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APPROXIMATION OF FIXED POINTS OF ASYMPTOTICALLY DEMICONTRACTIVE MAPPINGS IN ARBITRARY BANACH SPACES

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Abstract

Let E be a real Banach Space and K a nonempty closed convex (not necessarily bounded) subset of E. Iterative methods for the approximation of fixed points of asymptotically demicontractive mappings $T : K \to K$ are constructed using the more general modified Mann and Ishikawa iteration methods with errors.

Our results show that a recent result of Osilike [3] (which is itself a generalization of a theorem of Qihou [4]) can be extended from real q-uniformly smooth Banach spaces, $1 < q < \infty$, to arbitrary real Banach spaces, and to the more general Modified Mann and Ishikawa iteration methods with errors. Furthermore, the boundedness assumption imposed on the subset K in ([3, 4]) are removed in our present more general result. Moreover, our iteration parameters are independent of any geometric properties of the underlying Banach space.

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Contents

1. Introduction

Let E be an arbitrary real Banach space and let J denote the normalized duality mapping from E into 2^{E^*} given by $J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}$, where E^* denotes the dual space of E and $\langle\ ,\ \rangle$ denotes the generalized duality pairing. If E^* is strictly convex, then J is single-valued. In the sequel, we shall denote the single-valued duality mapping by j.

Let K be a nonempty subset of E. A mapping $T : K \to K$ is called *k-strictly asymptotically pseudocontractive* mapping, with sequence $\{k_n\} \subseteq [1, \infty)$, $\lim_n k_n = 1$ (see for example [3, 4]), if for all $x, y \in K$ there exists $j(x-y) \in J(x-y)$ and a constant $k \in [0, 1)$ such that

$$\text{(1.1)}\quad \langle (I-T^n)x - (I-T^n)y, j(x-y)\rangle \geq \frac{1}{2}(1-k)\,\|(I-T^n)x-(I-T^n)y\|^2 - \frac{1}{2}(k_n^2-1)\|x-y\|^2,$$

for all $n \in \mathbb{N}$. T is called an *asymptotically demicontractive* mapping with sequence $k_n \subseteq [0, \infty)$, $\lim_n k_n = 1$ (see for example [3, 4]) if $F(T) = \{x \in K : Tx = x\} \neq \emptyset$ and for all $x \in K$ and $x^* \in F(T)$, there exists $k \in [0, 1)$ and $j(x - x^*) \in J(x-x^*)$ such that

$$\text{(1.2)}\quad \langle x - T^n x, j(x-x^*)\rangle \geq \frac{1}{2}(1-k)\|x-T^n x\|^2 - \frac{1}{2}(k_n^2-1)\|x-x^*\|^2$$

for all $n \in \mathbb{N}$. Furthermore, T is *uniformly L-Lipschitzian*, if there exists a constant $L > 0$, such that

$$\text{(1.3)}\quad \|T^n x - T^n y\| \leq L\|x-y\|,$$

for all $x\,, y \in K$ and $n \in \mathbb{N}$.

It is clear that a k-strictly asymptotically pseudocontrative mapping with a nonempty fixed point set $F(T)$ is asymptotically demicontrative. The classes of k-strictly asymptotically pseudocontractive and asymptotically demicontractive maps were first introduced in Hilbert spaces by Qihou [4]. In a Hilbert space, j is the identity and it is shown in [3] that (1.1) and (1.2) are respectively equivalent to the inequalities:

$$\|T^n x - T^n y\| \leq k_n^2 \|x - y\|^2 + k\|(I - T^n)x - (I - T^n)y\|^2 \tag{1.4}$$

and

$$\|T^n x - T^n y\|^2 \leq k_n^2 \|x - y\|^2 + \|x - T^n x\|^2 \tag{1.5}$$

which are the inequalities considered by Qihou [4].

In [4], Qihou using the *modified Mann* iteration method introduced by Schu [5], proved convergence theorem for the iterative approximation of fixed points of k-strictly asymptotically pseudocontractive mappings and asymptotically demicontractive mappings in Hilbert spaces. Recently, Osilike [3], extended the theorems of Qihou [4] concerning the iterative approximation of fixed points of k-strictly asymptotically demicontractive mappings from Hilbert spaces to much more general real q-uniformly smooth Banach spaces, $1 < q < \infty$, and to the much more general *modified Ishikawa iteration method*. More precisely, he proved the following:

Theorem 1.1. *(Osilike [3, p. 1296]): Let $q > 1$ and let E be a real q-uniformly smooth Banach space. Let K be a closed convex and bounded subset of E and*

$T : K \to K$ a completely continuous uniformly L-Lipschitzian asymptotically demicontractive mapping with a sequence $k_n \subseteq [1, \infty)$ satisfying $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences satisfying the conditions.

(i) $0 \leq \alpha_n, \beta_n \leq 1, n \geq 1$;

(ii) $0 < \in \leq c_q \alpha_n^{q-1}(1 + L\beta_n)^q \leq \frac{1}{2}\{q(1-k)(1+L)^{-(q-2)}\} - \in$, *for all* $n \geq 1$ *and for some* $\in > 0$; *and*

(iii) $\sum_{n=0}^{\infty} \beta_n < \infty$.

Then the sequence $\{x_n\}$ generated from an arbitrary $x_1 \in K$ by

$$y_n = (1 - \beta_n)x_n + \beta_n T^n x_n, \; n \geq 1,$$
$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n y_n, \; n \geq 1$$

converges strongly to a fixed point of T.

