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APPLICATION OF A TRAPEZOID INEQUALITY TO NEUTRAL FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS IN BANACH SPACES

ALEXANDRU MIAHI BICA, VASILE AUREL CĂUŞ, AND SORIN MUREŞAN

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ORADEA,
STR. UNIVERSITATII NO.1
410087, ORADEA, ROMANIA
abica@uoradea.ro

vcaus@uoradea.ro

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ABSTRACT. A new approach for neutral Fredholm integro-differential equations in Banach spaces, using the Perov's fixed point theorem of existence, uniqueness and approximation is presented. The approximation of the solution and of its derivative is realized using the method of successive approximations and a trapezoidal quadrature rule in Banach spaces for Lipschitzian functions. The interest is focused on the error estimation.

Key words and phrases: Nonlinear neutral Fredholm integro-differential equations in Banach spaces, Perov's fixed point theorem, Method of successive approximations, Trapezoidal quadrature rule.

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1. INTRODUCTION

Consider the neutral Fredholm integro-differential equation

$$x(t) = \int_a^b f(t, s, x(s), x'(s))\, ds + g(t), \quad t \in [a, b] \tag{1.1}$$

where $f : [a, b] \times [a, b] \times X \times X \to X$ is continuous, X is a Banach space and $g \in C^1([a, b], X)$.

To obtain the existence, uniqueness and global approximation of the solution of (1.1) we will use Perov's fixed point theorem. To this purpose, we differentiate the equation (1.1) with respect to t and assume that $f(\cdot, s, u, v) \in C^1([a, b], X)$, $\forall s \in [a, b]$, $\forall u, v \in X$, where $x' = y$. Hence

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(1.1) reduces to the following system of Fredholm integral equations:

$$\left\{ \begin{array}{l} x(t) = \int_a^b f(t,s,x(s),y(s))\,ds + g(t) \\ \\ y(t) = \int_a^b \frac{\partial f}{\partial t}(t,s,x(s),y(s))\,ds + g'(t) \end{array} \right. , \quad t \in [a,b]. \tag{1.2}$$

The Perov fixed point theorem will be applied to the system (1.2) obtaining also the approximation of the solution of (1.1) and its derivative.

The Perov fixed point theorem appeared for the first time in [16] and was later used for two point boundary value problems of second order differential equations in [5]. The Perov fixed point theorem was also used in [1], [10], [18] and [19]. Bica and Muresan have used the Perov fixed point theorem for delay neutral integro-differential equations in [6] and [7]. In this paper the authors have constructed a method of approximating the solution of (1.1) and its derivative using a sequence of successive approximations and a trapezoidal quadrature rule from [8]. Some of the existing numerical methods applied to Fredholm integro-differential equations can be found in the papers [2], [3], [4], [9], [11], [12], [13], [14], [15], [17]. The tools utilised in these papers are: the tau method, direct methods, collocation methods, Runge-Kutta methods, wavelet methods and spline approximation.

In this paper, our interest will be focused on the error estimation of the method presented in the Section 3.

2. EXISTENCE, UNIQUENESS AND APPROXIMATION

Consider the following conditions:

(i) (continuity): $f \in C([a,b] \times [a,b] \times X \times X, X)$, $g \in C^1([a,b], X)$ and $f(\cdot,s,u,v) \in C^1([a,b], X)$,for any $s \in [a,b]$, $u,v \in X$

(ii) (Lipschitz conditions): there exist $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \eta \in \mathbb{R}_+^*$ such that for any $t, t', s, s_1, s_2 \in [a,b]$ and $u, v, u_1, u_2, v_1, v_2 \in X$, we have:

$$\|f(t,s,u_1,v_1) - f(t,s,u_2,v_2)\|_X \le \alpha_1 \|u_1 - u_2\|_X + \beta_1 \|v_1 - v_2\|_X , \tag{2.1}$$

$$\left\| \frac{\partial f}{\partial t}(t,s,u_1,v_1) - \frac{\partial f}{\partial t}(t,s,u_2,v_2) \right\|_X \le \alpha_2 \|u_1 - u_2\|_X + \beta_2 \|v_1 - v_2\|_X , \tag{2.2}$$

