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CERTAIN SUFFICIENCY CONDITIONS ON GAUSSIAN HYPERGEOMETRIC FUNCTIONS

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ABSTRACT. The author aims at finding certain conditions on a, b and c such that the normalized Gaussian hypergeometric function $zF(a, b; c; z)$ given by

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)(1,n)} z^n, \quad |z| < 1,$$

is in certain subclasses of analytic functions. A particular operator acting on $F(a, b; c; z)$ is also discussed.

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1. INTRODUCTION

As usual, let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

analytic in the open unit disk $\Delta = \{z : |z| < 1\}$, and $\mathcal{S}$ denote the subclass of $\mathcal{A}$ that are univalent in Δ. We begin with the following.

Definition 1.1 ([2])**.** Let $f \in \mathcal{A}$, $0 \leq k < \infty$, and $0 \leq \alpha < 1$. Then $f \in k - UCV(\alpha)$ if and only if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq k \left|\frac{zf''(z)}{f'(z)}\right| + \alpha. \tag{1.2}$$

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This class generalizes various other classes which are worthy of mention. The class $k-UCV(0)$, called the k-Uniformly convex is due to [11], and has its geometric characterization given in the following way: Let $0 \leq k < \infty$. The function $f \in \mathcal{A}$ is said to be k-uniformly convex in Δ, f is convex in Δ, and the image of every circular arc γ contained in Δ, with center ζ, where $|\zeta| \leq k$, is convex.

The class $0 - UCV(\alpha) = \mathcal{K}(\alpha)$ is the well-known class of convex functions of order α that satisfy the analytic conditions

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \alpha.$$

In particular, for $\alpha = 0$, f maps the unit disk onto the convex domain (for details, see [8]).

The class $1 - UCV(0) = UCV$ [9] describes geometrically the domain of values of the expression

$$p(z) = 1 + \frac{zf''(z)}{f'(z)}, \quad z \in \Delta,$$

as $f \in UCV$ if and only if p is in the conic region

$$\Omega = \{\omega \in \mathbb{C} : (\operatorname{Im}\omega)^2 < 2\operatorname{Re}\omega - 1\}.$$

The classes UCV and S_p are unified and studied using certain fractional calculus operator methods found in [18]. We refer to [10, 11, 12] and references therein for basic results related to this paper.

The Gaussian hypergeometric function $f(z) = zF(a, b; c; z)$, $z \in \Delta$, given by the series

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)(1, n)} z^n$$

is the solution of the homogenous hypergeometric differential equation

$$z(1-z)w''(z) + [c - (a+b+1)z]w'(z) - abw(z) = 0$$

and has rich applications in various fields such as conformal mappings, quasiconformal theory, continued fractions and so on.

Here a, b, c are complex numbers such that $c \neq 0, -1, -2, -3, \ldots$, $(a, 0) = 1$ for $a \neq 0$, and for each positive integer n, $(a, n) := a(a+1)(a+2)\cdots(a+n-1)$ is the Pochhammer symbol. In the case of $c = -k$, $k = 0, 1, 2, \ldots$, $F(a, b; c; z)$ is defined if $a = -j$ or $b = -j$ where $j \leq k$. In this situation, $F(a, b; c; z)$ becomes a polynomial of degree j in z. Results regarding $F(a, b; c; z)$ when $\operatorname{Re}(c-a-b)$ is positive, zero or negative are abundant in the literature. In particular when $\operatorname{Re}(c-a-b) > 0$, the function $F(a, b; c; z)$ is bounded. This and the zero balanced case $\operatorname{Re}(c-a-b) = 0$ are discussed in detail by many authors (for example, see [19, 25, 1]). For interesting results regarding $\operatorname{Re}(c-a-b) < 0$, see [26] and references therein.

The hypergeometric function $F(a, b; c; z)$ has been studied extensively by various authors and play an important role in Geometric Function Theory. It is useful in unifying various functions by giving appropriate values to the parameters a, b, and c. We refer to [3, 17, 29, 27, 20, 21, 25] and references therein for some important results. In particular, the close-to-convexity (in turn the univalency), convexity, starlikeness, (for details on these technical terms we refer to [8, 5]) and various other properties of these hypergeometric functions were examined based on the conditions on a, b, and c in [21].