In Theorem 1.1, c_q is a constant appearing in an inequality which characterizes q-uniformly smooth Banach spaces. In Hilbert spaces, $q = 2$, $c_q = 1$ and with $\beta_n = 0 \; \forall n$, Theorems 1 and 2 of Qihou [4] follow from Theorem 1.1 (see Remark 2 of [3]).

It is our purpose in this paper to extend Theorem 1.1 from real q-uniformly smooth Banach spaces to arbitrary real Banach spaces using the more general modified Ishikawa iteration method with errors in the sense of Xu [7] given by

$$y_n = a_n x_n + b_n T^n x_n + c_n u_n, \; n \geq 1,$$
$$x_{n+1} = a'_n x_n + b'_n T^n y_n + c'_n v_n, \; n \geq 1, \tag{1.6}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{c'_n\}$ are real sequences in $[0, 1]$. $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n$, $\{u_n\}$ and $\{v_n\}$ are bounded sequences in K. If we set $b_n = c_n = 0$ in (1.6) we obtain the *modified Mann iteration method with errors* in the sense of Xu [7] given by

(1.7) $$x_{n+1} = a'_n x_n + b'_n T^n x_n + c'_n v_n,\ n \geq 1.$$

2. Main Results

In the sequel, we shall need the following:

Lemma 2.1. *Let E be a normed space, and K a nonempty convex subset of E. Let $T : K \to K$ be uniformly L-Lipschitzian mapping and let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$ and $\{c'_n\}$ be sequences in $[0,1]$ with $a_n+b_n+c_n = a'_n+b'_n+c'_n = 1$. Let $\{u_n\}$, $\{v_n\}$ be bounded sequences in K. For arbitrary $x_1 \in K$, generate the sequence $\{x_n\}$ by*

$$\begin{aligned} y_n &= a_n x_n + b_n T^n x_n + c_n u_n,\ n \geq 1 \\ x_{n+1} &= a'_n x_n + b'_n T^n y_n + c'_n v_n,\ n \geq 0. \end{aligned}$$

Then

$$\begin{aligned} \|x_n - Tx_n\| \leq{} & \|x_n - T^n x_n\| + L(1+L)^2 \|x_{n+1} - T^{n-1} x_{n-1}\| \\ & + L(1+L)c'_{n-1}\|v_{n-1} - x_{n-1}\| + L^2(1+L)c_{n-1}\|u_{n-1} - x_n\| \\ & + Lc'_{n-1}\|x_{n-1} - T^{n-1}x_{n-1}\|. \end{aligned} \tag{2.1}$$

Proof. Set $\lambda_n = \|x_n - T^n x_n\|$. Then

$$\begin{aligned} \|x_n - Tx_n\| &\leq \|x_n - T^n x_n\| + L\|T^{n-1}x_n - x_n\| \\ &\leq \lambda_n + L^2\|x_n - x_{n-1}\| + L\|T^{n-1}x_{n-1} - x_n\| \\ &= \lambda_n + L^2\|a'_{n-1}x_n + b'_{n-1}T^{n-1}y_{n-1} + c'_{n-1}v_{n-1} - x_{n-1}\| \\ &\quad + L\|a'_{n-1}x_{n-1} + b'_{n-1}T^{n-1}y_{n-1} + c'_{n-1}v_{n-1} - T^{n-1}x_{n-1}\| \\ &= \lambda_n + L^2\|b'_{n-1}(T^{n-1}y_{n-1} - x_{n-1}) + c'_{n-1}(v_{n-1} - x_{n-1})\| \\ &\quad + L\|a'_{n-1}(x_{n-1} - T^{n-1}x_{n-1}) + b'_{n-1}(T^{n-1}y_{n-1} - T^{n-1}x_{n-1}) \\ &\quad + c'_{n-1}(v_{n-1} - T^{n-1}x_{n-1})\| \end{aligned}$$

$$
\begin{aligned}
&\leq \lambda_n + L^2\|T^{n-1}y_{n-1} - x_{n-1}\| + L^2 c'_{n-1}\|v_{n-1} - x_{n-1}\| \\
&\qquad + L\|x_{n-1} - T^{n-1}x_{n-1}\| + L^2\|y_{n-1} - x_{n-1}\| + Lc'_{n-1}\|v_{n-1} - x_{n-1}\| \\
&\qquad + Lc'_{n-1}\|x_{n-1} - T^{n-1}x_{n-1}\| \\
&= \lambda_n + L\lambda_{n-1} + L(1+L)c'_{n-1}\|v_{n-1} - x_{n-1}\| + L^2\|T^{n-1}y_{n-1} - x_{n-1}\| \\
&\qquad + L^2\|y_{n-1} - x_{n-1}\| + Lc'_{n-1}\|x_{n-1} - T^{n-1}x_{n-1})\| \\
&\leq \lambda_n + 2L\lambda_{n-1} + L(1+L)c'_{n-1}\|v_{n-1} - x_{n-1}\| + L^2(1+L)\|y_{n-1} - x_{n-1}\| \\
&\qquad + L^2\|T^{n-1}x_{n-1} - x_{n-1}\| + Lc'_{n-1}\|x_{n-1} - T^{n-1}x_{n-1})\| \\
&= \lambda_n + L(1+L)\lambda_{n-1} + L(1+L)c'_{n-1}\|v_{n-1} - x_{n-1}\| \\
&\qquad + L^2(1+L)\|b_{n-1}(T^{n-1}x_{n-1} - x_{n-1}) + c_{n-1}(u_{n-1} - x_{n-1})\| \\
&\qquad + Lc'_{n-1}\|x_{n-1} - T^{n-1}x_{n-1}\| \\
&\leq \lambda_n + L(L^2 + 2L + 1)\lambda_{n-1} + L(1+L)c'_{n-1}\|v_{n-1} - x_{n-1}\| \\
&\qquad + L^2(1+L)c_{n-1}\|u_{n-1} - x_{n-1}\| + Lc'_{n-1}\|x_{n-1} - T^{n-1}x_{n-1}\|,
\end{aligned}
$$

