Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 4, Issue 1, Article 5, 2002

NEW CONCEPTS OF WELL-POSEDNESS FOR OPTIMIZATION PROBLEMS WITH VARIATIONAL INEQUALITY CONSTRAINTS

[1]IMMA DEL PRETE, [1]M. BEATRICE LIGNOLA, AND [2]JACQUELINE MORGAN

[1]UNIVERSITÀ DEGLI STUDI DI NAPOLI "FEDERICO II"
DIPARTIMENTO DI MATEMATICA E APPLICAZIONI "R. CACCIOPPOLI"
VIA CLAUDIO, 21 - 80125 NAPOLI.
delprete@unina.it

lignola@unina.it

[2]UNIVERSITÀ DEGLI STUDI DI NAPOLI "FEDERICO II"
DIPARTIMENTO DI MATEMATICA E STATISTICA
COMPL. UNIVERSITARIO MONTE S. ANGELO,
VIA CINTIA - 80126 NAPOLI.
morgan@unina.it

Received 14 April, 2002; accepted 03 August, 2002
Communicated by A.M. Rubinov

ABSTRACT. In this note we present a new concept of well-posedness for Optimization Problems with constraints described by parametric Variational Inequalities or parametric Minimum Problems. We investigate some classes of operators and functions that ensure this type of well-posedness.

Key words and phrases: Variational Inequalities, Minimum Problems, Set-Valued Functions, Well-Posedness, Monotonicity, Hemicontinuity.

2000 *Mathematics Subject Classification.* 49J40, 49J53, 65K10.

1. INTRODUCTION

Let E be a reflexive Banach space with dual E^*, A be an operator from E to E^* and $K \subseteq E$ be a nonempty, closed, convex set. The Variational Inequality (VI), defined by the pair (A, K), consists of finding a point u_0 such that:

$$u_0 \in K \text{ and } \langle Au_0, u_0 - v\rangle \leq 0 \ \forall v \in K.$$

This problem, introduced by G. Stampacchia in [22], has been recently investigated by many authors including [2], [4], [8], [9] and [15].

If (X, τ) is a topological space, one can consider the parametric Variational Inequality $(VI)(x)$, defined by the pair $(A(x,\cdot), H(x))$, where, for all $x \in X$, $A(x,\cdot)$ is an operator from E to E^* and H is a set-valued function from X to E with nonempty and convex values.

ISSN (electronic): 1443-5756

035-02

The interest in this study is twofold: one is to study the behavior of perturbations of (VI), another is to consider the parameter x as a decision variable in a multilevel optimization problem. More precisely, the solution set to $(VI)(x)$ can be seen as the constraint set $T(x)$ of the following Optimization Problem with Variational Inequality Constraints:

$$\text{(OPVIC)} \qquad \inf_{x\in X} \inf_{u\in T(x)} f(x,u),$$

where $f : X \times E \to \mathbb{R} \cup \{+\infty\}$.

The problems *OPVIC* (often termed Mathematical Programming with Equilibrium Constraints *MPEC*) have been investigated by many authors (see for example [13], [14], [17], [19] and [21]) since they describe many economic or engineering problems (see for example [18]) such as:

- The price setting problem
- Price setting of telecommunication networks
- Yield management in airline industry
- Traffic management through link tolls.

Assuming that $(VI)(x)$ has a unique solution, a well-posedness concept for *OPVIC*, inspired from numerical methods, has been considered in [13]. However, in many applications, the problems $(VI)(x)$ do not always have a unique solution.

So, in this paper, motivated from a numerical method for Variational Inequalities (M. Fukushima [7]), we introduce and study, for $\alpha \geq 0$, the concepts of $\alpha-$well-posedness and $\alpha-$well-posedness in the generalized sense for a family of Variational Inequalities $(\mathbf{VI}) = \{(VI)(x),\ x \in X\}$ and for *OPVIC*. The particular case of variational inequalities arising from minimum problems is also considered.

The paper is organized as follows. In Section 2 we review some basic notions for variational inequalities and present some new results on $\alpha-$well-posedness for unparametric variational inequalities. Section 3 is devoted to introducing and investigating the concept of $\alpha-$well-posedness for parametric variational inequalities and Section 4 to parametric minimum problems. Finally, some new concepts of well-posedness for *OPVIC* is presented and investigated in Section 5.

2. DEFINITIONS AND BACKGROUND

In this section, some notions of *well-posedness* for variational inequalities (VI) introduced in [13] and in [15] and their connections with optimization problems are presented, together with equivalent characterizations.

Let E be a reflexive Banach space with dual E^*, σ be a convergence on E, and K be a nonempty, closed and convex subset of E.

Definition 2.1. [5, 23]. Let $h : K \to \mathbb{R} \cup \{+\infty\}$. The minimization problem (2.1):

$$\min_{v\in K} h(v) \tag{2.1}$$

is Tikhonov well-posed (resp. well-posed in the generalized sense) with respect to σ if there exists a unique solution u_0 to (2.1) and every minimizing sequence $\sigma-$converges to u_0 (resp. if (2.1) has at least a solution and every minimizing sequence has a subsequence $\sigma-$converging to a minimum point).

For an operator A from E to E^*, we consider the following Variational Inequality (VI) defined by the pair (A, K):

$$\text{find } u_0 \in K \text{ such that } \langle Au_0, u_0 - v\rangle \leq 0\ \forall v \in K.$$

Definition 2.2. [13, 15] Let $\alpha \geq 0$. A sequence $(u_n)_n$ is $\alpha-$*approximating* for (VI) if:

i) $u_n \in K \ \ \forall n \in \mathbb{N}$;
ii) there exists a sequence $(\varepsilon_n)_n$, $\varepsilon_n > 0$, decreasing to 0 such that

$$\langle Au_n, u_n - v\rangle - \frac{\alpha}{2}\|u_n - v\|^2 \leq \varepsilon_n \quad \forall v \in K \quad \forall n \in \mathbb{N}.$$

A variational inequality (VI) is termed $\alpha-$*well-posed* with respect to σ, if it has a unique solution u_0 and every $\alpha-$ approximating sequence $(u_n)_n$ $\sigma-$ converges to u_0. If σ is the strong convergence s (resp. the weak convergence w) on E, (VI) will be termed *strongly* $\alpha-$*well-posed* (resp. *weakly* $\alpha-$*well-posed*).

The above concept originated from the notion of Tikhonov well-posedness for the following minimization problem (2.2):

$$\min_{u \in K} g_\alpha(u), \tag{2.2}$$

where

$$g_\alpha(u) = \sup_{v \in K}\left(\langle Au, u - v\rangle - \frac{\alpha}{2}\|u - v\|^2\right).$$

Indeed, the following result holds:

Proposition 2.1. *Let $\alpha \geq 0$. The variational inequality problem $(\mathbf{VI})$ is $\alpha-$well-posed if and only if the minimization problem (2.2) is Tikhonov well-posed.*

Proof. If (VI) is $\alpha-$well-posed there exists a unique solution u_0 for $(\mathbf{VI})$, that is:

$$u_0 \in K \text{ and } g_0(u_0) = \sup_{v \in K}\langle Au_0, u_0 - v\rangle \leq 0$$

and, consequently, $g_\alpha(u_0) \leq g_0(u_0) \leq 0$. Since $g_\alpha(u) \geq 0$ for every $u \in K$, $g_\alpha(u_0) = 0$ and u_0 is a minimum point for g_α. In order to prove that (2.2) has a unique solution, consider $u' \in K$ such that $g_\alpha(u') = g_\alpha(u_0) = 0$. For every $v \in K$ consider the point $w = \lambda u' + (1 - \lambda)v$, $\lambda \in [0, 1]$, which belongs to K. Since $g_\alpha(u') = 0$ one has:

$$\langle Au', u' - w\rangle - \frac{\alpha}{2}\|u' - w\|^2 = (1 - \lambda)\langle Au', u' - v\rangle - \frac{\alpha}{2}(1 - \lambda)^2\|u' - v\|^2 \leq 0$$

which implies:

$$\langle Au', u' - v\rangle - \frac{\alpha}{2}(1 - \lambda)\|u' - v\|^2 \leq 0 \; \forall \lambda \; \in [0, 1].$$

So, when λ converges to 1, one gets:

$$\langle Au', u' - v\rangle \leq 0 \; \forall \; v \in K.$$

Then also u' solves (VI) and it must coincide with u_0.

As the family of minimizing sequences for (2.2) coincides with the family of $\alpha-$ approximating sequence for (VI), the first part is proved.

Now, assume that (2.2) is well-posed and u_α is the unique solution for (2.2), that is $u_\alpha \in K$ and $g_\alpha(u_\alpha) = 0$.

