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PREDICTOR-CORRECTOR METHODS FOR GENERALIZED GENERAL MULTIVALUED MIXED QUASI VARIATIONAL INEQUALITIES

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ABSTRACT. In this paper, a class of generalized general mixed quasi variational inequalities is introduced and studied. We prove the existence of the solution of the auxiliary problem for the generalized general mixed quasi variational inequalities, suggest a predictor-corrector method for solving the generalized general mixed quasi variational inequalities by using the auxiliary principle technique. If the bi-function involving the mixed quasi variational inequalities is skew-symmetric, then it is shown that the convergence of the new method requires the partially relaxed strong monotonicity property of the operator, which is a weak condition than cocoercivity. Our results can be viewed as an important extension of the previously known results for variational inequalities.

Key words and phrases: Variational inequalities, Auxiliary principle, Predictor-corrector method, Convergence.

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1. INTRODUCTION

In recent years, variational inequalities have been generalized and extended in many different directions using novel and innovative techniques to study wider classes of unrelated problems in mechanics, physics, optimization and control, nonlinear programming, economics, regional, structural, transportation, elasticity, and applied sciences, etc., see [1] – [8] and the references therein. An important and useful generalization of variational inequalities is called the general mixed quasi variational inequality involving the nonlinear bifunction. It is well-known that due to the presence of the nonlinear bifunction, projection method and its variant forms including the Wiener-Hopf equations, descent methods cannot be extended to suggest iterative methods for solving the general mixed quasi variational inequalities. In particular, it has been shown that if the nonlinear bifunction is proper, convex and lower semicontinuous with respect to the first argument, then the general mixed quasi variational inequalities are equivalent to the fixed-point

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problems. This equivalence has been used to suggest and analyze some iterative methods for solving the general mixed quasi variational inequalities. In this approach, one has to evaluate the resolvent of the operator, which is itself a difficult problem. To overcome these difficulties, Glowinski et al. [6] suggested another technique, which is called the auxiliary principle technique. Recently, Noor [1] extended the auxiliary principle technique to suggest and analyze a new predictor-corrector method for solving general mixed quasi variational inequalities. However, the main results in [1, Algorithm 3.1, Lemma 3.1 and Theorem 3.1] are wrong. Also, Algorithm 3.1 in [1] is based on the assumption that auxiliary problem has a solution, but the author did not show the existence of the solution for this auxiliary problem. On the other hand, in 1999, Huang et al. [7] modified and extended the auxiliary principle technique to study the existence of a solution for a class of generalized set-valued strongly nonlinear implicit variational inequalities and suggested some general iterative algorithms. Inspired and motivated by recent research going on in this fascinating and interesting field, in this paper, a class of generalized general mixed quasi variational inequalities is introduced and studied, which includes the general mixed quasi variational inequality as a special case. We prove the existence of the solution of the auxiliary problem for the generalized general mixed quasi variational inequalities, and suggest a predictor-corrector method for solving the generalized general mixed quasi variational inequalities by using the auxiliary principle technique. If the bi-function involving the mixed quasi variational inequalities is skew-symmetric, then it is shown that the convergence of the new method requires the partially relaxed strong monotonicity property of the operator, which is a weaker condition than cocoercivity. Our results extend, improve and modify the main results of Noor [1].

2. PRELIMINARIES

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$and$\|\cdot\|$, respectively. Let CB(H) be the family of all nonempty closed and bounded sets in H. Let K be a nonempty closed convex set in H. Let $\varphi(\cdot,\cdot): H\times H\to H$ be a nondifferentiable nonlinear bifunction. For given nonlinear operators $N(\cdot,\cdot): H\times H\to H$,$g: H\to H$ and two set-valued operators $T, V: H\to CB(H)$, consider the problem of finding $u\in H, w\in T(u), y\in V(u)$ such that

$$\langle N(w,y), g(v)-g(u)\rangle+\varphi(g(v),g(u))-\varphi(g(u),g(u))\geq 0, \forall g(v)\in H. \tag{2.1}$$

The inequality of type (2.1) is called the generalized general multivalued mixed quasi variational inequality.