completing the proof of Lemma 1. □

Lemma 2.2. *Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1+\delta_n)a_n + b_n,\ n \geq 1. \tag{2.2}$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$ then $\lim_{n\to\infty} a_n$ exists. If in addition $\{a_n\}$ has a subsequence which converges strongly to zero then $\lim_{n\to\infty} a_n = 0$.

Proof. Observe that

$$\begin{aligned} a_{n+1} &\leq (1+\delta_n)a_n + b_n \\ &\leq (1+\delta_n)[(1+\delta_{n-1})a_{n-1} + b_{n-1}] + b_n \\ &\leq \ldots \leq \prod_{j=1}^{n}(1+\delta_j)a_1 + \prod_{j=1}^{n}(1+\delta_j)\sum_{j=1}^{n} b_j \\ &\leq \ldots \leq \prod_{j=1}^{\infty}(1+\delta_j)a_1 + \prod_{j=1}^{\infty}(1+\delta_j)\sum_{j=1}^{\infty} b_j < \infty. \end{aligned}$$

Hence $\{a_n\}$ is bounded. Let $M > 0$ be such that $a_n \leq M,\ n \geq 1$. Then

$$a_{n+1} \leq (1+\delta_n)a_n + b_n \leq a_n + M\delta_n + b_n = a_n + \sigma_n$$

where $\sigma_n = M\delta_n + b_n$. It now follows from Lemma 2.1 of ([6, p. 303]) that $\lim_n a_n$ exists. Consequently, if $\{a_n\}$ has a subsequence which converges strongly to zero then $\lim_n a_n = 0$ completing the proof of Lemma 2.2. □

Lemma 2.3. *Let E be a real Banach space and K a nonempty convex subset of E. Let $T : K \to K$ be uniformly L-Lipschitzian asymptotically demicontractive mapping with a sequence $\{k_n\} \subseteq [1,\infty)$, such that $\lim_n k_n = 1$, and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{a_n\},\ \{b_n\},\ \{c_n\}$, $\{a'_n\},\{b'_n\},\{c'_n\}$ be real sequences in $[0,1]$ satisfying:*

(i) $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n$,

(ii) $\sum_{n=1}^{\infty} b'_n = \infty$, *(iii)* $\sum_{n=1}^{\infty}(b'_n)^2 < \infty$, $\sum_{n=1}^{\infty} c'_n < \infty$, $\sum_{n=1}^{\infty} b_n < \infty$, *and* $\sum_{n=1}^{\infty} c_n < \infty$.

Let $\{u_n\}$ *and* $\{v_n\}$ *be bounded sequences in* K *and let* $\{x_n\}$ *be the sequence generated from an arbitrary* $x_1 \in K$ *by*

$$\begin{aligned} y_n &= a_n x_n + b_n T^n x_n + c_n u_n, \ n \geq 1, \\ x_{n+1} &= a'_n + b'_n T^n y_n + c'_n v_n, \ n \geq 1, \end{aligned}$$

then $\liminf_n \|x_n - Tx_n\| = 0$.

Proof. It is now well-known (see e.g. [1]) that for all $x, y \in E$, there exists $j(x+y) \in J(x+y)$ such that

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle. \tag{2.3}$$

Let $x^* \in F(T)$ and let $M > 0$ be such that $\|u_n - x^*\| \leq M$, $\|v_n - x^*\| \leq M$, $n \geq 1$. Using (1.6) and (2.3) we obtain

$$\begin{aligned} &\|x_{n+1} - x^*\|^2 \\ &\quad = \|(1 - b'_n - c'_n)x_n + b'_n T^n y_n + c'_n v_n - x^*\|^2 \\ &\quad = \|(x_n - x^*) + b'_n(T^n y_n - x_n) + c'_n(v_n - x_n)\|^2 \\ &\quad \leq \|(x_n - x^*)\|^2 + 2\langle b'_n(T^n y_n - x_n) + c'_n(v_n - x_n), j(x_{n+1} - x^*)\rangle \\ &\quad = \|(x_n - x^*)\|^2 - 2b'_n\langle x_{n+1} - T^n x_{n+1}, j(x_{n+1} - x^*)\rangle \\ &\qquad + 2b'_n\langle x_{n+1} - T^n x_{n+1}, j(x_{n+1} - x^*)\rangle \\ &\qquad + 2b'_n\langle T^n y_n - x_n, j(x_{n+1} - x^*)\rangle + 2c'_n\langle v_n - x_n, j(x_{n+1} - x^*)\rangle \end{aligned}$$

$$
\begin{aligned}
&= \|(x_n - x^*)\|^2 - 2b_n'\langle x_{n+1} - T^n x_{n+1}, j(x_{n+1} - x^*)\rangle \\
&\quad + 2b_n'\langle x_{n+1} - x_n, j(x_{n+1} - x^*)\rangle \\
&\quad + 2b_n'\langle T^n y_n - T^n x_{n+1}, j(x_{n+1} - x^*)\rangle \\
&\quad + 2c_n'\langle v_n - x_n, j(x_{n+1} - x^*)\rangle. \qquad (2.4)
\end{aligned}
$$

