theorem 4.1.

Now, if we consider the quadrature

$$M_n^{(2)}(B,f;I_n):=\sum_{i=0}^{n-1}\left(\int_{t_i}^{t_{i+1}}B(s)\,ds\right)f\left(\frac{t_i+t_{i+1}}{2}\right), \tag{4.11}$$

then we also have

$$\int_a^b B(s)f(s)\,ds=M_n^{(2)}(B,f;I_n)+R_n^{(2)}(B,f;I_n), \tag{4.12}$$

and in this case the remainder satisfies the bound

$$\left\|R_n^{(2)}(B,f;I_n)\right\| \le K\times\begin{cases}\frac{1}{2^\beta(\beta+1)}\left|\left|\left|B\right|\right|\right|_{[a,b],\infty}\sum\limits_{i=0}^{n-1}h_i^{\beta+1}\\ \frac{1}{2^\beta(q\beta+1)^{\frac{1}{q}}}\left|\left|\left|B\right|\right|\right|_{[a,b],p}\left(\sum\limits_{i=0}^{n-1}h_i^{q\beta+1}\right)^{\frac{1}{q}},\quad p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \frac{1}{2^\beta}\left[\nu(h)\right]^\beta\left|\left|\left|B\right|\right|\right|_{[a,b],1}\end{cases} \tag{4.13}$$

$$\le K\times\begin{cases}\frac{1}{2^\beta(\beta+1)}(b-a)\left|\left|\left|B\right|\right|\right|_{[a,b],\infty}\left[\nu(h)\right]^\beta\\ \frac{(b-a)^{\frac{1}{q}}}{2^\beta(q\beta+1)^{\frac{1}{q}}}\left|\left|\left|B\right|\right|\right|_{[a,b],p}\left[\nu(h)\right]^\beta,\quad p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \frac{1}{2^\beta}\left|\left|\left|B\right|\right|\right|_{[a,b],1}\left[\nu(h)\right]^\beta,\end{cases}$$

provided B and f satisfy the hypothesis of Theorem 4.3.

Now, if we consider the equidistant partitioning of $[a,b]$,

$$E_n:t_i:=a+\left(\frac{b-a}{n}\right)\cdot i,\quad i=\overline{0,n},$$

then $M_n^{(1)}(B,f;E_n)$ becomes

$$M_n^{(1)}(B,f) := \sum_{i=0}^{n-1} B\left(a+\left(i+\frac{1}{2}\right)\cdot\frac{b-a}{n}\right)\int_{a+\frac{b-a}{n}\cdot i}^{a+\frac{b-a}{n}\cdot(i+1)} f(s)\,ds \tag{4.14}$$

and then

$$\int_a^b B(s)\,f(s)\,ds = M_n^{(1)}(B,f) + R_n^{(1)}(B,f), \tag{4.15}$$

where the remainder satisfies the bound

$$\left\|R_n^{(1)}(B,f)\right\| \le H\times\begin{cases}\dfrac{(b-a)^{\alpha+1}}{2^\alpha(\alpha+1)n^\alpha}\,|||f|||_{[a,b],\infty}\\[2ex] \dfrac{(b-a)^{\alpha+\frac{1}{q}}}{2^\alpha(\alpha+1)n^\alpha}\,|||f|||_{[a,b],p}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \dfrac{(b-a)^{\alpha}}{2^\alpha n^\alpha}\,|||f|||_{[a,b],1}.\end{cases} \tag{4.16}$$

Also, we have

$$\int_a^b B(s)\,f(s)\,ds = M_n^{(2)}(B,f) + R_n^{(2)}(B,f), \tag{4.17}$$

where

$$M_n^{(2)}(B,f) := \sum_{i=0}^{n-1}\left(\int_{a+\frac{b-a}{n}\cdot i}^{a+\frac{b-a}{n}\cdot(i+1)} B(s)\,ds\right) f\left(a+\left(i+\frac{1}{2}\right)\cdot\frac{b-a}{n}\right),$$

and the remainder $R_n^{(2)}(B,f)$ satisfies the estimate

$$\left\|R_n^{(2)}(B,f)\right\| \le K\times\begin{cases}\dfrac{(b-a)^{\beta+1}}{2^\beta(\beta+1)n^\beta}\,\|B\|_{[a,b],\infty}\\[2ex] \dfrac{(b-a)^{\beta+\frac{1}{q}}}{2^\beta(\beta+1)n^\beta}\,|||B|||_{[a,b],p}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \dfrac{(b-a)^{\beta}}{2^\beta n^\beta}\,|||B|||_{[a,b],1}.\end{cases} \tag{4.18}$$