$$\|f(t,s_1,u,v) - f(t,s_2,u,v)\|_X \le \gamma_1 |s_1 - s_2| , \tag{2.3}$$

$$\left\| \frac{\partial f}{\partial t}(t,s_1,u,v) - \frac{\partial f}{\partial t}(t,s_2,u,v) \right\|_X \le \gamma_2 |s_1 - s_2| , \tag{2.4}$$

$$\|f(t,s,u,v) - f(t',s,u,v)\|_X \le \delta_1 |t - t'| , \tag{2.5}$$

$$\left\| \frac{\partial f}{\partial t}(t,s,u,v) - \frac{\partial f}{\partial t}(t',s,u,v) \right\|_X \le \delta_2 |t - t'| , \tag{2.6}$$

$$\|g'(t) - g'(t')\|_X \le \eta \cdot |t - t'| . \tag{2.7}$$

We use Perov's fixed point theorem (see [16], [5] and [10]):

Theorem 2.1. *Let (X,d) be a complete generalized metric space such that $d(x,y) \in \mathbb{R}^n$ for $x,y \in X$. Suppose that there exists a function $A : X \to X$ such that:*

$$d(A(x), A(y)) \leq Q \cdot d(x,y)$$

for any $x, y \in X$, where $Q \in \mathcal{M}_n(\mathbb{R}_+)$. If all eigenvalues of Q lie in the open unit disc from $\mathbb{R}^2$ then $Q^m \to 0$ for $m \to \infty$ and the operator A has a unique fixed point $x^* \in X$. Moreover, the sequence of successive approximations $x_m = A(x_{m-1})$ converges to x^* in X for any $x_0 \in X$ and the following estimation holds:

$$d(x_m, x^*) \leq Q^m (I_n - Q)^{-1} \cdot d(x_0, x_1), \qquad \text{for each } m \in \mathbb{N}^* \tag{2.8}$$

where I_n is the unity matrix in $\mathcal{M}_n(\mathbb{R})$.

We recall the notion of generalized metric, which is a function $d : Y \times Y \to \mathbb{R}^n$ on a nonempty set Y with the properties:

a) $d(x,y) \geq 0$, for any $x, y \in Y$;
b) $d(x,y) = 0 \Leftrightarrow x = y$, for $x, y \in Y$;
c) $d(y,x) = d(x,y)$, for any $x, y \in Y$;
d) $d(x,z) \leq d(x,y) + d(y,z)$, for any $x, y, z \in Y$.

Here, the order relation on $\mathbb{R}^n$ is defined by

$$x \leq y \Leftrightarrow x_i \leq y_i, \quad \text{in} \quad \mathbb{R}, \text{ for each } i = \overline{1,n}, \quad x_i, y_i \in \mathbb{R}$$

for $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in \mathbb{R}^n$. The pair (Y,d) denote a generalized metric space.

In the following, we will use the notation

$$\|u\|_C = \max\{\|u(t)\|_X : t \in [a,b]\},$$

for $u \in C([a,b], X)$.

We consider the product space $Y = C([a,b], X) \times C([a,b], X)$ and define the generalized metric $d : Y \times Y \to \mathbb{R}^2$ by,

$$d((u_1, v_1), (u_2, v_2)) = (\|u_1 - u_2\|_C, \|v_1 - v_2\|_C),$$

for $(u_1, v_1), (u_2, v_2) \in Y$.

It is easy to prove that (Y,d) is a complete generalized metric space. We define the operator $A : Y \to Y$, $A = (A_1 A_2)$ by,

$$A_1(x,y)(t) = \int_a^b f(t,s,x(s),y(s))\, ds + g(t)$$

and

$$A_2(x,y)(t) = \int_a^b \frac{\partial f}{\partial t}(t,s,x(s),y(s))\, ds + gt(t), \quad t \in [a,b]. \tag{2.9}$$

The following result concerning the existence and uniqueness of the solution for the equation (1.1) holds.