The observation that $1 + z = F(-1, -1; 1; z)$ is convex in Δ and its normalized form $z(1 + z) = zF(-1, -1; 1; z)$ is not even univalent in Δ clearly exhibits that the normalized functions need not inherit the properties that non-normalized functions have. Even though, the starlikeness and close-to-convexity of the normalized hypergeometric functions $zF(a, b; c; z)$ are discussed in detail by many authors (see [21, 25, 16]), many results on the convexity of

$zF(a, b; c; z)$ do not seem to be available in the literature except the non-convexity condition discussed in [25], the convexity condition for $a = 1$ solved completely in [24], and a weaker condition for convexity given by [32]. There is also a sufficient condition for $F(a, b; c; z)$ to be in $k - UCV(0)$ given in [12], which gives the convexity condition when $k = 0$.

Theorem 1.1 ([12]). *Let $c \in \mathbb{R}$, and $a, b \in \mathbb{C}$. Let a, b and c satisfy the conditions $c > |a|+|b|+2$ and*

$$\frac{|ab|\Gamma(c)\Gamma(c-|a|-|b|-2)}{\Gamma(c-|a|)\Gamma(c-|b|)}(|ab|-|a|-|b|+2c-3) \le \frac{1}{2}. \tag{1.3}$$

Then $zF(a, b; c; z)$ is convex in Δ.

Remark 1.2. We note that for the case $a = 1$, the convexity condition for $zF(1, b; c; z)$ obtained in [24] does not require (1.3) and hence is stronger than Theorem 1.1.

Also, for $\tau \in \mathbb{C}\backslash\{0\}$ we introduce the class $P_\gamma^\tau(\beta)$, with $0 \le \gamma < 1$ and $\beta < 1$ as

$$P_\gamma^\tau(\beta) := \left\{ f \in \mathcal{A} : \left| \frac{(1-\gamma)\frac{f(z)}{z} + \gamma f'(z) - 1}{2\tau(1-\beta) + (1-\gamma)\frac{f(z)}{z} + \gamma f'(z) - 1} \right| < 1, \quad z \in \Delta \right\}.$$

We list a few particular cases of this class discussed in the literature.

(1) The class $P_1^\tau(\beta)$ is given in [4] and discussed for the operator $I_{a,b;c}(f)(z) = zF(a, b; c; z) * f(z)$ in [7].
(2) The class $P_\gamma^\tau(\beta)$ for $\tau = e^{i\eta}\cos\eta$ where $\pi/2 < \eta < \pi/2$ is given in [14] and discussed by many authors with reference to the Carlson–Schaffer operator $G_{b,c}(f)(z) = zF(1, b; c; z) * f(z)$ using duality techniques for various values of γ (for example, see [1, 6, 14, 15, 19, 22]).

To be more specific, the properties of certain integral transforms of the type

$$V_\lambda(f) = \int_0^1 \lambda(t)\frac{f(tz)}{t}dt, \quad f \in P_\gamma^{(e^{i\eta}\cos\eta)}(\beta)$$

with $\beta < 1$, $\gamma < 1$ and $|\eta| < \pi/2$, under suitable restrictions on $\lambda(t)$ was discussed by many authors [6, 14, 19, 22]. In particular, if

$$\lambda(t) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(b-c)}t^{b-1}(1-t)^{c-b-1},$$

then $V_{\lambda(f)}$ is the well known Carlson–Schaffer operator $G_{b,c}(f)(z)$.

2. MAIN RESULTS

If $f \in \mathcal{A}$ such that f has the power series expansion

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \ge 0 \tag{2.1}$$

then f is one main subclass of $\mathcal{S}$ and is denoted by $\mathcal{T}$. This class is due to H. Silverman [30] and has many interesting results (see [30] and [31]).

In the line of $k - UCV(\alpha)$, the following class was defined in [2].