5. APPLICATION FOR DIFFERENTIAL EQUATIONS IN BANACH SPACES

We recall that a family of operators $\mathcal{U} = \{U(t,s) : t \ge s\} \subset \mathcal{L}(X)$ with $t,s\in\mathbb{R}$ or $t,s\in\mathbb{R}_+$ is called an *evolution family* if:

(i) $U(t,t) = I$ and $U(t,s)\,U(s,\tau) = U(t,\tau)$ for all $t\ge s\ge\tau$; and
(ii) for each $x\in X$, the function $(t,s)\longmapsto U(t,s)\,x$ is continuous for $t\ge s.$

Here I is the identity operator in $\mathcal{L}(X)$.

An evolution family $\{U(t,s) : t\ge s\}$ is said to be *exponentially bounded* if, in addition,

(iii) there exist the constants $M\ge 1$ and $\omega>0$ such that

$$\|U(t,s)\| \le Me^{\omega(t-s)}, \quad t\ge s. \tag{5.1}$$

Evolution families appear as solutions for abstract Cauchy problems of the form

$$\dot{u}(t) = A(t)\,u(t)\,,\ u(s) = x_s,\ x_s \in \mathcal{D}(A(s))\,,\ t \geq s,\ t,s \in \mathbb{R} \text{ (or } \mathbb{R}_+\text{)}, \tag{5.2}$$

where the domain $\mathcal{D}(A(s))$ of the linear operator $A(s)$ is assumed to be dense in X. An evolution family is said to solve the abstract Cauchy problem (5.2) if for each $s \in \mathbb{R}$ there exists a dense subset $Y_s \subseteq D(A(s))$ such that for each $x_s \in Y_s$ the function

$$t \longmapsto u(t) := U(t,s)\,x_s : [s,\infty) \to X,$$

is differentiable, $u(t) \in D(A(t))$ for all $t \geq s$ and

$$\frac{d}{dt}u(t) = A(t)\,u(t)\,,\ \ t \geq s.$$

This later definition can be found in [15]. In this definition the operators $A(t)$ can be unbounded. The Cauchy problem (5.2) is called *well-posed* if there exists an evolution family $\{U(t,s) : t \geq s\}$ which solves it.

It is known that the well-posedness of (5.2) can be destroyed by a bounded and continuous perturbation [13]. Let $f : \mathbb{R} \to X$ be a locally integrable function. Consider the inhomogeneous Cauchy problem:

$$\dot{u}(t) = A(t)\,u(t) + f(t)\,,\ u(s) = x_s \in X,\ t \geq s,\ t,s \in \mathbb{R} \text{ (or } \mathbb{R}_+\text{)}. \tag{5.3}$$

A continuous function $t \longmapsto u(t) : [s,\infty) \to X$ is said to a *mild solution* of the Cauchy problem (5.3) if $u(s) = x_s$ and there exists an evolution family $\{U(t,\tau) : t \geq \tau\}$ such that

$$u(t) = U(t,s)\,x_s + \int_s^t U(t,\tau)\,f(\tau)\,d\tau,\ t \geq s,\ x_s \in X,\ t,s \in \mathbb{R} \text{ (or } \mathbb{R}_+\text{)}. \tag{5.4}$$

The following theorem holds.