For a suitable and appropriate choice of the operators N, g, φ and the space H, one can obtain a wide class of variational inequalities and complementarity problems, see [1]. Furthermore, problem (2.1) has important applications in various branches of pure and applied sciences.

Lemma 2.1. *For all $u, v\in H$, we have*

$$2\langle u,v\rangle=\|u+v\|^2-\|u\|^2-\|v\|^2. \tag{2.2}$$

Definition 2.1. For all $u_1, u_2, z\in H, x_1\in T(u_1), x_2\in T(u_2)$, an operator$N(\cdot,\cdot)$ is said to be:

(i) g-partially relaxed strongly monotone with respect to the first argument, if there exists a constant $\alpha>0$ such that

$$\langle N(x_1,\cdot)-N(x_2,\cdot), g(z)-g(u_2)\rangle\geq -\alpha\|g(u_1)-g(z)\|^2.$$

(ii) g-cocoercive with respect to the first argument, if there exists a constant $\mu > 0$ such that

$$\langle N(x_1,\cdot) - N(x_2,\cdot), g(u_1) - g(u_2)\rangle \geq \mu\|N(x_1,\cdot) - N(x_2,\cdot)\|^2.$$

(iii) T is said to be M-lipschitz continuous, if there exists a constant $\delta > 0$ such that

$$M(T(u_1), T(u_2)) \leq \delta\|u_1 - u_2\|,$$

where $M(\cdot,\cdot)$is the Hausdorff metric on CB(H).

We remark that if $N(x,\cdot) \equiv Tx$, then Definition 2.1 is exactly the Definition 2.1 of Noor and Memon [1]. If $z = u_1, N(x,\cdot) \equiv Tx$, g-partially relaxed strongly monotone with respect to the first argument of $N(\cdot,\cdot)$ is exactly g-monotone of T, and g- cocoercive implies g- partially relaxed strongly monotone [3]. This shows that g-partially relaxed strongly monotone with respect to the first argument of $N(\cdot,\cdot)$ is a weaker condition than g-cocoercive with respect to the first argument of $N(\cdot,\cdot)$.

Definition 2.2. For all $u, v \in H$,the bifunction $\varphi(\cdot,\cdot)$ is said to be skew-symmetric, if

$$\varphi(u,u) - \varphi(u,v) - \varphi(v,u) + \varphi(v,v) \geq 0.$$

Note that if the bifunction $\varphi(\cdot,\cdot)$ is linear in both arguments, then it is nonnegative.

In order to obtain our results, we need the following assumption.

Assumption 2.2. The mappings $N(\cdot,\cdot) : H \times H \to H$, $g : H \to H$ satisfy the following conditions:

(1) for all $w, y \in H$, there exists a constant $\tau > 0$ such that $\|N(w,y)\| \leq \tau(\|w\| + \|y\|)$;
(2) for a given $x \in H$, mapping $v \mapsto < x, g(v) >$ is convex;
(3) $\varphi(u,v)$is bounded, that is, there exists a constant $\gamma > 0$ such that

$$|\varphi(u,v)| \leq \gamma\|u\|\|v\|, \forall u, v \in H;$$

(4) $\varphi(u,v)$ is linear with respect to u.
(5) $\varphi(\cdot,\cdot)$ is continuous and $\varphi(g(\cdot),\cdot)$ is convex with respect to the first argument.

Remark 2.3. If $g \equiv I$, it is easy to see that the conditions (2), (5) in Assumption 2.2 can be easily satisfied.

We also need the following lemma.

Lemma 2.4. [4, 5]. *Let X be a nonempty closed convex subset of Hausdorff linear topological space E, $\phi, \psi : X \times X \to R$ be mappings satisfying the following conditions:*

(1) $\psi(x,y) \leq \phi(x,y), \forall x, y \in X$;
(2) *for each $x \in X$, $\phi(x,y)$ is upper semicontinuous with respect to y;*
(3) *for each $y \in X$, the set $\{x \in X : \psi(x,y) < 0\}$ is a convex set;*
(4) *there exist a nonempty compact set $K \subset X$ and $x_0 \in K$ such that $\psi(x_0,y) < 0$, for any $y \in X \setminus K$. Then there exists a $\overline{y} \in K$ such that $\phi(x,\overline{y}) \geq 0, \forall x \in X$.*

3. MAIN RESULTS

In this section, we give an existence theorem of a solution of the auxiliary problem for the generalized general set-valued quasi variational inequality (2.1). Based on this existence theorem, we suggest and analyze a new iterative method for solving the problem (2.1).

