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A NOTE ON AN INEQUALITY FOR THE GAMMA FUNCTION

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ABSTRACT. By means of the convex properties of function $\ln\Gamma(x)$, we obtain a new proof of a generalization of a double inequality on the Euler gamma function, obtained by Jozsef Sándor.

Key words and phrases: Gamma function, Inequalities, Convex function.

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The Euler gamma function $\Gamma(x)$ is defined for $x > 0$ by

$$\Gamma(x) = \int_0^{+\infty} e^{-t} t^{x-1} dt.$$

Recently, by using a geometrical method, C. Alsina and M.S. Tomas [1] have proved the folowing double inequality:

Theorem 1. *For all $x \in [0,1]$ and all nonnegative integers n, one has*

$$\frac{1}{n!} \le \frac{\Gamma(1+x)^n}{\Gamma(1+nx)} \le 1.$$

By using a representation theorem of the "digamma function" $\frac{\Gamma'(x)}{\Gamma(x)}$, J. Sándor [2] proved the following generalized result:

Theorem 2. *For all $a \ge 1$ and all $x \in [0,1]$, one has*

$$\frac{1}{\Gamma(1+a)} \le \frac{\Gamma(1+x)^a}{\Gamma(1+ax)} \le 1.$$

In this paper, by means of the convex properties of function $\ln\Gamma(x)$, for $0 < x < +\infty$, we will prove that

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Theorem 3. *For all* $a \geq 1$ *and all* $x > -\frac{1}{a}$*, one has*

$$\frac{\Gamma(1+x)^a}{\Gamma(1+ax)} \leq 1.$$

(i) *For all* $a \geq 1$ *and all* $x \in [0,1]$*, one has*

$$\frac{1}{\Gamma(1+a)} \leq \frac{\Gamma(1+x)^a}{\Gamma(1+ax)}.$$

(ii) *For all* $a \geq 1$ *and all* $x \geq 1$*, one has*

$$\frac{1}{\Gamma(1+a)} \geq \frac{\Gamma(1+x)^a}{\Gamma(1+ax)}.$$

(iii) *For all* $a \in [0,1]$ *and all* $x \in [0,1]$*, one has*

$$\frac{1}{\Gamma(1+a)} \geq \frac{\Gamma(1+x)^a}{\Gamma(1+ax)}.$$

(iv) *For all* $a \in [0,1]$ *and all* $x \geq 1$*, one has*

$$\frac{1}{\Gamma(1+a)} \leq \frac{\Gamma(1+x)^a}{\Gamma(1+ax)}.$$

Our method is elementary. We only need the following simple lemma, see [3].

Lemma 4.

(a) $\Gamma(x+1) = x\Gamma(x),$ *for* $0 < x < +\infty.$
(b) $\Gamma(n+1) = n!,$ *for* $n = 1, 2, \dots.$
(c) $\ln\Gamma(x)$ *is convex on* $(0, +\infty).$

Proof of Theorem 3. When $a = 1$, it is obvious.

When $a > 1$, by (c) of Lemma 4, we have

$$\Gamma\left(\frac{u}{p} + \frac{v}{q}\right) \leq \Gamma(u)^{\frac{1}{p}}\Gamma(v)^{\frac{1}{q}},$$

where $p > 1, q > 1, \frac{1}{p} + \frac{1}{q} = 1, u > 0, v > 0.$

Let $p = a$, $q = \frac{a}{a-1}$. Then

$$\Gamma\left(\frac{1}{a}u + \left(1 - \frac{1}{a}v\right)\right) \leq \Gamma(u)^{\frac{1}{a}}\Gamma(v)^{1-\frac{1}{a}},$$

for $u > 0, v > 0.$

Let $v = 1$, $u = ax + 1$. Note that $\Gamma(1) = 1$, $\frac{1}{a}u + (1 - \frac{1}{a}v) = x + 1.$

We obtain

$$\Gamma(x+1) \geq \Gamma(ax+1)^{\frac{1}{a}}, \qquad \text{for} \quad x = \frac{u-1}{a} > -\frac{1}{a}.$$

□

Remark 5. Theorem 3 is a generalization of the right side inequality of Theorem 2.

Proof of Theorem 3.

(i) Let

$$f(x) = \ln\Gamma(ax+1) - \ln\Gamma(1+a) - a\ln\Gamma(x+1).$$

Since $\Gamma(2) = 1$, We have $f(1) = 0.$

$$f'(x) = a\left(\frac{\Gamma'(ax+1)}{\Gamma(ax+1)} - \frac{\Gamma'(x+1)}{\Gamma(x+1)}\right)$$

Set $h(t) = \ln\Gamma(t)$. By (c) of the Lemma 4, $\ln\Gamma(x)$ is convex on $(0, +\infty)$. So $(\ln\Gamma(t))'' \geq 0$. That is $\left(\frac{\Gamma'(t)}{\Gamma(t)}\right)' \geq 0$. Therefore $\left(\frac{\Gamma'(t)}{\Gamma(t)}\right)$ is increasing. Because $a \geq 1$ and $x \in [0,1]$, one has $ax + 1 \geq x + 1$. So

$$\frac{\Gamma'(ax+1)}{\Gamma(ax+1)} \geq \frac{\Gamma'(x+1)}{\Gamma(x+1)}$$

Thus $f'(x) \geq 0$. In addition to $f(1) = 0$, we obtain that $f(x) \leq 0$, for $a \geq 1$ and $x \in [0,1]$.

So (i) is proved.

Note that

$$ax+1 \geq x+1, \quad \text{for} \quad a \geq 1 \quad \text{and} \quad x \geq 1;$$
$$ax+1 \leq x+1, \quad \text{for} \quad a \in [0,1] \quad \text{and} \quad x \in [0,1];$$
$$ax+1 \leq x+1, \quad \text{for} \quad a \in [0,1] \quad \text{and} \quad x \geq 1.$$

So (ii), (iii), (iv) are obvious. □

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