Definition 2.1 ([2]). Let $k-UCT(\alpha)$ be the class of functions $f(z)$ of the form (2.1) that satisfy the condition (1.2).

Using the analytic condition (1.2) and a Alexander type theorem, the following classes are defined in [2].

Definition 2.2 ([2]). Let $0 \leq k < \infty$, and $0 \leq \alpha < 1$. Then

(1) $f \in k - \mathcal{S}_p(\alpha)$ if and only if f has the form (1.1) and satisfies the condition

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq k\left|\frac{zf'(z)}{f'(z)} - 1\right| + \alpha. \tag{2.2}$$

(2) $f \in k - \mathcal{S}_pT(\alpha)$ if and only if f has the form (2.1) and satisfies the inequality given by the expression (2.2).

For $k = 0$, we obtain the well-known class of starlike functions of order α, which has the analytic characterization $\operatorname{Re}\frac{zf'(z)}{f(z)} > \alpha$ with $z \in \Delta$. In particular, for $\alpha = 0$, f maps the unit disk onto the starlike domain (for details, see [8]). We further note that, $1 - S_p(\alpha)$ is the well-known class discussed in [28]. We also need the following sufficient condition on the coefficients for the functions in the class $k - UCV(\alpha)$.

Lemma 2.1 ([2]). *A function $f(z)$ of the form (1.1) is in $k - UCV(\alpha)$ if it satisfies the condition*

$$\sum_{n=2}^{\infty} n\left[n(1+k) - (k+\alpha)\right] a_n \leq 1 - \alpha. \tag{2.3}$$

It was also found that the condition (2.3) is necessary and sufficient for f to be in $k-UCT(\alpha)$. Further that the condition

$$\sum_{n=2}^{\infty} \left[n(1+k) - (k+\alpha)\right] a_n \leq 1 - \alpha \tag{2.4}$$

is sufficient for f to be in $k - \mathcal{S}_p(\alpha)$ and it is both necessary and sufficient for f to be in $k - \mathcal{S}_pT(\alpha)$.

Another sufficient condition is also given for the class $k - UCV$ in [11] which is given by the following

Lemma 2.2 ([11]). *Let $f \in \mathcal{S}$ and be of the form (1.1). If for some k, $0 \leq k < \infty$, the inequality*

$$\sum_{n=2}^{\infty} n(n-1)|a_n| \leq \frac{1}{k+2}, \tag{2.5}$$

holds true, then $f \in k - UCV$. The number $1/(k+2)$ cannot be increased.

It is interesting to observe that sufficient conditions for $f \in k - S_p$, analogous to (2.5), cannot be obtained by replacing a_n by a_n/n as in many other situations.

Sufficiency conditions for $zF(a, b; c; z)$ to be in the class $k - UCV(\alpha)$ using the condition (2.1), and to be in the class $k - S_p(\alpha)$ using the condition (2.4) were obtained in [33] (see also [13]). In [11], it is proved that $zF(a, b; c; z)$ is in $k - UCV$ by applying the condition (2.5).

Theorem 2.3. *Let $f(z) \in \mathcal{S}$ and be of the form (1.1). If f is in $P_\gamma^\tau(\beta)$, then*

$$|a_n| \leq \frac{2|\tau|(1-\beta)}{1+\gamma(n-1)}. \tag{2.6}$$

The estimate is sharp.

It is easy to find the sufficient condition for $f(z)$ to be in $P_\gamma^\tau(\beta)$ under standard techniques. Hence we state the result without proof.

Theorem 2.4. *Let $f(z)$ be of the form (1.1). Then a sufficient condition for $f(z)$ to be in $P_\gamma^\tau(\beta)$ is*

$$\sum_{n=2}^{\infty}[1+\gamma(n-1)]|a_n| \le |\tau|(1-\beta). \tag{2.7}$$

This condition is also necessary if $f(z)$ is of the form (2.1) and $\tau = 1$.