With the same arguments used in the first part of this proof it can be proved that u_α solves also the variational inequality (VI) (this has been already proved in [7] with other arguments). In order to prove that u_α is the unique solution to (VI), let u' be another solution to (VI). Since $g_\alpha(u') \leq g_0(u') = 0$, the point u' should be a solution to (2.2), thus it has to coincide with u_α.

Then the result follows as in the first part. □

The gap function g_α, which provides an optimization problem formulation for (VI), is, for $\alpha = 0$, the gap function introduced by Auslender in [1], and, for $\alpha > 0$, the merit function introduced by Fukushima in [7] for numerical purposes.

As it is well known, when the set K is not bounded, the set T of the solutions to (VI) may be empty, even in finite dimensional spaces. This does not happen when the operator A satisfies some of the following well known properties.

Definition 2.3. The operator A is said to be:

- *monotone* on K if $\langle Au - Av, u - v\rangle \geq 0$ for every u and $v \in K$,
- *pseudomonotone* on K if for every u and $v \in K$ $\langle Au, u - v\rangle \leq 0 \Rightarrow \langle Av, u - v\rangle \leq 0$;
- *strongly monotone* on K (with modulus β) if $\langle Au - Av, u - v\rangle \geq \beta \|u - v\|^2$ for every u and $v \in K$;
- *hemicontinuous* on K if it is continuous from every segment of K to E^* endowed with the weak topology.

It is well known (see for example [2]) that the variational inequality (VI) has a unique solution if the operator A is strongly monotone and hemicontinuous, while there exists at least a solution for (VI) if the operator A is pseudomonotone and hemicontinuous and some coerciveness condition is satisfied (see for example [8]).

We recall some continuity properties for set-valued functions that will be used later on:

Definition 2.4. A set-valued function F from a topological space (X, τ) to a convergence space (Y, σ) (see [11]) is:

- *sequentially $\sigma-$lower semicontinuous* at $x \in X$ if, for every sequence $(x_n)_n$ $\tau-$converging to x and every $y \in F(x)$, there exists a sequence $(y_n)_n$ $\sigma-$converging to y such that $y_n \in F(x_n)$ $\forall n \in \mathbb{N}$;
- *sequentially $\sigma-$subcontinuous* at $x \in X$ if, for every sequence $(x_n)_n$ $\tau-$converging to $x,$ every sequence $(y_n)_n$, $y_n \in F(x_n)$ $\forall n \in \mathbb{N}$, has a $\sigma-$convergent subsequence;
- *sequentially $\sigma-$closed* at $x \in X$ if for every sequence $(x_n)_n$ $\tau-$converging to x, for every sequence $(y_n)_n$ $\sigma-$converging to y, $y_n \in F(x_n)$ $\forall n \in \mathbb{N}$, one has $y \in F(x)$.

We have chosen to deal with sequential continuity notions for set-valued functions since our well-posedness concepts are defined in a sequential way. However, for brevity, from now on the term *sequentially* will be omitted.

Let $\varepsilon > 0$. The following approximate solutions set, introduced in [15],

$$\mathcal{T}_{\alpha,\varepsilon} = \left\{ u \in K : \langle Au, u - v\rangle \leq \varepsilon + \frac{\alpha}{2} \|u - v\|^2 \;\; \forall v \in K \right\} \quad \text{for } \varepsilon > 0$$

can be used to provide a characterization of $\alpha-$well-posedness in line with [13, Prop. 2.3 bis] and [5].

Proposition 2.2. *Let $\alpha \geq 0$ and assume that the operator A is hemicontinuous and monotone on K and that (VI) has a unique solution. The variational inequality (VI) is strongly $\alpha-$well-posed if and only if*

$$\mathcal{T}_{\alpha,\varepsilon} \neq \emptyset \;\; \forall\, \varepsilon > 0 \quad \textit{and} \; \lim_{\varepsilon \to 0} diam(\mathcal{T}_{\alpha,\varepsilon}) = 0.$$

Proof. Assume that (VI) is strongly $\alpha-$well-posed and

$$\lim_{\varepsilon \to 0} diam \mathcal{T}_\alpha(\varepsilon) > 0.$$

Then there exists a positive number β such that, for every sequence $(\varepsilon_n)_n$ decreasing to 0, $\varepsilon_n > 0$, there exist two sequences $(y_n)_n$ and $(v_n)_n$ in K such that

$$y_n \in \mathcal{T}_{\alpha,\varepsilon_n}, \;\; v_n \in \mathcal{T}_{\alpha,\varepsilon_n} \text{ and } \|y_n - v_n\| > \beta \text{ for } n \text{ sufficiently large.}$$

Since (VI) is strongly $\alpha-$well-posed, the sequences $(y_n)_n$ and $(v_n)_n$ must converge to the unique solution u_0, so

$$\lim_n \|y_n - v_n\| = 0$$

which gives a contradiction.

Conversely, let $(y_n)_n$ be an $\alpha-$approximating sequence for (VI), that is $y_n \in \mathcal{T}_{\alpha,\varepsilon_n}$ for a sequence $(\varepsilon_n)_n$, $\varepsilon_n > 0$, decreasing to 0. Being $\lim_n diam\, \mathcal{T}_{\alpha,\varepsilon_n} = 0$, for every positive number β there exists a positive integer m such that $\|y_n - y_p\| < \beta \ \forall n \geq m$ and $p \geq m.$

Therefore $(y_n)_n$ is a Cauchy sequence and has to converge to a point $u_0 \in K.$ Since A is monotone one has:

$$\begin{aligned}\langle Av, u_0 - v\rangle &= \lim_n \langle Av, y_n - v\rangle \\ &\leq \liminf_n \langle Ay_n, y_n - v\rangle \\ &\leq \lim_n \frac{\alpha}{2}\|y_n - v\|^2 = \frac{\alpha}{2}\|u_0 - v\|^2 \ \forall v \ \in K.\end{aligned}$$

Since A is monotone and hemicontinuous, the following equivalence holds:

$$\langle Av, u_0 - v\rangle - \frac{\alpha}{2}\|u_0 - v\|^2 \leq 0 \ \ \forall v \in K \Leftrightarrow \langle Au_0, u_0 - v\rangle - \frac{\alpha}{2}\|u_0 - v\|^2 \leq 0 \ \ \forall v \in K.$$

In fact, assume that

$$\langle Av, u_0 - v\rangle - \frac{\alpha}{2}\|u_0 - v\|^2 \leq 0 \ \ \forall\, v \in K.$$

If v is a point of K, for every number $t \in [0,1]$ the point $v_t = tv + (1-t)u_0$ belongs to K, so:

$$\langle Av_t, u_0 - v_t\rangle - \frac{\alpha}{2}\|u_0 - v_t\|^2 = t\langle Av_t, u_0 - v\rangle - t^2\frac{\alpha}{2}\|u_0 - v\|^2 \leq 0 \ \forall t \in [0,1].$$

So one has:

$$\lim_{t\to 0}\left(\langle Av_t, u_0 - v\rangle - \frac{\alpha}{2}t\|u_0 - v\|^2\right) \leq 0$$

and, in light of the hemicontinuity of A:

$$\langle Au_0, u_0 - v\rangle - \frac{\alpha}{2}\|u_0 - v\|^2 \leq \langle Au_0, u_0 - v\rangle \leq 0 \quad \forall v \in K.$$

The converse is an easy consequence of the monotonicity of $A.$

So $g_\alpha(u_0) = 0$ and, arguing as in Proposition 2.1, it can be proved that u_0 coincides with the unique solution to (VI). This completes the proof. □

3. PARAMETRICALLY $\alpha-$WELL-POSED VARIATIONAL INEQUALITIES

In what follows we shall consider a topological space (X, τ), a convergence σ on E and, for every $x \in X$, a parametric variational inequality on E, $(VI)(x)$, defined by the pair $(A(x,\cdot), H(x))$, where A is an operator from $X \times E$ to E^* and H is a set-valued function from X to E which is assumed to be nonempty, convex and closed-valued. In many situations $H(x)$ is described by a finite number of inequalities: $H(x) = \{u \in E : g_i(x,u) \leq 0, \ \forall\, i = 1, \ldots, n\}$, where g_i is a real-valued function, for $i = 1, \ldots, n$, satisfying suitable assumptions.