Observe that

$$x_{n+1} - x_n = b_n'(T^n y_n - x_n) + c_n'(v_n - x_n).$$

Using this and (1.2) in (2.4) we have

$$
\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad \le \|(x_n - x^*)\|^2 - b_n'(1-k)\|x_{n+1} - T^n x_{n+1}\|^2 \\
&\qquad + b_n'(k_n^2 - 1)\|x_{n+1} - x^*\|^2 \\
&\qquad + 2(b_n')^2\langle T^n y_n - x_n, j(x_{n+1} - x^*)\rangle \\
&\qquad + 2b_n'\langle T^n y_n - T^n x_{n+1}, j(x_{n+1} - x^*)\rangle \\
&\qquad + 3c_n'\langle v_n - x_n, j(x_{n+1} - x^*)\rangle \\
&\quad \le \|(x_n - x^*)\|^2 - b_n'(1-k)\|x_{n+1} - T^n x_{n+1}\|^2 \\
&\qquad + (k_n^2 - 1)\|x_{n+1} - x^*\|^2 \\
&\qquad + 2(b_n')^2\|T^n y_n - x_n\|\|x_{n+1} - x^*\| \\
&\qquad + 2b_n'L\|x_{n+1} - y_n\|\|x_{n+1} - x^*\| \\
&\qquad + 3c_n'\|v_n - x_n\|\|x_{n+1} - x^*\| \\
&\quad = \|(x_n - x^*)\|^2 - b_n'(1-k)\|x_{n+1} - T^n x_{n+1}\|^2 \\
&\qquad + (k_n^2 - 1)\|x_{n+1} - x^*\|^2 + [2(b_n')^2\|T^n y_n - x_n\| \\
&\qquad + 2b_n'L\|x_{n+1} - y_n\| + 3c_n'\|v_n - x_n\|]\|x_{n+1} - x^*\|. \qquad (2.5)
\end{aligned}
$$

Observe that

$$\begin{aligned} \|y_n - x^*\| &= \|a_n(x_n - x^*) + b_n(T^n x_n - x^*) + c_n(u_n - x^*)\| \\ &\leq \|x_n - x^*\| + L\|x_n - x^*\| + M \\ &= (1+L)\|x_n - x^*\| + M, \end{aligned} \tag{2.6}$$

so that

$$\begin{aligned} \|T^n y_n - x_n\| &\leq L\|y_n - x^*\| + \|x_n - x^*)\| \\ &\leq L[(1+L)\|x_n - x^*\| + M] + \|x_n - x^*\| \\ &\leq [1 + L(1+L)]\|x_n - x^*\| + ML, \end{aligned} \tag{2.7}$$

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|a'_n(x_n - x^*) + b'_n(T^n y_n - x^*) + c'_n(v_n - x^*)\| \\ &\leq \|x_n - x^*\| + L\|y_n - x^*\| + M \\ &\leq \|x_n - x^*\| + L[(1+L)\|x_n - x^*\| + M] + M \\ &= [1 + L(1+L)]\|x_n - x^*\| + (1+L)M, \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} \|x_{n+1} - y_n\| &= \|a'_n(x_n - y_n) + b'_n(T^n y_n - y_n) + c'_n(v_n - y_n)\| \\ &\leq \|x_n - y_n\| + b'_n[\|T^n y_n - x^*\| + \|y_n - x^*\|] \\ &\quad + c'_n[\|v_n - x^*\| + \|y_n - x^*\|] \\ &= \|b_n(T^n x_n - x_n) + c_n(u_n - x_n)\| + b'_n[L\|y_n - x_n\| + \|y_n - x^*\|] \\ &\quad + c'_n M + c'_n\|y_n - x^*\| \\ &\leq b_n(1+L)\|x_n - x^*\| + c_n M + c_n\|x_n - x^*\| \\ &\quad + [b'_n(1+L) + c'_n]\|y_n - x^*\| + c'_n M \end{aligned}$$

$$\leq [b_n(1+L)+c_n]\|x_n - x^*\| + c_n M$$
$$+ [b'_n(1+L)+c'_n][(1+L)\|x_n - x^*\| + M] + c'_n M$$
$$\leq \{[b_n(1+L)+c_n] + [b'_n(1+L)+c'_n](1+L)\}\|x_n - x^*\|$$
$$+ M[b'_n(1+L)+2c'_n + c_n]. \tag{2.9}$$