Theorem 5.1. *Let $\mathcal{U} = \{U(\nu,\eta) : \nu \geq \eta\} \subset \mathcal{L}(X)$ be an evolution family and $f : \mathbb{R} \to X$ be a locally Bochner integrable and locally bounded function. We assume that for all $\nu \in \mathbb{R}$ (or $\mathbb{R}_+$) the function $\eta \longmapsto U(\nu,\eta) : [\nu,\infty) \to \mathcal{L}(X)$ is locally Hölder continuous (i.e. for all $a,b \geq \nu,\ a < b$, there exist $\alpha \in (0,1]$ and $H > 0$ such that*

$$\|U(\nu,t) - U(\nu,s)\| \leq H\,|t-s|^{\alpha}\,,\ \ \textit{for all } t,s \in [a,b]\,).$$

We use the notations in Section 4 for $a = 0$ and $b = t > 0$. The map $u(\cdot)$ from (5.4) can be represented as

$$u(t) = U(t,0)\,x_0 + \sum_{i=0}^{n-1} U(t,\xi_i) \int_{t_i}^{t_{i+1}} f(s)\,ds + R_n^{(1)}(\mathcal{U},f,I_n,\xi) \tag{5.5}$$

where the remainder $R_n^{(1)}(\mathcal{U},f,I_n,\xi)$ satisfies the estimate

$$\left\|R_n^{(1)}(\mathcal{U},f,I_n,\xi)\right\| \leq \frac{H}{\alpha+1}\,|||f|||_{[0,t],\infty} \sum_{i=0}^{n-1} \left[(t_{i+1}-\xi_i)^{\alpha+1} + (\xi_i - t_i)^{\alpha+1}\right].$$

Proof. It follows by representation (4.3) and the first estimate after it. □

Moreover, if n is a natural number, $i \in \{0,\ldots,n\}$, $t_i := \frac{t \cdot i}{n}$ and $\xi_i := \frac{(2i+1)t}{2n}$, then

$$u(t) = U(t,0)\,x_0 + \sum_{i=0}^{n-1} U\left(t, \frac{(2i+1)\,t}{2n}\right) \int_{\frac{t \cdot i}{n}}^{\frac{t\cdot(i+1)}{n}} f(s)\,ds + R_n^{(1)} \tag{5.6}$$

and the remainder $R_n^{(1)}$ satisfies the estimate

$$\left\|R_n^{(1)}\right\| \leq \frac{H}{\alpha+1} \cdot \frac{t^{\alpha+1}}{2^\alpha \cdot n^\alpha} |||f|||_{[0,t],\infty} . \tag{5.7}$$

The following theorem also holds.

Theorem 5.2. *Let $\mathcal{U} = \{U(\nu,\eta) : \nu \geq \eta\} \subset \mathcal{L}(X)$ be an exponentially bounded evolution family of bounded linear operators acting on the Banach space X and $f : \mathbb{R} \to X$ be a locally Hölder continuous function, i.e., for all $a, b \in \mathbb{R}$, $a < b$ there exist $\beta \in (0,1]$ and $K > 0$ such that (2.8) holds. We use the notations of Section 4 for $a = 0$ and $b = t > 0$. The map $u(\cdot)$ from (5.4) can be represented as*

$$u(t) = U(t,0)x_0 + \sum_{i=0}^{n-1} \left(\int_{t_i}^{t_{i+1}} U(t,\tau)\, d\tau \right) f(\xi_i) + R_n^{(2)}(\mathcal{U}, f, I_n, \xi) \tag{5.8}$$

where the remainder $R_n^{(2)}(\mathcal{U}, f, I_n, \xi)$ satisfies the estimate

$$\left\|R_n^{(2)}(\mathcal{U}, f, I_n, \xi)\right\| \leq \frac{KM}{\beta+1} e^{\omega t} \sum_{i=0}^{n-1} \left[(t_{i+1} - \xi_i)^{\beta+1} + (\xi_i - t_i)^{\beta+1} \right].$$

Proof. It follows from the first estimate in (4.7) for $B(s) := U(t,s)$, using the fact that

$$|||B(\cdot)|||_{[0,t],\infty} = \sup_{\tau\in[0,t]} \|U(t,\tau)\| \leq \sup_{\tau\in[0,t]} M e^{\omega(t-\tau)} \leq M e^{\omega t}.$$