Theorem 2.5. *Let a, b, c and γ satisfy any one of the following conditions such that $T_i(a,b,c,\gamma) \le |\tau|(1-\beta)$ for $i = 1,2,3$.*

(i) $a, b > 0$, $c > a + b$ *and*

$$T_1(a,b,c,\gamma) = \left(1+\frac{\gamma ab}{c}\right)\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

(ii) $-1 < a < 0$, $b > 0$, $c > 0$ *and*

$$T_2(a,b,c,\gamma) = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}\left(1+\frac{\gamma|ab|}{(c-a-b)}\right)+\frac{\gamma|ab|}{c}-\frac{\gamma(a,2)(b,2)}{(c,2)}.$$

(iii) $a, b \in \mathbb{C}\backslash\{0\}$, $c > |a| + |b|$ *and*

$$T_3(a,b,c,\gamma) = \gamma + \frac{\Gamma(c-|a|-|b|-1)\Gamma(c)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left(c-|a|-|b|-1+\gamma|ab|\right).$$

Then $zF(a,b;c;z)$ is in $P_\gamma^\tau(\beta)$.

Since $a = \overline{b}$ is useful in characterizing polynomials with positive coefficients when b is some negative integer, we give the corresponding result independently.

Corollary 2.6. *Let $a, b \in \mathbb{C}\backslash\{0\}$, $a = \overline{b}$, $c > 2\mathrm{Re}b$ and $T_4(a,b,c,\gamma) \le |\tau|(1-\beta)$ where*

$$T_4(a,b,c,\gamma) = \gamma + \frac{\Gamma(c-2\mathrm{Re}b-1)\Gamma(c)}{\Gamma(c-b)\Gamma(c-\overline{b})}\left(c-2\mathrm{Re}b-1+\gamma|b|^2\right).$$

Then $zF(\overline{b},b;c;z)$ is in $P_\gamma^\tau(\beta)$.

In the above theorem, if we take $a = 1$, we get the result for operator $G_{b,c}(f)(z)$ which we give independently as

Theorem 2.7. *Let $b > 0$ and*

$$\frac{(c+\gamma b)(c-1)}{c(c-b-1)} \le |\tau|(1-\beta).$$

Then the incomplete beta function $\phi(b;c;z) := zF(1,b;c;z)$ is in $P_\gamma^\tau(\beta)$.

When $f(z) = -\log(1-z)$, consider the operator of the form

$$G(a,b;c;z) = \int_0^z F(a,b;c;t)dt. \tag{2.8}$$

The sufficient condition for the operator $G(a,b;c;z)$ to be in $\mathcal{K}(\alpha)$ and $\mathcal{S}^*(\alpha)$ is given in [32] and extended to the class $k-UCV(\alpha)$ and $k-S_p(\alpha)$ in [33].

Theorem 2.8. *Let $0 < a \ne 1$, $0 < b \ne 1$ and $c > a+b+1$ such that $T(a,b,c,\gamma) \le 1+|\tau|(1-\beta)$ where*

$$T(a,b,c,\gamma) = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}\left(\gamma+\frac{(1-\gamma)(c-a-b)}{(a-1)(b-1)}\right)-\frac{(1-\gamma)(c-1)}{(a-1)(b-1)}. \tag{2.9}$$

Then $G(a,b;c;z)$ is in $P_\gamma^\tau(\beta)$.

Corollary 2.9. *Let $a=\bar{b}$, $0<b\neq 1$, and $c>2\mathrm{Re}b+1$ such that $T(\bar{b},b,c,\gamma)\leq 1+|\tau|(1-\beta)$ where*

$$T(\bar{b},b,c,\gamma)=\frac{\Gamma(c-2\mathrm{Re}b)\Gamma(c)}{\Gamma(c-\bar{b})\Gamma(c-b)}\left(\gamma+\frac{(1-\gamma)(c-2\mathrm{Re}b)}{|b-1|^2}\right)-\frac{(1-\gamma)(c-1)}{|b-1|^2}.$$

Then $G(\bar{b},b;c;z)$ is in $P_\gamma^\tau(\beta)$.

We note that an equivalent of Theorem 2.8 cannot be given for the Carlson–Schaffer operator $G_{b,c}(f)(z)=zF(1,b;c;z)*f(z)$ [3].