Substituting (2.7)-(2.9) in (2.5) we obtain,

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad \leq \|(x_n - x^*)\|^2 - b'_n(1-k)\|x_{n+1} - T^n x_{n+1}\|^2 \\
&\qquad + (k_n^2 - 1)\{[1+L(1+L)]\|x_{n+1} - x^*\| + M(1+L)\}^2 \\
&\qquad + \{(b'_n[[1+L(1+L)]\|x_n - x^*\| + ML] + 3c'_n[M + \|x_n - x^*\|] \\
&\qquad + 2b'_n L[[b_n(1+L)+c_n] + [b'_n(1+L+c'_n](1+L)]\|x_n - x^*\| \\
&\qquad + M[b'_n(1+L)+2c'_n + c_n]\}\{[1+L(1+L)]\|x_n - x^*\| + M(1+L)\} \\
&\quad \leq \|(x_n - x^*)\|^2 - b'_n(1-k)\|x_{n+1} - T^n x_{n+1}\|^2 \\
&\qquad + (k_n^2 - 1)\{[1+L(1+L)]^2\|x_{n+1} - x^*\|^2 \\
&\qquad + 2M(1+L)[1+L(1+L)]\|x_n - x^*\| + M^2(1+L)^2\} \\
&\qquad + 2(b'_n)^2[[1+L(1+L)]\|x_n - x^*\| + ML][[1+L(1+L)]\|x_n - x^*\| \\
&\qquad + M(1+L)] + 3c'_n[M + \|x_n - x^*\|][[1+L(1+L)]\|x_n - x^*\| \\
&\qquad + M(1+L)] + 2b'_n L\{[[b_n(1+L)+c_n] \\
&\qquad + [b'_n(1+L)+c'_n](1+L)]\|x_n - x^*\| \\
&\qquad + M[b'_n(1+L)+2c'_n + c_n]\}\{[1+L(1+L)]\|x_n - x^*\| + M(1+L)\}.
\end{aligned}$$

Since $\|x_n - x^*\| \leq 1 + \|x_n - x^*\|^2$, we have

$$\|x_{n+1} - x^*\|^2 \leq [1+\delta_n]\|x_n - x^*\|^2 + \sigma_n - b'_n(1-k)\|x_{n+1} - T^n x_{n+1}\|^2, \tag{2.10}$$

where

$$\delta_n = (k_n^2 - 1)\{[1 + L(1 + L)]^2 + 2M(1 + L)[1 + L(1 + L)]\}$$
$$+ 2(b'_n)^2\{[1 + L(1 + L)]^2 + M(1 + L)[1 + L(1 + L)]$$
$$+ ML[1 + L(1 + L)]\}$$
$$+ 3c'_n\{[1 + L(1 + L)] + M[1 + L(1 + L)] + M(1 + L)\}$$
$$+ 2b'_n L\{\{[b_n(1 + L) + c_n] + [b'_n(1 + L) + c'_n](1 + L)\}$$
$$\times \{[1 + L(1 + L)] + M(1 + L)\}$$
$$+ M[b'_n(1 + L) + 2c'_n + c_n][1 + L(1 + L)]\}$$

and

$$\sigma_n = (k_n^2 - 1)\{2M(1 + L)[1 + L(1 + L)] + M^2(1 + L)^2\}$$
$$+ 2(b'_n)^2\{[1 + L(1 + L)]M(1 + L) + ML[1 + L(1 + L)] + M^2L(1 + L)\}$$
$$+ 3c'_n\{M[1 + L(1 + L)] + M^2(1 + L) + M(1 + L)$$
$$+ 2b'_n L\{[[b_n(1 + L) + c_n] + [b'_n(1 + L) + c'_n](1 + L)][M(1 + L)]$$
$$+ M[b'_n(1 + L) + 2c'_n + c_n][[1 + L(1 + L)] + M(1 + L)]\}.$$

Since $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$, condition (iii) implies that $\sum_{n=1}^{\infty}\delta_n < \infty$ and $\sum_{n=1}^{\infty}\sigma_n < \infty$. From (2.10) we obtain

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\le [1 + \delta_n]\|x_n - x^*\| + \sigma_n \\ &\le \dots \le \prod_{j=1}^{n}[1 + \delta_j]\|x_1 - x^*\|^2 + \prod_{j=1}^{n}[1 + \delta_j]\sum_{j=1}^{n}\sigma_j \\ &\le \prod_{j=1}^{\infty}[1 + \delta_j]\|x_1 - x^*\|^2 + \prod_{j=1}^{\infty}[1 + \delta_j]\sum_{j=1}^{\infty}\sigma_j < \infty, \end{aligned}$$

since $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} \sigma_n < \infty$. Hence $\{\|x_n - x^*\|\}_{n=1}^{\infty}$ is bounded. Let $\|x_n - x^*\| \leq M,\ n \geq 1$. Then it follows from (2.10) that

$$\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 + M^2\delta_n + \sigma_n - b_n'(1-k)\|x_{n+1} - T^n x_{n+1}\|^2,\ n \geq 1 \tag{2.11}$$

Hence,

$$b_n'(1-k)\|x_{n+1} - T^n x_{n+1}\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \mu_n,$$

where $\mu_n = M^2\delta_n + \sigma_n$ so that,

$$(1-k)\sum_{j=1}^{n} b_j'\|x_{j+1} - T^j x_{j+1}\|^2 \leq \|x_1 - x^*\|^2 + \sum_{j=1}^{n} \mu_j < \infty,$$