Throughout this section we will consider the following family of variational inequalities:

$$(\mathbf{VI}) = \{(VI)(x), \ x \in X\}.$$

Let $\alpha \geq 0$ and $\varepsilon > 0$. In the sequel, we shall denote by T (resp. $T_{\alpha,\varepsilon}$) the map which associates to every $x \in X$ the solution set (resp. the approximate solution set) to $(VI)(x)$:

$$T(x) = \{u \in H(x) : \langle A(x,u), u - v\rangle \leq 0 \; \forall \; v \in \; H(x)\}$$

$$\text{(resp. } T_{\alpha,\varepsilon}(x) = \Big\{u \in H(x) : \langle A(x,u), u - v\rangle \leq \varepsilon + \frac{\alpha}{2}\|u-v\|^2 \;\; \forall v \in H(x)\Big\}).$$

Now, we introduce the notion of parametric $\alpha-$ well-posedness for the family $(\mathbf{VI})$.

Definition 3.1. Let $x \in X$ and $(x_n)_n$ be a sequence converging to x. A sequence $(u_n)_n$ is said to be $\alpha-$approximating for $(VI)(x)$ (with respect to $(x_n)_n$) if:

i) $u_n \in H(x_n) \;\; \forall n \in \mathbb{N}$,

ii) there exists a sequence $(\varepsilon_n)_n$, $\varepsilon_n > 0$, decreasing to 0 such that

$$\langle A(x_n,u_n), u_n - v\rangle - \frac{\alpha}{2}\|u_n - v\|^2 \leq \varepsilon_n \quad \forall v \in H(x_n) \quad \forall n \in \mathbb{N}.$$

Definition 3.2. The family of variational inequalities $(\mathbf{VI})$ is termed *parametrically $\alpha-$well-posed* with respect to σ if:

- for every $x \in X$, $(VI)(x)$ has a unique solution u_x;
- for every sequence $(x_n)_n$ converging to x, every $\alpha-$ approximating sequence $(u_n)_n$ for $(VI)(x)$ (with respect to $(x_n)_n$) $\sigma-$converges to u_x.

If σ is the strong convergence s (resp. the weak convergence w) on E, $(\mathbf{VI})$ will be termed *parametrically strongly $\alpha-$well-posed* (resp. *parametrically weakly $\alpha-$well-posed).*

Observe that for $\alpha = 0$ the above definition amounts to Definition 2.3 in [13].

Definition 3.3. The family of variational inequalities $(\mathbf{VI})$ is termed *parametrically* $\alpha-$well-posed in the *generalized* sense with respect to σ if, for every $x \in X$, $(VI)(x)$ has at least a solution and for every sequence $(x_n)_n$ converging to x, every $\alpha-$approximating sequence $(u_n)_n$ for $(VI)(x)$ (with respect to $(x_n)_n$) has a subsequence $\sigma-$convergent to a solution to $(VI)(x)$.

For a parametric variational inequality it is natural to consider the following parametric gap function $g_\alpha(x,u)$:

$$g_\alpha(x,u) = \sup_{v\in H(x)} \Big(\langle A(x,u), u - v\rangle - \frac{\alpha}{2}\|u-v\|^2\Big)$$

and with the same arguments as in Proposition 2.1 one can prove the following two propositions:

Proposition 3.1. *Let $\alpha \geq 0$ and $x \in X$. A point u_x solves the variational inequality $(VI)(x)$ if and only if :*

$$u_x \in H(x) \text{ and } g_\alpha(x,u_x) = \inf_{u\in H(x)} g_\alpha(x,u) = 0,$$

that is:

$$\langle A(x,u), u - v\rangle - \frac{\alpha}{2}\|u-v\|^2 \leq 0 \; \forall v \in H(x).$$

Proposition 3.2. *The family of variational inequality $(\mathbf{VI})$ is parametrically $\alpha-$well-posed (resp. parametrically-$\alpha-$well-posed in the generalized sense) with respect to σ if and only if, for every $x \in X$, the minimization problem*

$$\min_{u\in H(x)} g_\alpha(x,u) \tag{3.1}$$

is parametrically Tikhonov well-posed (resp. parametrically Tikhonov well-posed in the generalized sense) with respect to σ, that is: g_α is bounded from below, (3.1) has a unique solution *(resp. has at least a solution)* u_x *and for every sequence* $(x_n)_n$ *converging to* x, *every sequence* $(u_n)_n$ *such that*

$$\inf_{u \in H(x)} g_\alpha(x, u) \geq \liminf_n g_\alpha(x_n, u_n)$$

$\sigma-$*converges (resp. has a subsequence* $\sigma-$*convergent) to* u_x *(see Definition 2.3 in* [13]*).*

The connection between parametric $\alpha-$well-posedness and the convergence to 0 of the diameters of $T_{\alpha,\varepsilon}(x)$ is given by the following result.

Proposition 3.3. *Let* $\alpha \geq 0$. *If the family of variational inequalities* (VI) *is strongly parametrically* $\alpha-$*well-posed, then, for every* $x \in X$, *every sequence* $(x_n)_n$ *converging to* x *and every sequence* $(\varepsilon_n)_n$ *of positive real numbers decreasing to 0, one has:*

$$T_{\alpha,\varepsilon}(x) \neq \emptyset \;\; \forall \varepsilon > 0 \quad \textit{and} \;\; \lim_n diam(T_{\alpha,\varepsilon_n}(x_n)) = 0.$$

Proof. In light of the assumption, the set $T_{\alpha,\varepsilon}(x)$ is nonempty since $\{u_x\} = T(x) \subseteq T_{\alpha,\varepsilon}(x)$. Assume that $\lim_n diam(T_{\alpha,\varepsilon_n}(x_n) > 0$. Then there exist $\eta > 0$ and two sequences $(u_n)_n$ and $(y_n)_n$ such that $u_n \in T_{\alpha,\varepsilon_n}(x_n)$, $y_n \in T_{\alpha,\varepsilon_n}(x_n)$ and $\|y_n - u_n\| > \eta$, for n sufficiently large. But, being $(u_n)_n$ and $(y_n)_n$ sequences $\alpha-$ approximating for $(VI)(x)$ (with respect to $(x_n)_n$), they must converge to u_x, and this gives a contradiction. □

In order to achieve a similar result for generalized $\alpha-$ well-posedness, one can consider the non compactness measure μ, introduced by Kuratowski in [11]: if (S, d) is a metric space and B is a bounded subset of S, $\mu(B)$ is defined as the infimum of $\varepsilon > 0$ such that B can be covered by a finite number of open sets having diameter less than ε. The following proposition, whose proof is in line with previous results concerning generalized well-posedness for minimum problems (see [5]), gives the link between the noncompactness measure of $T_{\alpha,\varepsilon_n}(x)$ and the generalized $\alpha-$well-posedness, when the set-valued function H is constant:

Proposition 3.4. *Let* $\alpha \geq 0$. *Assume that for every* $u \in E$ *the operator* $A(\cdot, u)$ *is continuous from* X *to* (E^*, w) *and the set-valued function* H *is constant, that is* $H(x) = K$, *where* K *is a nonempty, closed convex subset of* E. *If the family of variational inequalities* $(\mathbf{VI})$ *is parametrically strongly* $\alpha-$*well-posed in the generalized sense, then, for every* $x \in X$, *every sequence* $(x_n)_n$ *converging to* x *and every sequence* $(\varepsilon_n)_n$ *of positive real numbers decreasing to 0, one has:*

$$T_{\alpha,\varepsilon}(x) \neq \emptyset \;\; \forall \varepsilon > 0 \;\; \textit{and} \;\; \lim_n \mu(T_{\alpha,\varepsilon_n}(x_n)) = 0.$$

Proof. Let $(\varepsilon_n)_n$ be a sequence of positive real numbers, let $x \in X$ and $(x_n)_n$ be a sequence converging to x.

We start by proving that $\lim_n h(T_{\alpha,\varepsilon_n}(x_n), T(x)) = 0$, where $h(T_{\alpha,\varepsilon_n}(x_n), T(x)) = h_n$ is the Hausdorff distance [11] between $T_{\alpha,\varepsilon_n}(x_n)$ and the set of solutions to $(VI)(x)$, that is:

$$h_n = \max\left\{ \sup_{u \in T_{\alpha,\varepsilon_n}(x_n)} d(u, T(x)), \sup_{v \in T(x)} d(T_{\alpha,\varepsilon_n}(x_n), v) \right\}.$$

By the assumptions, every $u \in T(x)$ belongs to $T_{\alpha,\varepsilon_n}(x_n)$, for n sufficiently large.