Theorem 2.2. *In the conditions (i), (2.1), (2.2), if $(\alpha_1 + \beta_2)(b-a) < 2$ and*

$$1 + (b-a)^4 (\alpha_1 \beta_2 - \alpha_2 \beta_1)^2 + 2\alpha_2 \beta_1 (b-a)^2 > (b-a)^2 \left(\alpha_1^2 + \beta_2^2\right) \tag{2.10}$$

then the operator A has a unique fixed point (x^*, y^*) such that

$$x^* \in C^1([a,b], X), \qquad y^* = (x^*)'$$

and x^* is the unique solution of the equation (1.1). Moreover, the sequence of the successive approximations given by,

$$(x_0(t), y_0(t)) = (g(t), g'(t)), \quad t \in [a,b] \tag{2.11}$$

$$x_{m+1}(t) = \int_a^b f(t, s, x_m(s), y_m(s))\, ds + g(t), \quad t \in [a,b] \tag{2.12}$$

and

$$y_{m+1}(t) = \int_a^b \frac{\partial f}{\partial t}(t, s, x_m(s), y_m(s))\, ds + g'(t), \quad t \in [a,b], \tag{2.13}$$

converges in Y to (x^*, y^*) and the following error estimation holds:

$$d((x_m, y_m), (x^*, y^*)) \le Q^m (I_n - Q)^{-1} \cdot d((x_0, y_0), (x_1, y_1)), \quad \textit{for any } m \in \mathbb{N}^*, \tag{2.14}$$

where

$$Q = (b-a) \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}.$$

Proof. From condition (i) we infer that $A(Y) \subset Y$. For $(u_1, v_1), (u_2, v_2) \in Y$, $t \in [a,b]$ we have

$$\begin{aligned} &\|A_1(u_1, v_1)(t) - A_1(u_2, v_2)(t)\|_X \\ &\qquad \le \int_a^b [\alpha_1 \|u_1(s) - u_2(s)\|_X + \beta_1 \|v_1(s) - v_2(s)\|_X] \\ &\qquad \le (b-a) \cdot [\alpha_1 \|u_1 - u_2\|_C + \beta_1 \|v_1 - v_2\|_C], \qquad \text{for any } t \in [a,b] \end{aligned}$$

and

$$\begin{aligned} &\|A_2(u_1, v_1)(t) - A_2(u_2, v_2)(t)\|_X \\ &\qquad \le (b-a) \cdot [\alpha_2 \|u_1 - u_2\|_C + \beta_2 \|v_1 - v_2\|_C], \qquad \text{for any } t \in [a,b]. \end{aligned}$$

We infer that

$$d(A(u_1, v_1), A(u_2, v_2)) \le Q \cdot d((u_1, v_1), (u_2, v_2)), \quad \text{for any } (u_1, v_1), (u_2, v_2) \in Y,$$

where

$$Q = \begin{pmatrix} (b-a)\alpha_1 & (b-a)\beta_1 \\ (b-a)\alpha_2 & (b-a)\beta_2 \end{pmatrix}.$$

It is easy to see that the eigenvalues of Q are real. The inequalities (2.10) and $(\alpha_1 + \beta_2)(b-a) < 2$ lead to $\mu_1, \mu_2 \in (-1, 1)$, where μ_1 and μ_2 are these eigenvalues. From the Perov fixed point theorem we infer that $Q^m \to 0$ for $m \to \infty$ and the operator A has a unique fixed point $(x^*, y^*) \in Y$. Then,

$$x^*(t) = \int_a^b f(t, s, x^*(s), y^*(s))\, ds + g(t), \quad \forall t \in [a,b]$$

and

$$y^*(t) = \int_a^b \frac{\partial f}{\partial t}(t, s, x^*(s), y^*(s))\, ds + g'(t), \quad \forall t \in [a,b].$$

Since $f(\cdot, s, u, v) \in C^1([a,b], X)$, for any $s \in [a,b]$, $u, v \in X$ and $g \in C^1([a,b], X)$ we infer that $x^* \in C^1([a,b], X)$. If we differentiate the first equality with respect to t we obtain