Hence,

$$\sum_{n=1}^{\infty} b_n'\|x_{n+1} - T^n x_{n+1}\|^2 < \infty,$$

and condition (ii) implies that $\liminf_{n\to\infty} \|x_{n+1} - T^n x_{n+1}\| = 0$. Observe that

$$\begin{aligned} \|x_{n+1} - T^n x_{n+1}\|^2 &= \|(1 - b_n' - c_n')x_n + b_n' T^n y_n + c_n' v_n - T^n x_{n+1}\|^2 \\ &= \|x_n - T^n x_n + b_n'(T^n y_n - x_n) + T^n x_n - T^n x_{n+1} \\ &\quad + c_n'(v_n - x_n)\|^2. \end{aligned} \tag{2.12}$$

For arbitrary $u,\ v \in E$, set $x = u + v$ and $y = -v$ in (2.3) to obtain

$$\|v + u\|^2 \geq \|u\|^2 + 2\langle v, j(u)\rangle. \tag{2.13}$$

From (2.12) and (2.13), we have

$$\begin{aligned}
&\|x_{n+1} - T^n x_{n+1}\|^2 \\
&\quad = \|x_n - T^n x_n + b'_n(T^n y_n - x_n) + T^n x_n - T^n x_{n+1} + c'_n(v_n - x_n)\|^2 \\
&\quad \geq \|x_n - T^n x_n\|^2 + 2\langle b'_n(T^n y_n - x_n) + T^n x_n - T^n x_{n+1} \\
&\qquad + c'_n(v_n - x_n), j(x_n - T^n x_n)\rangle.
\end{aligned}$$

Hence

$$\begin{aligned}
&\|x_n - T^n x_n\|^2 \\
&\quad \leq \|x_{n+1} - T^n x_{n+1}\|^2 + 2\|b'_n(T^n y_n - x_n) \\
&\qquad + T^n x_n - T^n x_{n+1} + c'_n(v_n - x_n)\|\|x_n - T^n x_n\| \\
&\quad \leq \|x_{n+1} - T^n x_{n+1}\|^2 + 2\{b'_n\|T^n y_n - x_n\| + L\|x_{n+1} - x_n\| \\
&\quad + c'_n\|v_n - x_n\|\}\|x_n - T^n x_n\| \\
&\quad \leq \|x_{n+1} - T^n x_{n+1}\|^2 + 2\{b'_n\|T^n y_n - x_n\| + Lb'_n\|T^n y_n - x_n\| \\
&\qquad + Lc'_n\|v_n - x_n\| + c'_n\|v_n - x_n\|\}\|x_n - T^n x_n\| \\
&\quad \leq \|x_{n+1} - T^n x_{n+1}\|^2 + 2(1+L)\|x_n - x^*\| \\
&\qquad \times \{(1+L)b'_n\|T^n y_n - x_n\| + (1+L)c'_n\|v_n - x_n\|\} \\
&\quad \leq \|x_{n+1} - T^n x_{n+1}\|^2 + 2(1+L)\|x_n - x^*\| \\
&\qquad \times \{(1+L)b'_n[[1 + L(1+L)]\|x_n - x^*\| + ML] \\
&\qquad + (1+L)c'_n[M + \|x_n - x^*\|], \quad \text{(using (2.6))} \\
&\quad \leq \|x_{n+1} - T^n x_{n+1}\|^2 \\
&\qquad + 2(1+L)M\{(1+L)b'_n[[1 + L(1+L)]M + ML] \\
&\qquad + (1+L)c'_n[M + M]\}, \quad \text{(since } \|x_n - x^*\| \leq M)
\end{aligned}$$

$$(2.14) \qquad = \|x_{n+1} - T^n x_{n+1}\|^2 + 2b'_n(1+L)^4 M^2 + 4c'_n(1+L)^2 M.$$

Since $\lim_{n\to\infty} b'_n = 0$, $\lim_{n\to\infty} c'_n = 0$ and $\liminf_{n\to\infty} \|x_{n+1} - T^n x_{n+1}\| = 0$, it follows from (2.14) that $\liminf_{n\to\infty} \|x_n - T^n x_n\| = 0$. It then follows from Lemma 1 that $\liminf_{n\to\infty} \|x_n - Tx_n\| = 0$, completing the proof of Lemma 2.3. □

Corollary 2.4. *Let E be a real Banach space and K a nonempty convex subset of E. Let $T : K \to K$ be a k-strictly asymptotically pseudocontractive map with $F(T) \neq \emptyset$ and sequence $\{k_n\} \subset [1,\infty)$ such that $\lim_n k_n = 1$, $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{c'_n\}$, $\{u_n\}$, and $\{v_n\}$ be as in Lemma 2.3 and let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in K$ by*

$$y_n = a_n x_n + b_n T^n x_n + c_n u_n,\ n \geq 1,$$
$$x_{n+1} = a'_n + b'_n T^n y_n + c'_n v_n,\ n \geq 1,$$

Then $\liminf_{n\to\infty} \|x_n - Tx_n\| = 0$.