For given $u \in H$, $w \in Tu$, $y \in Vu$, consider the problem of finding a unique $z \in H$ satisfying the auxiliary general mixed quasi variational inequality

$$\langle \rho N(w,y) + g(z) - g(u), g(v) - g(z)\rangle + \rho\varphi(g(v), g(z)) - \rho\varphi(g(z), g(z)) \geq 0, \tag{3.1}$$

for all $v \in H$, where $\rho > 0$ is a constant.

Remark 3.1. We note that if $z = u$, then clearly z is a solution of (2.1).

Theorem 3.2. *If Assumption 2.2 holds, $g : H \to H$ is invertible and Lipschitz continuous, and $0 < \rho\gamma < 1$, then $P(u, w, y)$ has a solution.*

Proof. Define $\phi, \psi : H \times H \to H$ by

$$\begin{aligned}\phi(v,z) = \langle g(v), g(v) - g(z)\rangle - \langle g(u), g(v) - g(z)\rangle + \rho\,\langle N(w,y), g(v) - g(z)\rangle \\ - \rho\varphi(g(z), g(z)) + \rho\varphi(g(v), g(z))\end{aligned}$$

and

$$\begin{aligned}\psi(v,z) = \langle g(z), g(v) - g(z)\rangle - \langle g(u), g(v) - g(z)\rangle + \rho\,\langle N(w,y), g(v) - g(z)\rangle \\ - \rho\varphi(g(z), g(z)) + \rho\varphi(g(v), g(z)),\end{aligned}$$

respectively. Now we show that the mappings ϕ, ψ satisfy all the conditions of Lemma 2.4.

Clearly, ϕ and ψ satisfy condition (1) of Lemma 2.4. It follows from Assumption 2.2(5) that $\phi(v, z)$ is upper semicontinuous with respect to z. By using Assumption 2.2 (2) and (5), it is easy to show that the set $\{v \in H | \psi(v, z) < 0\}$ is a convex set for each fixed $z \in H$ and so the conditions (2) and (3) of Lemma 2.4 hold.

Now let

$$\omega = \|g(u)\| + \rho\tau(\|w\| + \|y\|), K = \{z \in H : (1 - \rho\gamma)\|g(z)\| \le \omega\}.$$

Since $g : H \to H$ is invertible, K is a weakly compact subset of H. For any fixed $z \in H \setminus K$, take $v_0 \in K$ such that $g(v_0) = 0$. From Assumption 2.2, we have

$$\begin{aligned}\psi(v_0, z) &= -\langle g(z), g(z)\rangle + \langle g(u), g(z)\rangle + \rho\,\langle N(w,y), -g(z)\rangle - \rho\varphi(g(z), g(z)) \\ &\le -\|g(z)\|^2 + \|g(u)\|\|g(z)\| + \rho\tau(\|w\| + \|y\|)\|g(z)\| + \rho\gamma\|g(z)\|^2 \\ &= -\|g(z)\|(\|g(z)\| - \|g(u)\| - \rho\tau(\|w\| + \|y\|) - \rho\gamma\|g(z)\|) \\ &< 0.\end{aligned}$$

Therefore, the condition (4) of Lemma 2.4 holds. By Lemma 2.4, there exists a $\overline{z} \in H$ such that $\phi(v, \overline{z}) \ge 0$, for all $v \in H$, that is,

$$\begin{aligned}\langle g(v), g(v) - g(\overline{z})\rangle - \langle g(u), g(v) - g(\overline{z})\rangle + \rho\,\langle N(w,y), g(v) - g(\overline{z})\rangle \\ - \rho\varphi(g(\overline{z}), g(\overline{z})) + \rho\varphi(g(v), g(\overline{z})) \ge 0, \forall v \in H.\end{aligned} \tag{3.2}$$