Indeed $u \in T(x)$ if and only if $\langle A(x, u), u - v \rangle \leq 0 \; \forall \; v \in K$ and, consequently:

$$\langle A(x, u), u - v \rangle - \frac{\alpha}{2} \|u - v\|^2 \leq 0 \; \forall v \in K.$$

If

$$v \neq u, \; \langle A(x, u), u - v \rangle - \frac{\alpha}{2} \|u - v\|^2 < 0 = \lim_n \varepsilon_n$$

and in light of continuity of $A(\cdot, u)$ one gets

$$\langle A(x_n, u), u - v\rangle - \frac{\alpha}{2}\|u - v\|^2 < \varepsilon_n$$

for n sufficiently large.
If $v = u$, the result is obvious since

$$\langle A(x_n, u), u - v\rangle - \frac{\alpha}{2}\|u - v\|^2 = 0 < \varepsilon_n \text{ for every } n \in \mathbb{N}.$$

So, if $\limsup_n h(T_{\alpha,\varepsilon_n}(x_n), T(x)) > c > 0$, there exists a sequence $(u_n)_n$:

$$u_n \in T_{\alpha,\varepsilon_n}(x_n) \text{ and } d(\, u_n, T(x)) > c \text{ for } n \text{ sufficiently large.}$$

Since $(u_n)_n$ is $\alpha-$approximating, there is a subsequence $(u_{n_k})_k$ converging to $u_x \in T(x)$ and one gets:

$$0 = d(u_x, T(x)) \geq \limsup_k d(u_{n_k}, T(x)) > c,$$

which gives a contradiction.

In order to complete the proof, it takes only to observe that $T_{\alpha,\varepsilon_n}(x_n) \subseteq B(T(x), h_n)$ (the ball of radius h_n around $T(x)$) and $\mu(T(x)) = 0$, so the following inequality holds (see, for example [5]):

$$\mu(T_{\alpha,\varepsilon_n}(x_n)) \leq 2h_n + \mu(T(x)) = 2h_n.$$

□

The next lemma is in the spirit of the Minty's Lemma and will be used to characterize $\alpha-$well-posedness for parametric variational inequalities. The proof is omitted since it is similar to the proof given in Proposition 2.2 for unparametric variational inequalities.

Lemma 3.5. *Let $\alpha \geq 0$. If, for every $x \in X$, the operator $A(x, \cdot)$ is hemicontinuous and monotone on $H(x)$, then the following conditions are equivalent:*

i) $u_0 \in H(x)$ *and* $\langle A(x, u_0), u_0 - v\rangle - \frac{\alpha}{2}\|u_0 - v\|^2 \leq 0$ *for every* $v \in H(x)$,
ii) $u_0 \in H(x)$ *and* $\langle A(x, v), u_0 - v\rangle - \frac{\alpha}{2}\|u_0 - v\|^2 \leq 0$ *for every* $v \in H(x)$.

The next proposition proves that in finite dimensional spaces the parametric $\alpha-$well-posedness is equivalent to the uniqueness of solutions to $(VI)(x)$, for every $\alpha \geq 0$.

Proposition 3.6. *Let $\alpha \geq 0$ and $E = R^k$. If the following conditions hold:*

i) *the set-valued function H is lower semicontinuous, closed and subcontinuous;*
ii) *for every $x \in X$, $A(x, \cdot)$ is monotone and hemicontinuous;*
iii) *for every $u \in R^k$, $A(\cdot, u)$ is continuous on X;*
iv) *A is uniformly bounded on $X \times R^k$, that is there exists $k > 0$ such that for every converging sequence $(x_n, u_n)_n$ one has $\|A(x_n, u_n)\| \leq k$ for every $n \in \mathbb{N}$;*

then $(\mathbf{VI})$ *is parametrically $\alpha-$well-posed if and only if, for every $x \in X$, $(VI)(x)$ has a unique solution u_x.*

Proof. For $x \in X$, let $(x_n)_n$ be a sequence converging to x and $(u_n)_n$ be an $\alpha-$ approximating sequence (with respect to $(x_n)_n$), that is:

$$u_n \in H(x_n) \text{ and } \langle A(x_n, u_n), u_n - v\rangle \leq \varepsilon_n + \frac{\alpha}{2}\|u_n - v\|^2 \quad \forall v \in H(x_n),$$

where $(\varepsilon_n)_n$, $\varepsilon_n > 0$, is a sequence decreasing to 0.

Since H is closed and subcontinuous there exists a subsequence $(u_{n_k})_k$ of $(u_n)_n$ converging to a point $\widetilde{u}_x \in H(x)$. Moreover, in light of the lower semicontinuity of H, for every $v \in H(x)$ there exists a sequence $(v_n)_n$ converging to v such that $v_n \in H(x_n)$ for every $n \in \mathbb{N}$.

The monotonicity of $A(x_{n_k},\cdot)$ implies:

$$\begin{aligned}\langle A(x_{n_k},v),u_{n_k}-v\rangle &\leq \langle A(x_{n_k},u_{n_k}),u_{n_k}-v_{n_k}\rangle + \langle A(x_{n_k},u_{n_k}),v_{n_k}-v\rangle \\ &\leq \varepsilon_{n_k}+\frac{\alpha}{2}\left\|u_{n_k}-v_{n_k}\right\|^2+\left\|A(x_{n_k},u_{n_k})\right\|\left\|v_{n_k}-v\right\|\end{aligned}$$

for every $k\in\mathbb{N}$.

Since $A(\cdot,v)$ is continuous at x and A is uniformly bounded one has:

$$\langle A(x,v),\widetilde{u}_x-v\rangle \leq \frac{\alpha}{2}\left\|\widetilde{u}_x-v\right\|^2$$

and applying the previous lemma:

$$\langle A(x,\widetilde{u}_x),\widetilde{u}_x-v\rangle \leq \frac{\alpha}{2}\left\|\widetilde{u}_x-v\right\|^2.$$

But, from Proposition 3.1, this inequality is equivalent to:

$$\langle A(x,\widetilde{u}_x),\widetilde{u}_x-v\rangle \leq 0\ \forall v\in H(x)$$

that is $\widetilde{u}_x$ solves $(VI)(x)$.

Since $(VI)(x)$ has a unique solution, the point $\widetilde{u}_x$ must coincide with u_x and the whole sequence $(u_n)_n$ has to converge to u_x. □

A similar result could be obtained in infinite dimensional spaces if one modifies the assumptions: in iii) $A(\cdot,u)$ should be continuous from X to (E^*,s), but in i) H should be assumed to be $s-$lower semicontinuous, $w-$closed and $s-$subcontinuous, which unfortunately lead to the strong compactness of $H(x)$ for every $x\in X$.

Remark 3.7. If the set-valued function H is constant, that is $H(x)=K\ \forall x\in X$, the same result holds assuming that the set K is compact and convex, $A(x,\cdot)$ is monotone and hemicontinuous on K for every $x\in X$, and $A(\cdot,u)$ is continuous on X for every $u\in K$. Indeed, arguing as in Proposition 3.6, for every $v\in K$ one has:

$$\begin{aligned}\langle A(x_{n_k},v),\widetilde{u}-v\rangle &= \langle A(x_{n_k},v),\widetilde{u}-u_{n_k}\rangle + \langle A(x_{n_k},v),u_{n_k}-v\rangle \\ &\leq \langle A(x_{n_k},v),\widetilde{u}-u_{n_k}\rangle + \langle A(x_{n_k},u_{n_k}),u_{n_k}-v\rangle \\ &\leq \langle A(x_{n_k},v),\widetilde{u}-u_{n_k}\rangle + \varepsilon_{n_k}+\frac{\alpha}{2}\left\|u_{n_k}-v\right\|^2,\end{aligned}$$

and for k converging to $+\infty$ the result follows.

Example 3.1. If E is an infinite dimensional space, the previous result may fail to be true when K is only weakly compact, that is: there are variational inequalities with a unique solution which are not $\alpha-$well-posed. Indeed, the following example (already considered in [5]) holds: let E be a separable Hilbert space with an ortonormal basis $(e_n)_n$, B be the unitary closed ball of E. Consider the operator $\bigtriangledown h(u)$, where $h(u)=\sum_n\frac{\langle u,e_n\rangle}{n^2}$ and the variational inequality (VI) defined by: $v\in B$ and $\langle\bigtriangledown h(u),u-v\rangle\leq 0\ \forall\ v\in B$.

It has as unique solution $u_0=0$, but $(e_n)_n$ is an approximating (and consequently $\alpha-$ approximating for every $\alpha>0$) sequence that does not strongly converge to 0.

The next result and the following remark, concerning $\alpha-$well-posedness in the generalized sense, can be easily proved with the same arguments as in Proposition 3.6 and Remark 3.7.