$y^* = (x^*)'$. Then x^* is the unique solution of (1.1). From the relations (2.12) and (2.13) and from (2.9) we infer that the sequences given in (2.12), (2.13) fulfil the recurrence relation

$$(x_{m+1}, y_{m+1}) = A((x_m, y_m)), \quad \forall m \in \mathbb{N}.$$

Now, the inequality (2.14) follows from the estimation (2.8). Since $Q^m \to 0$ for $m \to \infty$ in $\mathcal{M}_2(\mathbb{R})$ we infer that

$$\lim_{m\to\infty} d((x_m, y_m), (x^*, y^*)) = (0, 0).$$

This proves the theorem. □

3. THE MAIN RESULT

To compute the terms of the sequence of successive approximations we use in the calculus of integrals in (2.12), (2.13) the trapezoidal quadrature rule for Lipschitzian functions from [8]:

$$\int_a^b F(x)\,dx = \frac{(b-a)}{2n}\left[F(a) + 2\sum_{i=1}^{n-1} F\left(a + \frac{i(b-a)}{n}\right) + F(b)\right] + R_n(F) \tag{3.1}$$

with

$$\|R_n(F)\|_X \le \frac{L(b-a)^2}{4n}, \tag{3.2}$$

where L is the Lipschitz constant of $F : [a,b] \to X$.

In this respect consider the uniform partition of $[a,b]$,

$$\Delta : a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b \tag{3.3}$$

with $t_i = a + \frac{i(b-a)}{n}$, $\quad i = \overline{0,n}$ and compute $x_m(t_i), y_m(t_i)$, $\quad i = \overline{0,n}$, $\quad m \in \mathbb{N}^*$.

From (2.12) and (2.13) we have

$$x_{m+1}(t_i) = \int_a^b f(t_i, s, x_m(s), y_m(s))\,ds + g(t_i) \tag{3.4}$$

and

$$y_{m+1}(t_i) = \int_a^b \frac{\partial f}{\partial t}(t_i, s, x_m(s), y_m(s))\,ds + g'(t_i), \quad \text{for any } i = \overline{0,n}, \quad m \in \mathbb{N}. \tag{3.5}$$

We define the functions

$$F_{m,i}, G_{m,i} : [a,b] \to X, \qquad m \in \mathbb{N}, \qquad i = \overline{0,n}$$

by

$$\begin{cases} F_{m,i}(s) = f(t_i, s, x_m(s), y_m(s)) \\ G_{m,i}(s) = \frac{\partial f}{\partial t}(t_i, s, x_m(s), y_m(s)) \end{cases}, \text{ for any } s \in [a,b]. \tag{3.6}$$

Definition 3.1. A set $Z \subset C([a,b], X)$ is equally Lipschitz if there exists $L \ge 0$ such that for any $h \in Z$,

$$\|h(t) - h(t')\|_X \le L \cdot |t - t'|, \qquad \text{for each } t, t' \in [a,b].$$

Theorem 3.1. *The subsets*

$$\{\{F_{m,i}\}_{m\in\mathbb{N}}, \quad i = \overline{0,n}\} \subset C([a,b], X)$$

and

$$\{\{G_{m,i}\}_{m\in\mathbb{N}}, \quad i = \overline{0,n}\} \subset C([a,b], X)$$

defined in (3.6), are equally Lipschitz, if the conditions (i) and (2.1) – (2.7) are true.

Proof. Let

$$\mu = \|g'\|_C = \max\{\|g'(t)\|_X : t \in [a,b]\}.$$

For $m \in \mathbb{N}^*$ and $i = \overline{0,n}$ we have

$$\begin{aligned}
&\|F_{m,i}(s_1) - F_{m,i}(s_2)\|_X \\
&\quad \le \gamma_1 |s_1 - s_2| + \alpha_1 \|x_m(s_1) - x_m(s_2)\|_X + \beta_1 \|y_m(s_1) - y_m(s_2)\|_X \\
&\quad \le \gamma_1 |s_1 - s_2| + \alpha_1 [\delta_1 (b-a) + \mu] \cdot |s_1 - s_2| \\
&\qquad\qquad + \beta_1 [\delta_2 (b-a) \cdot |s_1 - s_2| + \|g'(s_1) - g'(s_2)\|_X] \\
&\quad \le [\gamma_1 + \alpha_1\mu + \beta_1\eta + (b-a)(\alpha_1\delta_1 + \beta_1\delta_2)] \cdot |s_1 - s_2|,
\end{aligned}$$