We give here another sufficiency condition for $G(a,b;c;z)$ to be in $k-UCV(0)$ using the sufficiency condition (2.5) of $k-UCV(0)$ given in [11]. A simple computation of applying (2.5) in the series representation of $G(a,b;c;z)$ gives the following result immediately. We omit the proof.

Theorem 2.10. *Let $a>-1$, $b>-1$ and $c>a+b+2$ such that for all $0\leq k<\infty$,*

$$\frac{(a+1)(b+1)}{(c+1)}\cdot\frac{\Gamma(c-a-b-1)\Gamma(c+1)}{\Gamma(c-a)\Gamma(c-b)}\leq\frac{1}{k+2}. \tag{2.10}$$

Then $zF(a,b;c;z)$ is in $k-UCV(0)=:k-UCV$.

The following results are immediate.

Corollary 2.11. *Let $b>-1$, $a=\bar{b}$ and $c>2+\mathrm{Re}b$ such that for all $0\leq k<\infty$,*

$$\frac{|b+1|^2}{(c+1)}\cdot\frac{\Gamma(c-\mathrm{Re}b-1)\Gamma(c+1)}{\Gamma(c-\bar{b})\Gamma(c-b)}\leq\frac{1}{k+2}.$$

Then $zF(\bar{b},b;c;z)$ is in $k-UCV(0)=k-UCV$.

Corollary 2.12. *Let $b>-1$ and $c>b+3$ such that for all $0\leq k<\infty$,*

$$\frac{2(b+1)}{(c+1)}\cdot\frac{c(c-1)}{(c-b-1)(c-b-2)}\leq\frac{1}{k+2}.$$

Then the incomplete function $\phi(b;c;z)$ is in $k-UCV(0)=k-UCV$. In particular, when $k=0$, $\phi(b;c;z)$ is convex in Δ.

3. PROOFS OF THEOREMS 2.3, 2.5 AND 2.8

We need the following result and we state this as

Lemma 3.1. *Let $a,b\in\mathbb{C}\backslash\{0\}$, $c>0$. Then we have the following:*

(i) *For $a,b>0$, $c>a+b+1$,*

$$\sum_{n=0}^{\infty}\frac{(n+1)(a,n)(b,n)}{(c,n)(1,n)}=\frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}\left[\frac{ab}{c-1-a-b}+1\right]. \tag{3.1}$$

(ii) *For $a\neq 1$, $b\neq 1$ and $c\neq 1$ with $c>\max\{0,a+b-1\}$,*

$$\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n)(1,n+1)}=\frac{1}{(a-1)(b-1)}\left[\frac{\Gamma(c+1-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}-(c-1)\right]. \tag{3.2}$$

(iii) *For $a\neq 1$ and $c\neq 1$ with $c>\max\{0,2\,\mathrm{Re}\,a-1\}$,*

$$\sum_{n=0}^{\infty}\frac{|(a,n)|^2}{(c,n)(1,n+1)}=\frac{1}{|a-1|^2}\left[\frac{\Gamma(c+1-2\,\mathrm{Re}\,a)\Gamma(c)}{\Gamma(c-a)\Gamma(c-\overline{a})}-(c-1)\right]. \tag{3.3}$$

The results in this lemma are part of Lemma 3.1 given in [23] and we omit details.

Proof of Theorem 2.3. Since $f \in P_{\gamma}^{\tau}(\beta)$, we have

$$1+\frac{1}{\tau}\left\{(1-\gamma)\frac{f(z)}{z}+\gamma f'(z)-1\right\}=\frac{1+(1-2\beta)w(z)}{1-w(z)},$$

where $w(z)$ is analytic in Δ and satisfies the condition $w(0)=0$, $|w(z)|<1$ for $z\in\Delta$. Hence we have

$$\frac{1}{\tau}\left((1-\gamma)\frac{f(z)}{z}+\gamma f'(z)-1\right)=w(z)\left\{2(1-\beta)+\frac{1}{\tau}\left((1-\gamma)\frac{f(z)}{z}+\gamma f'(z)-1\right)\right\}.$$