Proposition 3.8. *Let $E=R^k$ and $\alpha\geq 0$. If the assumptions of Proposition 3.6 hold, then the family* $(\mathbf{VI})$ *is parametrically $\alpha-$well-posed in the generalized sense.*

Proof. Since under assumption i) the set $H(x)$ is compact, $(VI)(x)$ has at least a solution for every $x\in X$ (see for example [10] or [2]), so the result can be easily proved as in Proposition 3.6. □

The previous proposition says nothing else that, under conditions i) to iv), in finite dimensional spaces, the parametric $\alpha-$well-posedness in the generalized sense is equivalent to the existence of solutions.

Remark 3.9. If the set-valued function K is constant, that is $H(x) = K \ \forall x \in X$, the same result holds assuming that the set K is compact and convex, for every $x \in X$ $A(x, \cdot)$ is monotone and hemicontinuous on H, and, for every $u \in K$ $A(\cdot, u)$ is continuous on X.

The following propositions furnish classes of operators for which the corresponding variational inequalities are parametrically $\alpha-$well-posed or parametrically $\alpha-$well-posed in the generalized sense.

Proposition 3.10. *Assume that the following conditions are satisfied:*

i) *the operator A is strongly monotone on E in the variable u, uniformly with respect to x, that is:*

$$\exists\, \beta > 0 \ \text{ such that } \ \langle A(x,u) - A(x,v), u - v\rangle \geq \beta \left\|u - v\right\|^2 \ \forall x \in X, \ \forall\, u \in E, \ \forall v \in E;$$

ii) *for every $u \in E$, $A(\cdot, u)$ is continuous from (X, τ) to (E^*, s);*
iii) *for every $x \in X$, $A(x, \cdot)$ is hemicontinuous on $H(x)$;*
iv) *A is uniformly bounded on $X \times E$;*
v) *the set-valued function H is $w-$closed, $w-$subcontinuous and $s-$lower semicontinuous.*

Then $(\mathbf{VI})$ *is parametrically strongly $\alpha-$well-posed for every α such that $0 \leq \alpha \leq 2\beta$.*

Proof. First of all, for every $x \in X$, the variational inequality $(VI)(x)$ has a unique solution u_x (see, for example, [10] or [2]).

To prove that, for $0 \leq \alpha \leq 2\beta$, every $\alpha-$approximating sequence is strongly convergent, let $x \in X$, $(x_n)_n$ be a sequence converging to x and $(u_n)_n$ be an $\alpha-$approximating sequence for $(\mathbf{VI})$ with respect to $(x_n)_n$.

Since H is $w-$closed and $w-$subcontinuous, the sequence $(u_n)_n$ has a subsequence, still denoted by $(u_n)_n$, which weakly converges to $\widetilde{u}_x \in H(x)$. To prove that $\widetilde{u}_x = u_x$, consider a point $v \in H(x)$ and a sequence $(v_n)_n$ strongly converging to v such that $v_n \in H(x_n)$ for every $n \in \mathbb{N}$ (such sequence exists in virtue of the lower semicontinuity of H). One has, for every $n \in \mathbb{N}$:

$$\begin{aligned}\langle A(x_n, v), u_n - v\rangle &\leq \langle A(x_n, u_n), u_n - v\rangle - \beta \left\|u_n - v\right\|^2 \\ &= \langle A(x_n, u_n), u_n - v_n\rangle + \langle A(x_n, u_n), v_n - v\rangle - \beta \left\|u_n - v\right\|^2 \\ &\leq \varepsilon_n + \frac{\alpha}{2} \left\|u_n - v_n\right\|^2 - \beta \left\|u_n - v\right\|^2 + \left\|A(x_n, u_n)\right\| \left\|v_n - v\right\|.\end{aligned}$$

Since $\frac{\alpha}{2} \leq \beta$, one gets:

$$\langle A(x_n, v), u_n - v\rangle \leq \varepsilon_n + \beta \left(\left\|v_n - v\right\|^2 + 2\left\|u_n - v\right\| \left\|v_n - v\right\|\right) + \left\|A(x_n, u_n)\right\| \left\|v_n - v\right\|$$

and in light of assumptions ii) and iv):

$$\langle A(x, v), \widetilde{u}_x - v\rangle \leq 0.$$

The last inequality, for the arbitrarity of v, implies, by Minty's Lemma (see, for example, [2]), that $\widetilde{u}_x$ solves $(VI)(x)$, so $\widetilde{u}_x = u_x$.

To prove that the sequence $(u_n)_n$ strongly converges to u_x, let $(w_n)_n$ be a sequence strongly converging to u_x, $w_n \in H(x_n)\ \forall\, n \in \mathbb{N}$ (such a sequence exists since H is $s-$lower semicontinuous). Observe that:

$$\begin{aligned}\beta\|u_n-u_x\|^2 &\le \langle A(x_n,u_n)-A(x_n,u_x),u_n-u_x\rangle\\ &= \langle A(x_n,u_n),u_n-w_n\rangle+\langle A(x_n,u_n),w_n-u_x\rangle-\langle A(x_n,u_x),u_n-u_x\rangle\\ &\le \varepsilon_n+\frac{\alpha}{2}\|u_n-w_n\|^2+\|A(x_n,u_n)\|\,\|w_n-u_x\|\\ &\qquad -\langle A(x_n,u_x),u_n-u_x\rangle\ \forall\, n\in\mathbb{N}.\end{aligned}$$

Since $\|w_n-u_n\|^2 \le (\|w_n-u_x\|+\|u_n-u_x\|)^2$, one gets, for every $n \in \mathbb{N}$:

$$\begin{aligned}0 &\le \left(\beta-\frac{\alpha}{2}\right)\|u_n-u_x\|^2\\ &\le \varepsilon_n+\frac{\alpha}{2}\|u_x-w_n\|^2+\alpha\|u_n-u_x\|\,\|u_x-w_n\|\\ &\qquad +\|A(x_n,u_n)\|\,\|w_n-u_x\|-\langle A(x_n,u_x),u_n-u_x\rangle\end{aligned}$$

and this implies that $\lim_n \|u_n-u_x\| = 0$. So, we have proved that every weakly converging subsequence of $(u_n)_n$ is also strongly converging to the unique solution for $(VI)(x)$. Then the whole sequence $(u_n)_n$ strongly converges to u_x. □

Remark 3.11. If the set-valued function H is constant, that is $H(x) = K\ \forall x \in X$, the same result can be established assuming that:

i) the operator A is strongly monotone in the variable u on E (with modulus β), uniformly with respect to x;
ii) for every $u \in K$, $A(\cdot,u)$ is continuous from (X,τ) to (E^*,s);
iii) for every $x \in X$, $A(x,\cdot)$ is hemicontinuous on $H(x)$;
iv) the set K is convex, closed and bounded.

For what concerning parametric $\alpha-$well-posedness in the generalized sense, we have the following result for $\alpha = 0$:

Proposition 3.12. *Assume that the following conditions are satisfied:*

i) *for every $x \in X$, $A(x,\cdot)$ is monotone on $H(x)$;*
ii) *for every $u \in H$, $A(\cdot,u)$ is continuous from (X,τ) to (E^*,s);*
iii) *for every $x \in X$, $A(x,\cdot)$ is hemicontinuous on $H(x)$;*
iv) *A is uniformly bounded on $X \times E$;*
v) *the set-valued function H is $w-$closed, $w-$subcontinuous and $s-$lower semicontinuous.*

Then $(\mathbf{VI})$ *is parametrically weakly well-posed in the generalized sense.*

Proof. First of all, for every $x \in X$, the variational inequality $(VI)(x)$ has at least a solution (see, for example, [10] or [2]), since under our assumptions the set $H(x)$ is compact with respect to the weak convergence.

Let $x \in X$, $(x_n)_n$ be a sequence converging to x, and $(u_n)_n$ be an approximating sequence for $(\mathbf{VI})$ with respect to $(x_n)_n$.

Since H is $w-$closed and $w-$subcontinuous, the sequence $(u_n)_n$ has a subsequence, still denoted by $(u_n)_n$, which weakly converges to $u_x \in H(x)$. To prove that $u_x \in T(x)$, consider a point $v \in H(x)$, a sequence $(v_n)_n$ strongly converging to v such that $v_n \in H(x_n)$ for every

$n \in \mathbb{N}$ (such sequence exists in virtue of the lower semicontinuity of H). Since:

$$\begin{aligned}\langle A(x_n, v), u_n - v\rangle &\leq \langle A(x_n, u_n), u_n - v\rangle \\ &= \langle A(x_n, u_n), u_n - v_n\rangle + \langle A(x_n, u_n), v_n - v\rangle \\ &\leq \varepsilon_n + \langle A(x_n, u_n), v_n - v\rangle \\ &\leq \varepsilon_n + \|A(x_n, u_n)\| \, \|v_n - v\| \ \ \forall n \in \mathbb{N}\end{aligned}$$

and assumptions ii) and iv) hold, one gets:

$$\langle A(x, v), u_x - v\rangle \leq 0 \; \forall v \in H(x),$$

that, for the Minty's Lemma, is equivalent to say that u_x solves $(VI)(x)$. □

Remark 3.13. If the set-valued function H is constant, that is $H(x) = K, \forall\, x \in X$, the same result can be established assuming that:

i) the operator $A(x, \cdot)$ is hemicontinuous on H;
ii) the operator $A(x, \cdot)$ is monotone;
iii) for every $u \in K$, $A(\cdot, u)$ is continuous on X;
iv) the set K is convex, closed and bounded.