for any $s_1, s_2 \in [a,b]$ and

$$\begin{aligned}
&\|G_{m,i}(s_1) - G_{m,i}(s_2)\|_X \\
&\quad \le \gamma_2 |s_1 - s_2| + \alpha_2 \|x_m(s_1) - x_m(s_2)\|_X + \beta_2 \|y_m(s_1) - y_m(s_2)\|_X \\
&\quad \le [\gamma_2 + \alpha_2\mu + \beta_2\eta + (b-a)(\alpha_2\delta_1 + \beta_2\delta_2)] \cdot |s_1 - s_2|,
\end{aligned}$$

for any $s_1, s_2 \in [a,b]$. Moreover, for any $i = \overline{0,n}$ we have,

$$\|F_{0,i}(s_1) - F_{0,i}(s_2)\|_X \le (\gamma_1 + \alpha_1\mu + \beta_1\eta) \cdot |s_1 - s_2|, \quad \text{for each } s_1, s_2 \in [a,b]$$

and

$$\|G_{0,i}(s_1) - G_{0,i}(s_2)\|_X \le (\gamma_2 + \alpha_2\mu + \beta_2\eta) \cdot |s_1 - s_2|, \quad \text{for each } s_1, s_2 \in [a,b].$$

Let

$$L_1 = \gamma_1 + \alpha_1\mu + \beta_1\eta + (b-a)(\alpha_1\delta_1 + \beta_1\delta_2),$$

$$L_2 = \gamma_2 + \alpha_2\mu + \beta_2\eta + (b-a)(\alpha_2\delta_1 + \beta_2\delta_2).$$

From the above, we infer that for any $i = \overline{0,n}$, we have,

$$\|F_{m,i}(s_1) - F_{m,i}(s_2)\|_X \le L_1 \cdot |s_1 - s_2|, \quad \text{for each } s_1, s_2 \in [a,b]$$

and

$$\|G_{m,i}(s_1) - G_{m,i}(s_2)\|_X \le L_2 \cdot |s_1 - s_2|, \quad \text{for each } s_1, s_2 \in [a,b] \text{ and } m \in \mathbb{N}.$$

This concludes the proof of the theorem. □

Applying in (3.4), (3.5) the quadrature rule (3.1) – (3.2) we obtain the numerical method:

$$x_0(t_i) = g(t_i),\ y_0(t_i) = g'(t_i), \qquad \text{for } i = \overline{0,n} \tag{3.7}$$

$$\begin{aligned}
x_m(t_i) = g(t_i) + \frac{(b-a)}{2n} \cdot \Bigg[& f(t_i, a, x_{m-1}(a), y_{m-1}(a)) \\
& + 2\sum_{j=1}^{n-1} f(t_i; t_j, x_{m-1}(t_j), y_{m-1}(t_j)) + f(t_i, b, x_{m-1}(b), y_{m-1}(b)) \Bigg] \\
& + R_{m,i}, \quad \text{for } i = \overline{0,n} \text{ and } m \in \mathbb{N}^*
\end{aligned} \tag{3.8}$$

and

$$y_m(t_i) = g'(t_i) + \frac{(b-a)}{2n} \cdot \left[\frac{\partial f}{\partial t}(t_i, a, x_{m-1}(a), y_{m-1}(a)) \right. \quad (3.9)$$

$$\left. + 2\sum_{j=1}^{n-1} \frac{\partial f}{\partial t}(t_i; t_j, x_{m-1}(t_j), y_{m-1}(t_j)) + \frac{\partial f}{\partial t}(t_i, b, x_{m-1}(b), y_{m-1}(b)) \right]$$

$$+ R_{m,i}, \quad \text{for } i = \overline{0,n} \text{ and } m \in \mathbb{N}^*.$$

with the remainder estimations

$$\|R_{m,i}\|_X \le \frac{L_1(b-a)^2}{4n}, \qquad \text{for any } m \in \mathbb{N}^* \text{ and } i = \overline{0,n}, \quad (3.10)$$

$$\|\omega_{m,i}\|_X \le \frac{L_2(b-a)^2}{4n}, \qquad \text{or any } m \in \mathbb{N}^* \text{ and } i = \overline{0,n}. \quad (3.11)$$