For arbitrary $t \in (0, 1)$ and $v \in H$, let $g(x_t) = tg(v) + (1 - t)g(\overline{z})$. Replacing v by x_t in (3.2), we obtain

$$\begin{aligned}0 &\le \langle g(x_t), g(x_t) - g(\overline{z})\rangle - \langle g(u), g(x_t) - g(\overline{z})\rangle + \rho\,\langle N(w,y), g(x_t) - g(\overline{z})\rangle \\ &\quad - \rho\varphi(g(\overline{z}), g(\overline{z})) + \rho\varphi(g(x_t), g(\overline{z})) \\ &= t(\langle g(x_t), g(v) - g(\overline{z})\rangle - \langle g(u), g(v) - g(\overline{z})\rangle) + \rho t\,\langle N(w,y), g(v) - g(\overline{z})\rangle \\ &\quad + \rho t\varphi(g(v) - g(\overline{z}), g(\overline{z})).\end{aligned}$$

Hence

$$\begin{aligned}\langle g(x_t), g(v) - g(\overline{z})\rangle - \langle g(u), g(v) - g(\overline{z})\rangle + \rho\,\langle N(w,y), g(v) - g(\overline{z})\rangle \\ + \rho\varphi(g(v), g(\overline{z})) - \rho\varphi(g(\overline{z}), g(\overline{z})) \ge 0\end{aligned}$$

and so

$$\langle g(x_t), g(v) - g(\overline{z})\rangle \geq \langle g(u), g(v) - g(\overline{z})\rangle - \rho \langle N(w,y), g(v) - g(\overline{z})\rangle - \rho\varphi(g(v), g(\overline{z})) + \rho\varphi(g(\overline{z}), g(\overline{z})).$$

Letting $t \to 0$, we have

$$\langle g(\overline{z}), g(v) - g(\overline{z})\rangle \geq \langle g(u), g(v) - g(\overline{z})\rangle - \rho \langle N(w,y), g(v) - g(\overline{z})\rangle - \rho\varphi(g(v), g(\overline{z})) + \rho\varphi(g(\overline{z}), g(\overline{z})).$$

Therefore, $\overline{z} \in H$ is a solution of the auxiliary problem $P(u, w, y)$. This completes the proof. □

By using Theorem 3.2, we now suggest the following iterative method for solving the generalized general set-valued quasi variational inequality (2.1).

Algorithm 3.1. For given $u_0 \in H, \xi_0 \in Tu_0, \eta_0 \in Vu_0$, compute the approximate solution u_{n+1} by the iterative scheme

$$x_n \in T(w_n) : \|x_{n+1} - x_n\| \leq M(T(w_{n+1}), T(w_n)),$$

$$y_n \in V(w_n) : \|y_{n+1} - y_n\| \leq M(V(w_{n+1}), V(w_n)),$$

$$\text{(3.3)} \quad \langle \rho N(x_n, y_n) + g(u_{n+1}) - g(w_n), g(v) - g(u_{n+1})\rangle + \rho\varphi(g(v), g(u_{n+1})) - \rho\varphi(g(u_{n+1}), g(u_{n+1})) \geq 0, \quad \forall v \in H,$$

and

$$\xi_n \in T(u_n) : \|\xi_{n+1} - \xi_n\| \leq M(T(u_{n+1}), T(u_n)),$$

$$\eta_n \in V(u_n) : \|\eta_{n+1} - \eta_n\| \leq M(V(u_{n+1}), V(u_n)),$$

$$\text{(3.4)} \quad \langle \beta N(\xi_n, \eta_n) + g(w_n) - g(u_n), g(v) - g(w_n)\rangle + \beta\varphi(g(v), g(w_n)) - \beta\varphi(g(w_n), g(w_n)) \geq 0, \quad \forall v \in H,$$

where $\rho > 0, \beta > 0$ are constants.