4. PARAMETRICALLY $\alpha-$WELL-POSED MINIMUM PROBLEMS

In this section we consider variational inequalities arising from parametric minimum problems and we investigate, for $\alpha > 0$, the links between parametric $\alpha-$well-posedness of such problems and parametric $\alpha-$ well-posedness of the corresponding variational inequalities. The case $\alpha = 0$ can be found in [13].

Let h be a function from $X \times E$ to $\mathbb{R} \cup \{+\infty\}$ and H be a set-valued function from X to E, which is assumed to be nonempty, convex and closed-valued. If, for every $x \in X$, the function $h(x, \cdot)$ is Gâteaux differentiable, bounded from below and convex on $H(x)$, the minimum problem:

$$((P)\,(x)) \qquad \inf_{u \in H(x)} h(x, u)$$

is equivalent to the following variational inequality problem:

$$((VI)(x)) \qquad \text{find } u \in H(x) \text{ such that } \quad \left\langle \frac{\partial h}{\partial u}(x, u), u - v\right\rangle \leq 0 \quad \forall v \in H(x),$$

where $\frac{\partial h}{\partial u}$ is the derivative of the function h with respect to the variable u (see [2]). Then, it is natural to introduce the notion of parametric $\alpha-$well-posedness for a family of minimization problems $\mathbf{P} = \{\ (P)\,(x)\ ,\ x \in X\}$ and compare it with the parametric $\alpha-$well-posedness for the family $\mathbf{VI} = \{\ (VI)(x),\ \ x \in X\}$.

Definition 4.1. Let $x \in X$, $(x_n)_n$ be a sequence converging to x; the sequence $(u_n)_n$ is termed $\alpha-$minimizing for $(P)\,(x)$ (with respect to $(x_n)_n$) if:

i) $u_n \in H(x_n)\; \forall\, n \in \mathbb{N}$,
ii) there exists a sequence $(\varepsilon_n)_n$, $\varepsilon_n > 0$, decreasing to 0 such that:

$$h(x_n, u_n) \leq h(x_n, v) + \frac{\alpha}{2}\|u_n - v\|^2 + \varepsilon_n \quad \forall\, v \in H(x_n) \text{ and } \forall n \in \mathbb{N}.$$

Definition 4.2. The family of minimum problems $\mathbf{P}$ is called *parametrically $\alpha-$well-posed, with respect to σ*, if:

i) for every $x \in X$, $h(x, \cdot)$ is bounded from below,
ii) for every $x \in X$, $(P)\,(x)$ has a unique solution u_x,

iii) for every sequence $(x_n)_n$ converging to a point x, every $\alpha-$minimizing sequence $(u_n)_n$ for $(P)(x)$ (with respect to $(x_n)_n$) $\sigma-$converges to u_x.

Definition 4.3. The family of minimum problems $\mathbf{P}$ is called *parametrically $\alpha-$well-posed in the generalized sense, with respect to σ*, if:

i) for every $x \in X$, $h(x, \cdot)$ is bounded from below,
ii) for every $x \in X$, $(P)(x)$ has at least a solution u_x,
iii) for every sequence $(x_n)_n$ converging to a point x, every $\alpha-$minimizing sequence $(u_n)_n$ for $(P)(x)$ (with respect to $(x_n)_n$) has a subsequence $\sigma-$convergent to a solution for $(P)(x)$.

The following two propositions give, under suitable assumptions, the equivalence between parametric $\alpha-$well-posedness for a minimization problem and the corresponding variational inequality.

Proposition 4.1. *Assume that, for all $x \in X$, the function $h(x, \cdot)$ is bounded from below, convex and Gâteaux differentiable on $H(x)$ and the family of problems $\mathbf{P}$ is parametrically $\alpha-$well-posed (resp. in the generalized sense) with respect to σ. Then the family of variational inequalities defined by*

$$((VI)(x)) \qquad \textit{find } u \in H(x) \textit{ such that } \left\langle \frac{\partial h}{\partial u}(x, u), u - v \right\rangle \leq 0 \ \ \forall v \in H(x),$$

is parametrically $\alpha-$well-posed (resp. in the generalized sense) with respect to σ.

Proof. Under the above assumptions, for all $x \in X$, the problems $(VI)(x)$ and $(P)(x)$ have the same solutions. Consider a point $x \in X$, a sequence $(x_n)_n$ converging to x and an $\alpha-$approximating sequence $(u_n)_n$ for $(VI)(x)$, with respect to $(x_n)_n$, that is:

$$u_n \in H(x_n) \text{ and } \left\langle \frac{\partial h}{\partial u}(x_n, u_n), u_n - v \right\rangle - \frac{\alpha}{2} \|u_n - v\|^2 \leq \varepsilon_n \quad \forall v \in H(x_n) \ \ \forall n \in \mathbb{N},$$

where $(\varepsilon_n)_n$, $\varepsilon_n > 0$, decreases to 0. Since $h(x_n, \cdot)$ is convex one has:

$$h(x_n, u_n) - h(x_n, v) \leq \left\langle \frac{\partial h}{\partial u}(x_n, u_n), u_n - v \right\rangle \leq \frac{\alpha}{2} \|u_n - v\|^2 + \varepsilon_n \ \ \forall v \in H(x_n) \ \ \forall n \in \mathbb{N},$$

that is $(u_n)_n$ is $\alpha-$minimizing for $(P)(x)$ (with respect to $(x_n)_n$) and the result then follows. □

Proposition 4.2. *Let E be a real Hilbert space. Assume that, for all $x \in X$, the function $h(x, \cdot)$ is lower semicontinuous, bounded from below and Gâteaux differentiable on $H(x)$ and the family of variational inequalities $(\mathbf{VI})$ is parametrically strongly $0-$well-posed . If the range $H(X)$ is a bounded subset of E, then the family of minimum problems $\mathbf{P}$ is strongly parametrically $\alpha-$well-posed for every $\alpha > 0$.*

Proof. Under the assumptions above, every solution to $(P)(x)$ has to coincide with the unique solution to $(VI)(x)$, $\forall x \in X$.

Consider $x \in X$, a sequence $(x_n)_n$ converging to x and an α-minimizing sequence $(u_n)_n$ for $(P)(x)$, with respect to $(x_n)_n$, that is:

$$u_n \in H(x_n) \quad \text{and} \quad h(x_n, u_n) \leq h(x_n, v) + \frac{\alpha}{2} \|u_n - v\|^2 + \varepsilon_n \ \ \forall\, v \in H(x_n) \ \ \forall n \in \mathbb{N},$$

where $(\varepsilon_n)_n$, $\varepsilon_n > 0$, is a sequence decreasing to 0.

For every $n \in \mathbb{N}$ define a new function f_n on E by:

$$f_n(v) = h(x_n, v) + \frac{\alpha}{2} \|u_n - v\|^2$$

and observe that f_n is lower semicontinuous, bounded from below, Gâteaux differentiable on $H(x_n)$ and $f_n(u_n) = h(x_n, u_n)$.