□

Moreover, if n is a natural number, $i \in \{0, \ldots, n\}$, $t_i := \frac{t\cdot i}{n}$ and $\xi_i := \frac{(2i+1)t}{2n}$ then

$$u(t) = U(t,0)x_0 + \sum_{i=0}^{n-1} \left(\int_{\frac{t\cdot i}{n}}^{\frac{t\cdot(i+1)}{n}} U(t,\tau)\, d\tau \right) f\left(\frac{(2i+1)t}{2n} \right) + R_n^{(2)} \tag{5.9}$$

and the remainder $R_n^{(2)}$ satisfies the estimate

$$\left\|R_n^{(2)}\right\| \leq \frac{KM}{\beta+1} e^{\omega t} \cdot \frac{t^{\beta+1}}{2^\beta \cdot n^\beta}. \tag{5.10}$$

6. SOME NUMERICAL EXAMPLES

1. Let $X = \mathbb{R}^2$, $x = (\xi,\eta) \in \mathbb{R}^2$, $\|x\|_2 = \sqrt{\xi^r + \eta^2}$. We consider the linear 2-dimensional system

$$\begin{cases} \dot{u}_1(t) = \left(-1 - \sin^2 t\right) u_1(t) + (-1 + \sin t \cos t) u_2(t) + e^{-t}; \\ \dot{u}_2(t) = (1 + \sin t \cos t) u_1(t) + (-1 - \cos^2 t) u_2(t) + e^{-2t}; \\ u_1(0) = u_2(0) = 0. \end{cases} \tag{6.1}$$

If we denote

$$A(t) := \begin{pmatrix} -1 - \sin^2 t & -1 + \sin t \cos t \\ 1 + \sin t \cos t & -1 - \cos^2 t \end{pmatrix}, \quad f(t) = \left(e^{-t}, e^{-2t}\right), \quad x = (0,0)$$

and we identify (ξ, η) with $\begin{pmatrix} \xi \\ \eta \end{pmatrix}$, then the above system is a Cauchy problem. The evolution family associated with $A(t)$ is

$$U(t,s) = P(t) P^{-1}(s), \quad t \geq s, \ t, s \in \mathbb{R},$$

where

$$P(t) = \begin{pmatrix} e^{-t}\cos t & e^{-2t}\sin t \\ -e^{-t}\sin t & e^{-2t}\cos t \end{pmatrix}, \quad t \in \mathbb{R}. \tag{6.2}$$

The exact solution of the system (6.1) is $u = (u_1, u_2)$, where

$$u_1(t) = \left(e^{-t}\cos t\right) E_1(t) + \left(e^{-2t}\sin t\right) E_2(t)$$

$$u_2(t) = -\left(e^{-t}\sin t\right) E_1(t) + \left(e^{-2t}\cos t\right) E_2(t), \quad t \in \mathbb{R},$$

and

$$\begin{aligned} E_1(t) &= \sin t + \frac{1}{2}e^{-t}(\cos t + \sin t) - \frac{1}{2}, \\ E_2(t) &= \sin t + \frac{1}{2}e^{t}(\sin t - \cos t) + \frac{1}{2}, \end{aligned}$$

see [2, Section 4] for details. The function $t \mapsto A(t)$ is bounded on $\mathbb{R}$ and therefore there exist $M \geq 1$ and $\omega > 0$

$$\|U(t,s)\| \leq Me^{\omega|t-s|}, \quad \text{for all } t, s \in \mathbb{R}.$$

Let $\xi \geq 0$ be fixed and $t, s \geq \xi$. Then there exists a real number μ between t and s such that

$$\|U(\xi,t) - U(\xi,s)\| = |t-s| \, \|U(\xi,\mu) A(\mu)\| \leq Me^{\omega\mu} \, |||A(\cdot)|||_{\infty} \cdot |t-s|,$$

that is, the function $\eta \mapsto U(\xi,\eta)$ is locally Lipschitz continuous on $[\xi, \infty)$.