Proof. From (1.1) we obtain

$$\begin{aligned}
&\|(I-T^n)x - (I-T^n)y\|\|x-y\| \\
&\quad \geq \frac{1}{2}\{(1-k)\|(I-T^n)x - (I-T^n)y\|^2 - (k_n^2-1)\|x-y\|^2\} \\
&\quad = \frac{1}{2}[\sqrt{1-k}\|(I-T^n)x - (I-T^n)y\| \\
&\qquad + \sqrt{k_n^2-1}\|x-y\|][\sqrt{1-k}\|(I-T^n)x - (I-T^n)y\| - \sqrt{k^2-1}\|x-y\|]
\end{aligned}$$

$$\geq \frac{1}{2}[\sqrt{1-k}\|(I-T^n)x-(I-T^n)y\|]$$
$$[\sqrt{1-k}\|(I-T^n)x-(I-T^n)y\| - \sqrt{k^2-1}\|x-y\|]$$

so that

$$\frac{1}{2}\sqrt{1-k}[\sqrt{1-k}\|(I-T^n)x-(I-T^n)y\|] - \sqrt{k^2-1}\|x-y\| \leq \|x-y\|.$$

Hence

$$\|(I-T^n)x-(I-T^n)y\| \leq [\frac{2+\sqrt{\{(1-k)(k_n^2-1)\}}}{1-k}]\|x-y\|.$$

Furthermore,

$$\begin{aligned}\|T^n x - T^n y\| - \|x-y\| &\leq \|(I-T^n)x-(I-T^n)y\| \\ &\leq [\frac{2+\sqrt{\{(1-k)(k_n^2-1)\}}}{1-k}]\|x-y\|,\end{aligned}$$

from which it follows that

$$\|T^n x - T^n y\| \leq [1+\frac{2+\sqrt{\{(1-k)(k_n^2-1)\}}}{1-k}]\|x-y\|.$$

Since $\{k_n\}$ is bounded, let $k_n \leq D$, $\forall\, n \geq 1$. Then

$$\begin{aligned}\|T^n x - T^n y\| &\leq [1+\frac{2+\sqrt{\{(1-k)(D^2-1)\}}}{1-k}]\|x-y\| \\ &\leq L\|x-y\|,\end{aligned}$$

where

$$L = 1 + \frac{2 + \sqrt{\{(1-k)(D^2-1)\}}}{1-k}.$$

Hence T is uniformly L-Lipschitzian. Since $F(T) \neq \emptyset$, then T is uniformly L-Lipschitzian and asymptotically demicontractive and hence the result follows from Lemma 2.3. □

Remark 2.1. *It is shown in [3] that if E is a Hilbert space and $T : K \to K$ is k-asymptotically pseudocontractive with sequence $\{k_n\}$ then*

$$\|T^n x - T^n y\| \leq \frac{D + \sqrt{k}}{1 - \sqrt{k}} \|x - y\| \, \forall\, x,\, y \in K, \text{ where } k_n \leq D, \, \forall\, n \geq 1.$$

Theorem 2.5. *Let E be a real Banach space and K a nonempty closed convex subset of E. Let $T : K \to K$ be a completely continuous uniformly L-Lipschitzian asymptotically demicontractive mapping with sequence $\{k_n\} \subset [1, \infty)$ such that $\lim_n k_n = 1$ and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}, \{u_n\}$, and $\{v_n\}$ be as in Lemma 2.3. Then the sequence $\{x_n\}$ generated from an arbitrary $x_1 \in K$ by*

$$\begin{aligned} y_n &= a_n x_n + b_n T^n x_n + c_n u_n, \; n \geq 1, \\ x_{n+1} &= a'_n x_n + b'_n T^n y_n + c'_n v_n, \; n \geq 1, \end{aligned}$$

converges strongly to a fixed point of T.

Proof. From Lemma 2.3, $\liminf_n \|x_n - T^n x_n\| = 0$, hence there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim_n \|x_{n_j} - Tx_{n_j}\| = 0$.

Since $\{x_{n_j}\}$ is bounded and T is completely continuous, then $\{Tx_{n_j}\}$ has a subsequence $\{Tx_{j_k}\}$ which converges strongly. Hence $\{x_{n_{j_k}}\}$ converges strongly. Suppose $\lim_{k\to\infty} x_{n_{j_k}} = p$. Then $\lim_{k\to\infty} Tx_{n_{j_k}} = Tp$. $\lim_{k\to\infty} \|x_{n_{j_k}} - Tx_{n_{j_k}}\|$ $= \|p - Tp\| = 0$ so that $p \in F(T)$. It follows from (2.11) that

$$\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 + \mu_n$$

Lemma 2.2 now implies $\lim_{n\to\infty} \|x_n - p\| = 0$ completing the proof of Theorem 2.5. □

Corollary 2.6. *Let E be an arbitrary real Banach space and K a nonempty closed convex subset of E. let $T : K \to K$ be a k-strictly asymptotically pseudocontractive mapping with $F(T) \neq \emptyset$ and sequence $\{k_n\} \subset [1, \infty)$ such that $\lim_n k_n = 1$, and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{c'_n\}$, $\{u_n\}$, and $\{v_n\}$ be as in Lemma 2.3. Then the sequence $\{x_n\}$ generated from an arbitrary $x_1 \in K$ by*

$$\begin{aligned} y_n &= a_n x_n + b_n T^n x_n + c_n u_n, \; n \geq 1, \\ x_{n+1} &= a'_n x_n + b'_n T^n y_n + c'_n v_n, \; n \geq 1, \end{aligned}$$

converges strongly to a fixed point of T.