Using (1.1) and $w(z)=\sum_{n=1}^{\infty}b_n z^n$ we have

$$\left[2(1-\beta)+\frac{1}{\tau}\left(\sum_{n=2}^{\infty}[1+\gamma(n-1)]a_n z^{n-1}\right)\right]\left[\sum_{n=1}^{\infty}b_n z^n\right]=\frac{1}{\tau}\sum_{n=2}^{\infty}[1+\gamma(n-1)]a_n z^{n-1}.$$

Equating the coefficients of the above expression, we observe that the coefficient a_n in the right hand side of the above expression depends only on $a_2,\dots,a_{n-1}$ and the left hand side of the above expression. This gives

$$\left[2(1-\beta)+\frac{1}{\tau}\left(\sum_{n=2}^{k-1}[1+\gamma(n-1)]a_n z^{n-1}\right)\right]w(z)$$
$$=\frac{1}{\tau}\sum_{n=2}^{k}[1+\gamma(n-1)]a_n z^{n-1}+\sum_{n=k+1}^{\infty}d_n z^{n-1}.$$

Using $|w(z)|<1$, this reduces to the inequality

$$\left|2(1-\beta)+\frac{1}{\tau}\left(\sum_{n=2}^{k-1}[1+\gamma(n-1)]a_n z^{n-1}\right)\right|$$
$$>\left|\frac{1}{\tau}\sum_{n=2}^{k}[1+\gamma(n-1)]a_n z^{n-1}+\sum_{n=k+1}^{\infty}d_n z^{n-1}\right|.$$

Squaring the above inequality and integrating around $|z|=r$, $0<r<1$, and letting $r\to 1$ we obtain

$$4(1-\beta)^2\geq\frac{1}{|\tau|^2}[1+\gamma(n-1)]^2|a_n|^2$$

which gives the desired result. Equality holds for the function

$$f(z)=\frac{1}{\gamma z^{1-\frac{1}{\gamma}}}\int_0^z w^{1-\frac{1}{\gamma}}\left[1+\frac{2(1-\beta)\tau w^{n-1}}{1-2^{n-1}}\right]dw.$$

□

Proof of Theorem 2.5. Clearly $zF(a,b;c;z)$ has the series representation of the form (1.1) where

$$a_n=\frac{(a,n-1)(b,n-1)}{(c,n-1)(1,n-1)}.$$

Hence it suffices to prove that

$$\sum_{n=2}^{\infty}[1+\gamma(n-1)]|a_n|\leq|\tau|(1-\beta).$$

It is easy to see that

$$S := \sum_{n=2}^{\infty}[1+\gamma(n-1)]a_n = \sum_{n=1}^{\infty}\frac{(a,n)(b,n)}{(c,n)(1,n)} + \gamma\frac{ab}{c}\sum_{n=2}^{\infty}\frac{(a+1,n-2)(b+1,n-2)}{(c+1,n-2)(1,n-2)}. \tag{3.4}$$

Case 1 (i). Let $a, b > 0$ and $c > a+b$. An easy computation using hypothesis (i) of the theorem and

$$F(a,b;c;1) = \sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n)(1,n)} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},$$

where $a, b > 0$ and $c > a+b$, gives the required result.

Case 2 (ii). Let $-1 < a < 0$, $b > 0$ and $c > 0$. Then (3.4) gives

$$\begin{aligned} S &= \frac{|ab|}{c}\sum_{n=0}^{\infty}\frac{(a+1,n)(b+1,n)}{(c+1,n)(1,n+1)} + \gamma\frac{|ab|}{c}\sum_{n=0}^{\infty}\frac{(a+1,n)(b+1,n)}{(c+1,n)(1,n)} \\ &= \frac{|ab|}{c}\sum_{n=0}^{\infty}\frac{(a+1,n)(b+1,n)}{(c+1,n)(1,n+1)} + \gamma\frac{|ab|}{c}\cdot\frac{(a+1)(b+1)}{c+1}\sum_{n=1}^{\infty}\frac{(a+2,n)(b+2,n)}{(c+2,n)(1,n+1)}. \end{aligned}$$