For the convergence analysis of Algorithm 3.1, we need the following result.

Lemma 3.3. *Let $u \in H$, $x \in Tu$, $y \in Vu$ be the exact solution of (2.1) and u_{n+1} be the approximate solution obtained from Algorithm 3.1. If the operator $N(\cdot,\cdot)$ is g- partially relaxed strongly monotone with respect to the first and second argument with constants $a > 0, b > 0$, respectively, the bifunction $\varphi(\cdot,\cdot)$ is skew-symmetric and the conditions in Theorem 3.2 are satisfied, then*

$$\text{(3.5)} \quad \|g(u_{n+1}) - g(u)\|^2 \leq \|g(u_n) - g(u)\|^2 - (1 - 2\rho(a+b))\|g(u_{n+1}) - g(u_n)\|^2.$$

Proof. Let $u \in H, x \in Tu, y \in Vu$ be a solution of (2.1). Then

$$\text{(3.6)} \quad \langle \rho N(x,y), g(v) - g(u)\rangle + \rho\varphi(g(v), g(u)) - \rho\varphi(g(u), g(u)) \geq 0, \forall v \in H,$$

$$\text{(3.7)} \quad \langle \beta N(x,y), g(v) - g(u)\rangle + \beta\varphi(g(v), g(u)) - \beta\varphi(g(u), g(u)) \geq 0, \forall v \in H,$$

where $\rho > 0, \beta > 0$ are constants. Now taking $v = u_{n+1}$ in (3.6) and $v = u$ in (3.3), we have

$$\text{(3.8)} \quad \langle \rho N(x,y), g(u_{n+1}) - g(u)\rangle + \rho\varphi(g(u_{n+1}), g(u)) - \rho\varphi(g(u), g(u)) \geq 0,$$

$$\langle \rho N(x_n, y_n) + g(u_{n+1}) - g(w_n), g(u) - g(u_{n+1})\rangle + \rho\varphi(g(u), g(u_{n+1})) - \rho\varphi(g(u_{n+1}), g(u_{n+1})) \geq 0. \tag{3.9}$$

Adding (3.8) and (3.9), we have (3.10)

$$\begin{aligned} \langle g(u_{n+1}) - g(w_n), g(u) - g(u_{n+1})\rangle &\geq \rho \langle N(x_n, y_n) - N(x, y), g(u_{n+1}) - g(u)\rangle + \rho\{\varphi(g(u), g(u)) \\ &\quad - \varphi(g(u), g(u_{n+1})) - \varphi(g(u_{n+1}), g(u)) + \varphi(g(u_{n+1}), g(u_{n+1}))\} \\ &\geq \rho \langle N(x_n, y_n) - N(x_n, y), g(u_{n+1}) - g(u)\rangle \\ &\quad + \rho \langle N(x_n, y) - N(x, y), g(u_{n+1}) - g(u)\rangle \\ &\geq -\rho(a+b)\|g(w_n) - g(u_{n+1})\|^2, \end{aligned} \tag{3.10}$$

where we have used the fact that $N(\cdot,\cdot)$is g- partially relaxed strongly monotone with respect to the first and second argument with constants $a > 0, b > 0$, respectively, and the bifunction $\varphi(\cdot,\cdot)$ is skew-symmetric. Setting $u = g(u) - g(u_{n+1}), v = g(u_{n+1}) - g(w_n)$ in (2.2), we obtain

$$\langle g(u_{n+1}) - g(w_n), g(u) - g(u_{n+1})\rangle = \frac{1}{2}\{\|g(u) - g(w_n)\|^2 - \|g(u_{n+1}) - g(w_n)\|^2 - \|g(u) - g(u_{n+1})\|^2\}. \tag{3.11}$$

Combining (3.10) and (3.11), we have

$$\|g(u_{n+1}) - g(u)\|^2 \leq \|g(w_n) - g(u)\|^2 - (1 - 2\rho(a+b))\|g(u_{n+1}) - g(w_n)\|^2, \tag{3.12}$$