Using (6.2), it follows

$$U(t,s) = \begin{pmatrix} a_{11}(t,s) & a_{12}(t,s) \\ a_{21}(t,s) & a_{22}(t,s) \end{pmatrix},$$

where

$$\begin{aligned} a_{11}(t,s) &= e^{(s-t)}\cos t\cos s + e^{2(s-t)}\sin t\sin s; \\ a_{12}(t,s) &= -e^{(s-t)}\cos t\sin s + \frac{1}{2}e^{2(s-t)}\sin t\cos s; \\ a_{21}(t,s) &= -e^{(s-t)}\sin t\cos s + e^{2(s-t)}\cos t\sin s; \\ a_{22}(t,s) &= e^{(s-t)}\sin t\sin s + \frac{1}{2}e^{2(s-t)}\cos t\cos s. \end{aligned}$$

Then from (5.6) we obtain the following approximating formula for $u(\cdot)$:

$$u_1(t) = -\sum_{i=0}^{n-1}\left[a_{11}\left(t, \frac{(2i+1)t}{2n}\right)\left(e^{-\frac{t(i+1)}{n}} - e^{-\frac{ti}{n}}\right) + \frac{1}{2}a_{12}\left(t, \frac{(2i+1)t}{2n}\right)\left(e^{-\frac{2t(i+1)}{n}} - e^{-\frac{2ti}{n}}\right)\right] + R_{1,n}^{(1)}$$

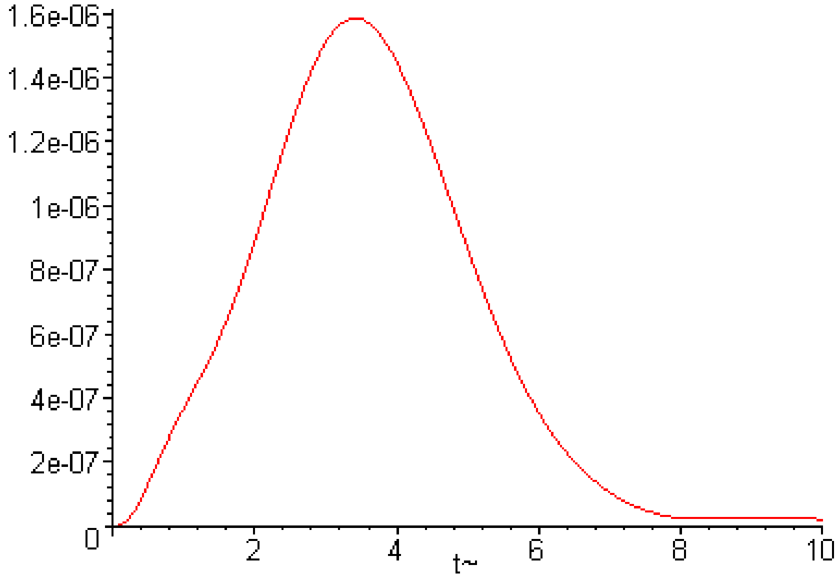

Figure 6.1: The behaviour of the error $\varepsilon_n(t) := \left\| \left(R^{(1)}_{1,n}, R^{(1)}_{2,n} \right) \right\|_2$ *for* $n = 200$.

and

$$u_2(t) = -\sum_{i=0}^{n-1} \left[a_{21}\left(t, \frac{(2i+1)t}{2n}\right)\left(e^{-\frac{t(i+1)}{n}} - e^{-\frac{ti}{n}}\right) + \frac{1}{2}a_{22}\left(t, \frac{(2i+1)t}{2n}\right)\left(e^{-\frac{2t(i+1)}{n}} - e^{-\frac{2ti}{n}}\right)\right] + R^{(1)}_{2,n},$$

where the remainder $R^{(1)}_n = \left(R^{(1)}_{1,n}, R^{(1)}_{2,n}\right)$ satisfies the estimate (5.7) with $\alpha = 1$, $H = Me^{\omega t}\,|||A(\cdot)|||_\infty$ and $|||f|||_{[0,t],\infty} \leq 2$.