Using (3.2), we easily get that the above expression is equivalent to

$$\begin{aligned} &\frac{|ab|}{c}\left\{\frac{1}{|ab|}\cdot\frac{\Gamma(c-a-b)\Gamma(c+1)}{\Gamma(c-a)\Gamma(c-b)} - \frac{c}{|ab|}\right\} \\ &\quad + \gamma\frac{|ab|}{c}\cdot\frac{(a+1)(b+1)}{(c+1)}\left\{\frac{1}{(a+1)(b+1)}\cdot\frac{\Gamma(c-a-b-1)\Gamma(c+2)}{\Gamma(c-a)\Gamma(c-b)} - \frac{(c+1)}{(a+1)(b+1)} - 1\right\} \end{aligned}$$

which by hypothesis (ii) of the theorem gives the result.

Case 3 (iii). Let $a, b \in \mathbb{C}\backslash\{0\}$, $c > |a|+|b|$. Since $|(a,n)| \le (|a|,n)$, we have from (3.4),

$$\begin{aligned} S &:= \sum_{n=2}^{\infty}[1+\gamma(n-1)]|a_n| \\ &= \sum_{n=0}^{\infty}[1+\gamma(n+1)]|a_{n+2}| \\ &\le \frac{|ab|}{c}\sum_{n=0}^{\infty}\frac{(|a|+1,n)(|b|+1,n)}{(c+1,n)(1,n+1)} + \gamma\sum_{n=0}^{\infty}(n+1)\frac{(|a|,n+1)(|b|,n+1)}{(c,n+1)(1,n+1)}. \end{aligned}$$

The right hand side of the above expression can be written as

$$\frac{|ab|}{c}\sum_{n=0}^{\infty}\frac{(|a|+1,n)(|b|+1,n)}{(c+1,n)(1,n+1)} + \gamma\sum_{n=1}^{\infty}(n+1)\frac{(|a|,n)(|b|,n)}{(c,n)(1,n)} - \gamma\sum_{n=1}^{\infty}\frac{(a,n)(b,n)}{(c,n)(1,n)}. \tag{3.5}$$

Now using (3.2) we get the first part of the expression (3.5) as

$$\frac{|ab|}{c}\sum_{n=0}^{\infty}\frac{(|a|+1,n)(|b|+1,n)}{(c+1,n)(1,n+1)} = \frac{\Gamma(c-|a|-|b|)\Gamma(c)}{\Gamma(c-|a|)}\Gamma(c-|b|) - 1.$$

Similarly using (3.1) we get the second part of the expression (3.5) as

$$\gamma\sum_{n=1}^{\infty}(n+1)\frac{(|a|,n)(|b|,n)}{(c,n)(1,n)}=\gamma\frac{\Gamma(c-|a|-|b|)\Gamma(c)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left(\frac{|ab|}{c-1-|a|-|b|}+1\right).$$

Since the third part of the expression (3.5) is $zF(a,b;c;1)-1$, combining these three parts and using hypothesis (iii) of the theorem we obtain the required result.

□

Proof of Theorem 2.8. Clearly we have

$$G(a,b;c;z)=z+\sum_{n=2}^{\infty}\frac{(a,n-1)(b,n-1)}{(c,n-1)(1,n)}z^n=:z+\sum_{n=2}^{\infty}A_nz^n,$$

and it suffices to prove that

$$\sum_{n=2}^{\infty}[1+\gamma(n-1)]|A_n|\le 1+|\tau|(1-\beta). \tag{3.6}$$

The left hand side of the above inequality can be expressed as

$$(1-\gamma)\sum_{n=1}^{\infty}\frac{(a,n)(b,n)}{(c,n)(1,n+1)}+\gamma\sum_{n=1}^{\infty}\frac{(a,n)(b,n)}{(c,n)(1,n)}$$

which by using (3.2) and $F(a,b;c;1)$ gives (2.9). □

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