Similarly, we have

$$\begin{aligned} \|g(u) - g(w_n)\|^2 &\leq \|g(u_n) - g(u)\|^2 - (1 - 2\beta(a+b))\|g(u_n) - g(w_n)\|^2, \\ &\leq \|g(u_n) - g(u)\|^2, 0 < \beta < 1/2(a+b). \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} \|g(u_{n+1}) - g(w_n)\|^2 &= \|g(u_{n+1}) - g(u_n) + g(u_n) - g(w_n)\|^2 \\ &= \|g(u_{n+1}) - g(u_n)\|^2 + \|g(u_n) - g(w_n)\|^2 \\ &\quad + 2\langle g(u_{n+1}) - g(u_n), g(u_n) - g(w_n)\rangle. \end{aligned} \tag{3.14}$$

Combining (3.12) – (3.14), we have

$$\|g(u_{n+1}) - g(u)\|^2 \leq \|g(u_n) - g(u)\|^2 - (1 - 2\rho(a+b))\|g(u_{n+1}) - g(u_n)\|^2.$$

The required result. □

Theorem 3.4. *Let H be finite dimensional, $g : H \to H$ be invertible, g^{-1} is Lipschitz continuous and $0 < \rho < \frac{1}{2}(a+b)$. Let $\{u_n\}, \{\xi_n\}, \{\eta_n\}$ be the sequences obtained from Algorithm 3.1, $u \in H$ be the exact solution of (2.1) and the conditions in Lemma 3.3 are satisfied, then $\{u_n\}, \{\xi_n\}$, and $\{\eta_n\}$ strongly converge to a solution of (2.1).*

Proof. Let $u \in H$ be a solution of (2.1). Since $0 < \rho < \frac{1}{2}(a+b)$, from (3.5), it follows that the sequence $\{\|g(u) - g(u_n)\|\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. Furthermore, we have

$$\Sigma(1 - 2\rho(a+b))\|g(u_{n+1}) - g(u_n)\|^2 \leq \|g(u_0) - g(u)\|^2,$$

which implies that

$$\lim_{n\to\infty} \|g(u_{n+1}) - g(u_n)\| = 0. \tag{3.15}$$

Let $\hat{u}$ be the cluster point of $\{u_n\}$ and the subsequence $\{u_{n_j}\}$ of the sequence $\{u_n\}$ converge to $\hat{u}$, which implies $\{u_{n_j}\}$ is a Cauchy sequence in H. By (3.4), we know that both $\{\xi_{n_j}\}$ and $\{\eta_{n_j}\}$ are Cauchy sequences in H. Let $\xi_{n_j} \to \hat{x}$ and $\eta_{n_j} \to \hat{y}$. Since

$$d(\hat{x}, T(\hat{u})) \leq \|\hat{x} - \xi_{n_j}\| + M(T(u_{n_j}), T(\hat{u})) \to 0, \quad n_j \to \infty.$$

So we can obtain $\hat{x} \in T(\hat{u})$. Similarly, we can obtain $\hat{y} \in V(\hat{u})$. Replacing w_n by u_{n_j} in (3.3) and (3.4), the limit $n_j \to \infty$ and using (3.14), we have

$$\langle N(\hat{x}, \hat{y}), g(v) - g(\hat{u})\rangle + \varphi(g(v), g(\hat{u})) - \varphi(g(\hat{u}), g(\hat{u})) \geq 0, \qquad \forall v \in H,$$

which implies that $\hat{u} \in H,\ \hat{x} \in T\hat{u},\ \hat{y} \in V\hat{u}$ is a solution of (2.1), and

$$\|g(u_{n+1} - g(u))\|^2 \leq \|g(u_n) - g(u)\|^2.$$

Thus it follows from the above inequality that the sequence$\{u_n\}$ has exactly one cluster point $\hat{u}$ and $\lim_{n\to\infty} g(u_n) = g(\hat{u})$. Since g is invertible and g^{-1} is Lipschitz continuous, $\lim_{n\to\infty} u_n = \hat{u}$. The required result. □

Remark 3.5. Lemma 3.3 and Theorem 3.4 improve and modify the main results of Noor [1].

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