Since $f_n(u_n) \leq f_n(v) + \varepsilon_n \quad \forall\, v \in H(x_n)$, from Ekeland Theorem (see [6]), for every $n \in \mathbb{N}$ there exists $u'_n \in H(x_n)$ such that:

$$\|u'_n - u_n\| < \sqrt{\varepsilon_n} \text{ and}$$
$$\left\langle \frac{\partial f_n}{\partial u}(u'_n), u'_n - v \right\rangle \leq \sqrt{\varepsilon_n}\, \|u'_n - v\|\, \forall v \in H(x_n)\ \forall n \in \mathbb{N}.$$

Therefore:

$$\begin{aligned}\left\langle \frac{\partial h}{\partial u}(x_n, u'_n), u'_n - v \right\rangle &= \left\langle \frac{\partial f_n}{\partial u}(u'_n), u'_n - v \right\rangle - \alpha \langle u_n - u'_n, u'_n - v\rangle \\ &\leq \sqrt{\varepsilon_n}\, \|u'_n - v\|\, (1+\alpha) \quad \forall v \in H(x_n).\end{aligned}$$

Since the set-valued function H has a bounded range, the sequence $(u'_n)_n$ is 0-approximating for $(VI)(x)$ and the result follows. □

Corollary 4.3. *Let E be a real Hilbert space. Assume that, for all $x \in X$, the function $h(x,\cdot)$ is lower semicontinuous, convex, bounded from below and Gâteaux differentiable on $H(x)$ and the range $H(X)$ is a bounded subset of E. Then the family of variational inequalities (**VI**) is parametrically strongly $\alpha-$well-posed (resp. in the generalized sense) with respect to σ, if and only if the minimum problem $\mathbf{P}$ is parametrically strongly $\alpha-$well-posed (resp. in the generalized sense) with respect to σ.*

Corollary 4.4. *Let E be a real Hilbert space. Assume that, for all $x \in X$, the function $h(x,\cdot)$ is lower semicontinuous, convex, bounded from below and Gâteaux differentiable on $H(x)$ and the range $H(X)$ is a bounded subset of E. Then the family of variational inequalities (**VI**) is parametrically strongly $0-$well-posed (resp. in the generalized sense) if and only if it is parametrically strongly $\alpha-$well-posed (resp. in the generalized sense) for (every) $\alpha > 0$.*

5. $\alpha-$WELL-POSEDNESS FOR *OPVIC*

In this section we consider a convergence σ on E and the problem introduced in Section 1:

(*OPVIC*) $$\inf_{x \in X} \inf_{u \in T(x)} f(x,u),$$

where $f : X \times E \to \mathbb{R} \cup \{+\infty\}$ is bounded from below, H is a set-valued function from X to E, and, for every $x \in X$, $A(x,\cdot)$ is an operator from E to E^*, while $T(x)$ is the set of solutions to the parametric variational inequality $(VI)(x)$ defined by the pair $(A(x,\cdot), H(x))$.

In order to obtain sufficient conditions for $\alpha-$well-posedness of *OPVIC* we shall assume also that the function f satisfies a coercivity condition: namely, we say that f is *equicoercive* on $(X \times E, (\tau \times \sigma))$ if every sequence $(x_n, u_n)_n$, such that $f(x_n, u_n) \leq k\ \forall n \in \mathbb{N}$, has a $(\tau \times \sigma)-$convergent subsequence.

Definition 5.1. Let $\alpha \geq 0$. A sequence $(x_n, u_n)_n$ is said to be $\alpha-$*approximating* for *OPVIC* if:

i) $\liminf_n f(x_n, u_n) \leq \inf_{(x,u) \in X \times E, u \in T(x)} f(x,u)$;

ii) there exists a sequence $(\varepsilon_n)_n$, $\varepsilon_n > 0$, decreasing to 0, such that $u_n \in T_{a,\varepsilon_n}(x_n)\ \forall n \in \mathbb{N}$, that is:

$$u_n \in H(x_n) \text{ and } \langle A(x_n, u_n), u_n - v\rangle - \frac{\alpha}{2}\|u_n - v\|^2 \leq \varepsilon_n \quad \forall\, v \in H(x_n).$$

Observe that for $\alpha = 0$ the above definition amounts to Definition 3.1 in [13] for *OPVIC* with variational inequalities having a unique solution.

Definition 5.2. An optimization problem with variational inequality constraints *OPVIC* is termed $\alpha-$*well-posed* with respect to $(\tau \times \sigma)$, if it has a unique solution (x_0, u_0) towards which every $\alpha-$approximating sequence $(x_n, u_n)_n$ $(\tau \times \sigma)-$converges.

Definition 5.3. An optimization problem with variational inequality constraints *OPVIC* is termed $\alpha-$*well-posed in the generalized sense* with respect to $(\tau \times \sigma)$, if *OPVIC* has at least a solution and every $\alpha-$approximating sequence $(x_n, u_n)_n$ has a subsequence $\tau \times \sigma-$convergent to a solution for *OPVIC*.

Remark 5.1. We point out that the set $T(x)$ of solutions to $(VI)(x)$ is not assumed to be always a singleton. In this situation many different types of "approximating" sequences could be considered instead of the ones considered in Definition 5.1 (see [20], where the well-posedness of MinSup problems is investigated).

In order to give sufficient conditions for the $\alpha-$well-posedness or $\alpha-$well-posedness in the generalized sense of *OPVIC*, we will distinguish the following situations:

- for every $x \in X$ $(VI)(x)$ has a unique solution;
- there exists $x \in X$ such that $(VI)(x)$ has not a unique solution.

First Case: for every $x \in X$ $(VI)(x)$ has a unique solution

Since this case for $\alpha = 0$ has been already investigated in [13], assume that $\alpha > 0$.

Theorem 5.2. *If* $(\mathbf{VI})$ *is parametrically* $\alpha-$*well-posed with respect to* σ, f *is sequentially lower semicontinuous and equicoercive on* $(X \times E, (\tau \times \sigma))$ *and OPVIC has a unique solution, then OPVIC is* $\alpha-$*well-posed with respect to* $(\tau \times \sigma)$.

Proof. Let $(x_n, u_n)_n$ be a sequence $\alpha-$approximating for *OPVIC*. Being f equicoercive, there exists a subsequence of $(x_n, u_n)_n$, still denoted by $(x_n, u_n)_n$, which $(\tau \times \sigma)-$converges to a point (x_0, u_0).

Since the sequence $(u_n)_n$ is $\alpha-$approximating for $(VI)(x_0)$ with respect to $(x_n)_n$ and $(\mathbf{VI})$ is parametrically $\alpha-$well-posed with respect to σ, the point u_0 must belong to $T(x_0)$. Therefore, in light of condition i) in Definition 5.1 and lower semicontinuity of f, one has:

$$f(x_0, u_0) \leq \inf_{(x,u) \in X \times E, u \in T(x)} f(x, u),$$

that is (x_0, u_0) is the unique solution to *OPVIC*. Since every $(\tau \times \sigma)-$convergent subsequence of $(x_n, u_n)_n$ converges to the unique solution for *OPVIC*, the whole sequence $(x_n, u_n)_n$ $(\tau \times \sigma)-$converges to it. ☐

Bearing in mind the proof of Proposition 3.10, a sufficient condition for the strongly $\alpha-$well-posedness of *OPVIC* with explicit assumptions on the data can be established.

Theorem 5.3. *Assume that* f *is sequentially lower semicontinuous and equicoercive on* $(X \times E, (\tau \times w))$*, and OPVIC has a unique solution. If the following assumptions are satisfied:*

i) *the operator* A *is strongly monotone on* E *in the variable* u*, uniformly with respect to* x*, that is:*

$$\exists \beta > 0 \text{ such that } \langle A(x,u) - A(x,v), u - v \rangle \geq \beta \|u - v\|^2 \, \forall\, x \in X, \forall\, u \in E, \forall\, v \in E;$$

ii) *for every* $u \in E$, $A(\cdot, u)$ *is continuous from* (X, τ) *to* (E^*, s);

iii) *for every* $x \in X$, $A(x, \cdot)$ *is hemicontinuous on* $H(x)$;

iv) A *is uniformly bounded on* $X \times E$;

v) *the set-valued function* H *is* $w-$*closed,* $w-$*subcontinuous,* $s-$*lower semicontinuous and convex-valued.*

Then OPVIC is $\alpha-$*well-posed with respect to* $(\tau \times s)$*, for every* $\alpha \leq 2\beta$.

Now we do not assume that *OPVIC* has a unique solution. With the same arguments as in Theorem 5.2 one can prove:

Theorem 5.4. *If* $(\mathbf{VI})$ *is parametrically* $\alpha-$*well-posed with respect to* $\sigma,$ f *is sequentially lower semicontinuous and equicoercive on* $(X \times E, (\tau \times \sigma))$ *and OPVIC has at least a solution, then OPVIC is* $\alpha-$*well-posed in the generalized sense with respect to* $(\tau \times \sigma)$.

In finite dimensional spaces one obtains:

Corollary 5.5. *Assume that* f *is sequentially lower semicontinuous and equicoercive on* $X \times R^k$*, OPVIC has at least a solution and, for every* $x \in X$, $(VI)(x)$ *has a unique solution.*

If the following assumptions are satisfied:

i) *the set-valued function* H *is closed, lower semicontinuous, subcontinuous and convex-valued;*
ii) *for every* $x \in X$, $A(x, \cdot)$ *is monotone and hemicontinuous on* $H(x)$*;*
iii) *for every* $u \in R^k$, $A(\cdot, u)$ *is continuous on* X;
iv) A *is uniformly bounded on* $X \times R^k$;

then OPVIC is $\alpha-$*well-posed in the generalized sense. If the set-valued function* H *is constant, that is* $H(x) = K \,\forall\, x \in X,$ *the same result holds assuming ii), iii) and the set* K *compact and convex.*

Second Case: there exists $x \in X$ such that $(VI)(x)$ does not have a unique solution.