Proof. As shown in Corollary 2.4, T is uniformly L-Lipschitzian and since $F(T) \neq \emptyset$ then T is asymptotically demicontractive and the result follows from Theorem 2.5. □

Remark 2.2. *If we set $b_n = c_n = 0$, $\forall\, n \geq 1$ in Lemma 2.3, Theorem 2.5 and Corollaries 2.4 and 2.6, we obtain the corresponding results for the modified Mann iteration method with errors in the sense of Xu [7].*

Remark 2.3. *Theorem 2.5 extends the results of Osilike [3] (which is itself a generalization of a theorem of Qihou [4]) from real q-uniformly smooth Banach space to arbitrary real Banach space.*

Furthermore, our Theorem 2.5 is proved without the boundedness condition imposed on the subset K in ([3, 4]) and using the more general modified Ishikawa Iteration method with errors in the sense of Xu [7]. Also our iteration parameters $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{c'_n\}$, $\{u_n\}$, and $\{v_n\}$ are completely independent of any geometric properties of underlying Banach space.

Remark 2.4. *Prototypes for our iteration parameters are:*

$$\begin{aligned} b'_n &= \frac{1}{3(n+1)},\ c'_n = \frac{1}{3(n+1)^2},\ a'_n = 1-(b'_n+c'_n), \\ b_n &= c_n = \frac{1}{3(n+1)^2},\ a_n = 1-\frac{1}{3(n+1)^2},\ n \geq 1. \end{aligned}$$

The proofs of the following theorems and corollaries for the Ishikawa iteration method with errors in the sense of Liu [2] are omitted because the proofs follow by a straightforward modifications of the proofs of the corresponding results for the Ishikawa iteration method with errors in the sense of Xu [7].

Theorem 2.7. *Let E be a real Banach space and let $T : E \to E$ be a uniformly L-Lipschitzian asymptotically demicontractive mapping with sequence $\{k_n\} \subset [1, \infty)$ such that $\lim_n k_n = 1$, and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{u_n\}$ and $\{v_n\}$ be sequences in E such that $\sum_{n=1}^{\infty} \|u_n\| < \infty$ and $\sum_{n=1}^{\infty} \|v_n\| < \infty$, and let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, 1]$ satisfying the conditions:*

(i) $0 \leq \alpha_n, \beta_n \leq 1, n \geq 1$;

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$

(iii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$ and $\sum_{n=1}^{\infty} \beta_n < \infty$.

Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in E$ by

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n T^n x_n + u_n, \; n \geq 1, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T^n y_n + v_n, \; n \geq 1, \end{aligned}$$

Then $\liminf_{n\to\infty} \|x_n - Tx_n\| = 0.$

Corollary 2.8. *Let E be a real Banach space and let $T : E \to E$ be a k-strictly asymptotically pseudocontractive map with $F(T) \neq \emptyset$ and sequence $\{k_n\} \subset [1,\infty)$ such that $\lim_n k_n = 1$, and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{u_n\}$, $\{v_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ be as in Theorem 2.7 and let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in E$ by*

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n T^n x_n + u_n, \; n \geq 1, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T^n y_n + v_n, \; n \geq 1, \end{aligned}$$

Then $\liminf_{n\to\infty} \|x_n - Tx_n\| = 0.$

Theorem 2.9. *Let E, T, $\{u_n\}$, $\{v_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ be as in Theorem 2.7. If in addition $T : E \to E$ is completely continuous then the sequence $\{x_n\}$ generated from an arbitrary $x_1 \in E$ by*

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n T^n x_n + u_n, \; n \geq 1, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T^n y_n + v_n, \; n \geq 1, \end{aligned}$$

converges strongly to a fixed point of T.

Corollary 2.10. *Let E, T, $\{u_n\}$, $\{v_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ be as in Corollary 2.8. If in addition T is completely continuous, then the sequence $\{x_n\}$ generated from an arbitrary $x, y \in E$ by*

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n T^n x_n + u_n,\ n \geq 1, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T^n y_n + v_n,\ n \geq 1, \end{aligned}$$

converges strongly to a fixed point of T.

Remark 2.5. *(a) If K is a nonempty closed convex subset of E and $T : K \to K$, then Theorems 2.7 and 2.9 and Corollaries 2.8 and 2.10 also hold provided that in each case the sequence $\{x_n\}$ lives in K.*

(b) If we set $\beta_n = 0$, $\forall\, n \geq 1$ in Theorems 2.7 and 2.9 and Corollaries 2.8 and 2.10, we obtain the corresponding results for modified Mann iteration method with errors in the sense of Liu [2].

References

[1] S.S. CHANG, Some problems and results in the study of nonlinear analysis, *Nonlinear Analysis*, **30** (1997), 4197–4208.

[2] L. LIU, Ishikawa and Mann iteration processes with errors for nonlinear strongly accretive mappings in Banach spaces, *J. Math. Anal. Appl.*, **194** (1995), 114–125.

[3] M.O. OSILIKE, Iterative approximations of fixed points of asympotically demicontractive mappings, *Indian J. Pure Appl. Math.*, **29**(12) (1998), 1291–1300.

[4] L. QIHOU, Convergence theorems of the sequence of iterates for asymptotically demicontractive and hemicontractive mappings, *Nonlinear Analysis*, **26**(11) (1996), 1835–1842.

[5] J. SCHU, Iterative construction of fixed points of asymptotically nonexpansive mappings, *J. Math. Anal. Appl.*, **158** (1991), 407–413.

[6] K.K. TAN AND H.K. XU, Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, *J. Math. Anal. Appl.*, **178** (1993), 301–308.

[7] Y. XU, Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive operator equations, *J. Math. Anal. Appl.*, **224** (1998), 91–101.