Figure 6.1 contains the behaviour of the error $\varepsilon_n(t) := \left\| \left(R^{(1)}_{1,n}, R^{(1)}_{2,n} \right) \right\|_2$ for $n = 200$.

2. Let $X = \mathbb{R}$ and $U(t,s) := \frac{t+1}{s+1}$, $t \geq s \geq 0$. It is clear that the family $\{U(t,s) : t \geq s \geq 0\} \subset \mathcal{L}(\mathbb{R})$ is an exponentially bounded evolution family which solves the Cauchy problem

$$\dot{u}(t) = \frac{1}{t+1}u(t), \quad u(s) = x_s \in \mathbb{R}, \ t \geq s \geq 0.$$

Consider the inhomogeneous Cauchy problem

$$\text{(6.3)} \qquad \begin{cases} \dot{u}(t) = \frac{1}{t+1}u(t) + \cos[\ln(t+1)], & t \geq 0 \\ u(0) = 0. \end{cases}$$

The solution of (6.3) is given by

$$u(t) = \int_0^t \frac{t+1}{\tau+1}\cos(\ln(\tau+1))\,d\tau = (t+1)\sin[\ln(t+1)], \ t \geq 0.$$

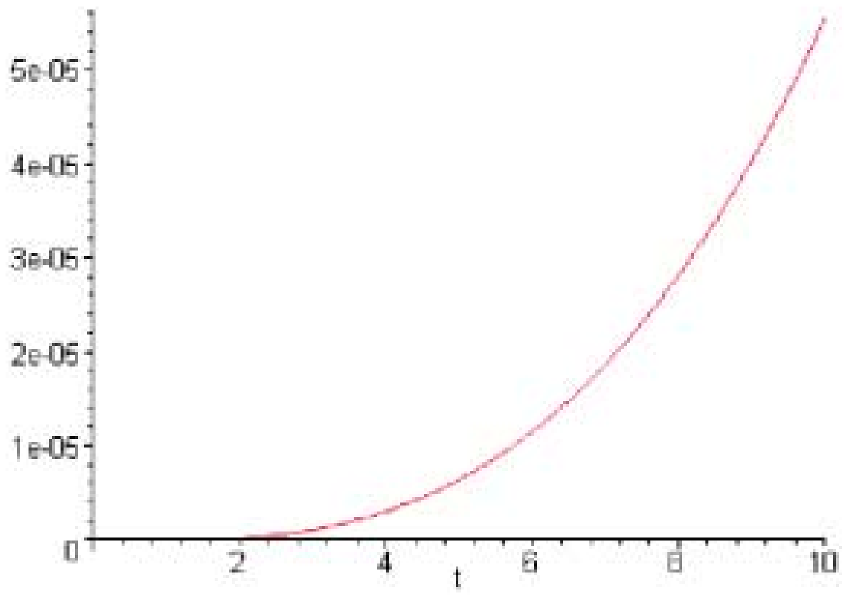

Figure 6.2: The behaviour of the error $\varepsilon_n(t) := |R_n|$ for $n = 400$.

From (5.9) we obtain the approximating formula for $u(\cdot)$ as,

$$u(t) = (t+1)\sum_{i=0}^{n-1} \ln\left[\frac{n+ti+t}{n+ti}\right]\cos\left\{\ln\left[1+\frac{(2i+1)\,t}{2n}\right]\right\} + R_n,$$

where R_n satisfies the estimate (5.10) with $K = M = \omega = \beta = 1$. Indeed,

$$\frac{t+1}{s+1} \leq e^t, \quad \text{for all } t \geq s \geq 0$$

and

$$|\cos[\ln(t+1)] - \cos[\ln(s+1)]| = |t-s|\left|\frac{1}{c+1}\sin[\ln(c+1)]\right| \leq |t-s|$$

for all $t \geq s \geq 0$, where c is some real number between s and t.

The following Figure 6.2 contains the behaviour of the error $\varepsilon_n(t) := |R_n|$ for $n = 400$.

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