Theorem 5.6. *Let* $\alpha \geq 0.$ *If* $(\mathbf{VI})$ *is parametrically* $\alpha-$ *well-posed in the generalized sense with respect to* $\sigma,$ f *is sequentially lower semicontinuous and equicoercive on* $(X \times E, (\tau \times \sigma))$ *and OPVIC has at least a solution, then OPVIC is* $\alpha-$*well-posed in the generalized sense with respect to* $(\tau \times \sigma)$.

Proof. Let $(x_n, u_n)_n$ be a sequence $\alpha-$approximating for *OPVIC*. From the equicoercivity of f, there exists a subsequence of $(x_n, u_n)_n$, still denoted by $(x_n, u_n)_n$, which $(\tau \times \sigma)-$converges to a point (x_0, u_0).

Since the sequence $(u_n)_n$ is $\alpha-$approximating for $(\mathbf{VI})$ with respect to $(x_n)_n$ and $(\mathbf{VI})$ is parametrically $\alpha-$well-posed in the generalized sense with respect to σ, $(u_n)_n$ has a subsequence $(u_{n_k})_{n_k}$ $\sigma-$ converging to a solution u_0 to $(VI)(x_0)$. Therefore, from condition i) in Definition 5.1 and in light of the lower semicontinuity of f, one has:

$$f(x_0, u_0) \leq \inf_{(x,u) \in X \times E, u \in T(x)} f(x, u),$$

that is (x_0, u_0) is a solution to *OPVIC*. □

Theorem 5.7. *Under the same assumptions of Theorem 5.6, if, moreover, OPVIC has a unique solution, then OPVIC is* $\alpha-$*well-posed with respect to* $(\tau \times \sigma)$.

Proof. Following the proof of the previous theorem, every $\alpha-$approximating sequence $(x_n, u_n)_n$ for *OPVIC* has a subsequence which $(\tau \times \sigma)-$converges to the unique solution (x_0, u_0). This is sufficient to conclude that the whole sequence $(x_n, u_n)_n$ $(\tau \times \sigma)-$converges to (x_0, u_0). □

When the variational inequality arises from a minimization problem, *OPVIC* is nothing else than a bilevel optimization problem, also called strong Stackelberg problem (see [16]):

$$\inf_{x \in X} \inf_{u \in M(x)} f(x, u),$$

where

$$M(x) = Argmin\, h(x, \cdot) = \left\{ u \in H(x) : h(x, u) \leq \inf_{u' \in H(x)} h(x, u') \right\}.$$

Theorem 5.8. *Assume that f is sequentially lower semicontinuous, equicoercive on $(X \times E, (\tau \times w))$ and OPVIC has a unique solution. If the following assumptions are satisfied:*

i) *for every $x \in X$, the function $h(x, \cdot)$ is lower semicontinuous, bounded from below, convex and Gâteaux differentiable on $H(x)$;*
ii) *the set-valued function H is $w-$closed, $w-$subcontinuous, $s-$lower semicontinuous, convex-valued and the range $H(X)$ is a bounded subset of E;*
iii) *for every $u \in E$, $\frac{\partial h}{\partial u}(\cdot, u)$ is continuous on X;*
iv) *for every $x \in X$, $\frac{\partial h}{\partial u}(x, \cdot)$ is hemicontinuous on $H(x)$;*
v) *$\frac{\partial h}{\partial u}$ is uniformly bounded on $X \times E$;*

then OPVIC is $\alpha-$well-posed with respect to $(\tau \times s)$.

REFERENCES

[1] A. AUSLENDER, *Optimisation: Méthodes Numériques*, Masson, Paris, 1976.

[2] C. BAIOCCHI AND A. CAPELO, *Variational and Quasivariational Inequalities, Applications to Free Boundary Problems*, John Wiley and Sons, New-York, 1984.

[3] T. BASAR AND G.J. OLSDER, *Dynamic Noncooperative Games*, Academic Press, New York, Second Edition, 1995.

[4] J.P. CROUZEIX, Pseudomonotone Variational Inequality problems: Existence of Solutions, *Math. Programming*, Ser. A, **78**(3) (1997), 305–314.

[5] A.L. DONTCHEV AND T. ZOLEZZI, Well-posed optimization problems, *Lectures Notes in Math.*, Springer Verlag, Vol. **1543**, (1993).

[6] I. EKELAND AND R. TEMAM, Convex Analysis and Variational Problems, *Studies in Mathematics and Applications*, North Holland, 1976.

[7] M. FUKUSHIMA, Equivalent differentiable optimization problem and descent method for symmetric variational inequalities, *Mathematical Programming*, **53** (1992), 99–110.

[8] N. HADJISAVVAS AND S. SCHAIBLE, Quasimonotonicity and pseudo monotonicity in variational inequalities and equilibrium problems, generalized convexity, generalized monotonicity: recent results (Luminy, 1996) *Nonconvex Optim. Appl.*, 27, Kluwer Acad. Publ., Dordrecht, 1998, 257–275.

[9] P.T. HARKER AND J.S. PANG, Finite-dimensional variational inequality and nonlinear complementarity problems: a survey of theory, algorithms and application, *Mathematical Programming*, **48** (1990), 161–220.

[10] D. KINDERLEHRER AND G. STAMPACCHIA, *An Introduction to Variational Inequality and their Applications*, Academic Press, New-York, 1980.

[11] K. KURATOWSKI, *Topology*, Vol. I and II., Academic Press, New York-London, 1968.

[12] M.B. LIGNOLA AND J. MORGAN, Existence of solutions to generalized bilevel programming problem, in *Multilevel Programming-Algorithms and Applications*, Eds A. Migdalas, P.M. Pardalos and P. Varbrand, Kluwer Academic Publishers, (1998), 315–332.

[13] M.B. LIGNOLA AND J. MORGAN, Well-posedness for optimization problems with constraints defined by variational inequality having a unique solution, *J. Global Optim.*, **16**(1) (2000), 57–67.

[14] M.B. LIGNOLA AND J. MORGAN, Existence of solutions to bilevel variational problems in Banach spaces, in *Equilibrium Problems, Non Smooth Optimization and Variational Models*, Kluwer Academic Publishers, 2001.

[15] M.B. LIGNOLA AND J. MORGAN, Approximate solutions and $\alpha-$well-posedness for variational inequalities and Nash equilibria, in *Decision and Control in Management Science*, Kluwer Academic Publishers, 2002, 367–378.

[16] P. LORIDAN AND J. MORGAN, Weak via Strong Stackelberg problem: new results. in hierarchical and bilevel programming, *J. Global Optim.*, **8**(3) (1996), 263–287.

[17] Z.Q. LUO, J.S. PANG, D. RALPH AND S.Q. WU, Exact penalization and stationarity conditions of mathematical programs with equilibrium constraints, *Mathematical Programming*, **75** (1996), 19–76.

[18] P. MARCOTTE AND G. SAVARD, A bilevel programming approach to optimal price setting, in *Decision and Control in Management Science*, Kluwer Academic Publishers, 2002, 97–120.

[19] P. MARCOTTE AND D.L. ZHU, Exact and inexact penalty methods for the generalized bilevel programming problem, *Mathematical Programming,* **74** (1996), 141–157.

[20] J. MORGAN, Constrained well-posed two-level optimization problem, in *Non-smooth Optimization and Related Topics,* F.H. Clarke, V.F. Demyanov, F. Giannessi, Eds., Plenum Press, New-York, 1989, 307–325.

[21] J. OUTRATA, M. KOCVARA AND J. ZOWE, Nonsmooth approach to optimization problems with equilibrium constraints. Theory, applications and numerical results, in *Nonconvex Optimization and its Applications*, 28, Kluwer Academic Publishers, Dordrecht, (1998).

[22] G. STAMPACCHIA, Variational inequalities, in *Theory and Applications of Monotone Operators* (Proc. NATO Advanced Study Inst. 1968 Venice), Edizioni “Oderisi”, Gubbio, 1969, 101–192.

[23] A.N. TYKHONOV, On the stability of the functional optimization problem, *U.S.S.R. Computational Math. and Math. Phys.*, **6**(4) (1966), 26–33.

[24] T. ZOLEZZI, Wellposed criteria in optimization with application to the calculus of variations, *Nonlinear Anal. TMA*, **25** (1995), 437–453.