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A NOTE ON AN INEQUALITY FOR THE GAMMA FUNCTION

BAOGUO JIA

School of Mathematics and Scientific Computer
Zhongshan University
Guangzhou, China, 510275.

*E*Mail: mcsjbg@zsu.edu.cn

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Abstract

By means of the convex properties of function $\ln \Gamma(x)$, we obtain a new proof of a generalization of a double inequality on the Euler gamma function, obtained by Jozsef Sándor.

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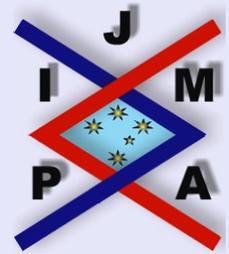
The Euler gamma function $\Gamma(x)$ is defined for $x > 0$ by

$$\Gamma(x) = \int_0^{+\infty} e^{-t} t^{x-1} dt.$$

Recently, by using a geometrical method, C. Alsina and M.S. Tomas [1] have proved the following double inequality:

Theorem 1. For all $x \in [0, 1]$ and all nonnegative integers n , one has

$$\frac{1}{n!} \leq \frac{\Gamma(1+x)^n}{\Gamma(1+nx)} \leq 1.$$



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By using a representation theorem of the “digamma function” $\frac{\Gamma'(x)}{\Gamma(x)}$, J. Sándor [2] proved the following generalized result:

Theorem 2. For all $a \geq 1$ and all $x \in [0, 1]$, one has

$$\frac{1}{\Gamma(1+a)} \leq \frac{\Gamma(1+x)^a}{\Gamma(1+ax)} \leq 1.$$

In this paper, by means of the convex properties of function $\ln \Gamma(x)$, for $0 < x < +\infty$, we will prove that

Theorem 3. For all $a \geq 1$ and all $x > -\frac{1}{a}$, one has

$$\frac{\Gamma(1+x)^a}{\Gamma(1+ax)} \leq 1.$$

(i) For all $a \geq 1$ and all $x \in [0, 1]$, one has

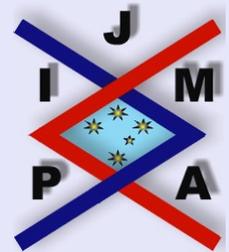
$$\frac{1}{\Gamma(1+a)} \leq \frac{\Gamma(1+x)^a}{\Gamma(1+ax)}.$$

(ii) For all $a \geq 1$ and all $x \geq 1$, one has

$$\frac{1}{\Gamma(1+a)} \geq \frac{\Gamma(1+x)^a}{\Gamma(1+ax)}.$$

(iii) For all $a \in [0, 1]$ and all $x \in [0, 1]$, one has

$$\frac{1}{\Gamma(1+a)} \geq \frac{\Gamma(1+x)^a}{\Gamma(1+ax)}.$$



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(iv) For all $a \in [0, 1]$ and all $x \geq 1$, one has

$$\frac{1}{\Gamma(1+a)} \leq \frac{\Gamma(1+x)^a}{\Gamma(1+ax)}.$$

Our method is elementary. We only need the following simple lemma, see [3].

Lemma 4.

(a) $\Gamma(x+1) = x\Gamma(x)$, for $0 < x < +\infty$.

(b) $\Gamma(n+1) = n!$, for $n = 1, 2, \dots$

(c) $\ln \Gamma(x)$ is convex on $(0, +\infty)$.

Proof of Theorem 3. When $a = 1$, it is obvious.

When $a > 1$, by (c) of Lemma 4, we have

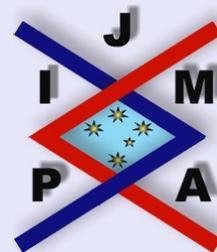
$$\Gamma\left(\frac{u}{p} + \frac{v}{q}\right) \leq \Gamma(u)^{\frac{1}{p}} \Gamma(v)^{\frac{1}{q}},$$

where $p > 1, q > 1, \frac{1}{p} + \frac{1}{q} = 1, u > 0, v > 0$.

Let $p = a, q = \frac{a}{a-1}$. Then

$$\Gamma\left(\frac{1}{a}u + \left(1 - \frac{1}{a}v\right)\right) \leq \Gamma(u)^{\frac{1}{a}} \Gamma(v)^{1-\frac{1}{a}},$$

for $u > 0, v > 0$.



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Let $v = 1$, $u = ax + 1$. Note that $\Gamma(1) = 1$, $\frac{1}{a}u + (1 - \frac{1}{a}v) = x + 1$.

We obtain

$$\Gamma(x + 1) \geq \Gamma(ax + 1)^{\frac{1}{a}}, \quad \text{for } x = \frac{u - 1}{a} > -\frac{1}{a}.$$

□

Remark 1. *Theorem 3 is a generalization of the right side inequality of Theorem 2.*

Proof of Theorem 3.

(i) Let

$$f(x) = \ln \Gamma(ax + 1) - \ln \Gamma(1 + a) - a \ln \Gamma(x + 1).$$

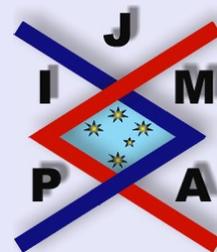
Since $\Gamma(2) = 1$, We have $f(1) = 0$.

$$f'(x) = a \left(\frac{\Gamma'(ax + 1)}{\Gamma(ax + 1)} - \frac{\Gamma'(x + 1)}{\Gamma(x + 1)} \right)$$

Set $h(t) = \ln \Gamma(t)$. By (c) of the Lemma 4, $\ln \Gamma(x)$ is convex on $(0, +\infty)$. So $(\ln \Gamma(t))'' \geq 0$. That is $\left(\frac{\Gamma'(t)}{\Gamma(t)}\right)' \geq 0$. Therefore $\left(\frac{\Gamma'(t)}{\Gamma(t)}\right)$ is increasing. Because $a \geq 1$ and $x \in [0, 1]$, one has $ax + 1 \geq x + 1$. So

$$\frac{\Gamma'(ax + 1)}{\Gamma(ax + 1)} \geq \frac{\Gamma'(x + 1)}{\Gamma(x + 1)}$$

Thus $f'(x) \geq 0$. In addition to $f(1) = 0$, we obtain that $f(x) \leq 0$, for $a \geq 1$ and $x \in [0, 1]$.



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So (i) is proved.

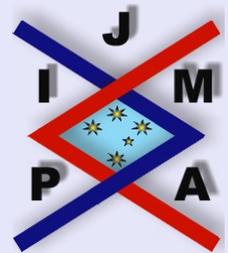
Note that

$$ax + 1 \geq x + 1, \quad \text{for } a \geq 1 \quad \text{and } x \geq 1;$$

$$ax + 1 \leq x + 1, \quad \text{for } a \in [0, 1] \quad \text{and } x \in [0, 1];$$

$$ax + 1 \leq x + 1, \quad \text{for } a \in [0, 1] \quad \text{and } x \geq 1.$$

So (ii), (iii), (iv) are obvious. □



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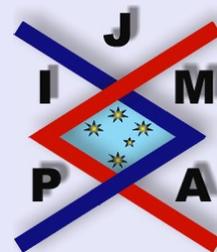
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