These lead to the following algorithm:

$$x_0(t_i) = g(t_i), y_0(t_i) = g'(t_i),$$

$$x_1(t_i) = g(t_i) + \frac{(b-a)}{2n} \cdot \left[f(t_i, a, g(a), g'(a)) \right. \quad (3.12)$$

$$\left. + 2\sum_{j=1}^{n-1} f(t_i; t_j, g(t_j), g'(t_j)) + f(t_i, b, g(b), g'(b)) \right] + R_{1,i}$$

$$= \overline{x_1}(t_i) + R_{1,i},$$

$$y_1(t_i) = g'(t_i) + \frac{(b-a)}{2n} \cdot \left[\frac{\partial f}{\partial t}(t_i, a, g(a), g'(a)) \right. \quad (3.13)$$

$$\left. + 2\sum_{j=1}^{n-1} \frac{\partial f}{\partial t}(t_i; t_j, g(t_j), g'(t_j)) + \frac{\partial f}{\partial t}(t_i, b, g(b), g'(b)) \right] + \omega_{1,i}$$

$$= \overline{y_1}(t_i) + \omega_{1,i},$$

$$x_2(t_i) = g(t_i) + \frac{(b-a)}{2n} \cdot \left[f(t_i, t_0, \overline{x_1}(t_0) + R_{1,0}, \overline{y_1}(t_0) + \omega_{1,0}) \right. \quad (3.14)$$

$$+ 2\sum_{j=1}^{n-1} f(t_i; t_j, \overline{x_1}(t_j) + R_{1,j}, \overline{y_1}(t_j) + \omega_{1,j}) + f(t_i, t_n, \overline{x_1}(t_n)$$

$$\left. + R_{1,n}, \overline{y_1}(t_n) + \omega_{1,n}) \right] + R_{2,i}$$

$$
\begin{aligned}
&= g(t_i) + \frac{(b-a)}{2n} \cdot \Bigg[f(t_i, t_0, \overline{x_1}(t_0), \overline{y_1}(t_0)) \\
&\qquad + 2\sum_{j=1}^{n-1} f(t_i; t_j, \overline{x_1}(t_j), \overline{y_1}(t_j)) + f(t_i, t_n, \overline{x_1}(t_n), \overline{y_1}(t_n)) \Bigg] + \overline{R_{2,i}} \\
&= \overline{x_2}(t_i) + \overline{R_{2,i}},
\end{aligned}
$$

$$
\begin{aligned}
y_2(t_i) &= g'(t_i) + \frac{(b-a)}{2n} \cdot \Bigg[\frac{\partial f}{\partial t}(t_i, t_0, \overline{x_1}(t_0) + R_{1,0}, \overline{y_1}(t_0) + \omega_{1,0}) \\
&\qquad + 2\sum_{j=1}^{n-1} \frac{\partial f}{\partial t}(t_i; t_j, \overline{x_1}(t_j) + R_{1,j}, \overline{y_1}(t_j) + \omega_{1,j}) + \frac{\partial f}{\partial t}(t_i, t_n, \overline{x_1}(t_n) \\
&\qquad + R_{1,n}, \overline{y_1}(t_n) + \omega_{1,n}) \Bigg] + \omega_{2,i} \\
&= g'(t_i) + \frac{(b-a)}{2n} \cdot \Bigg[\frac{\partial f}{\partial t}(t_i, t_0, \overline{x_1}(t_0), \overline{y_1}(t_0)) \\
&\qquad + 2\sum_{j=1}^{n-1} \frac{\partial f}{\partial t}(t_i; t_j, \overline{x_1}(t_j), \overline{y_1}(t_j)) + \frac{\partial f}{\partial t}(t_i, t_n, \overline{x_1}(t_n), \overline{y_1}(t_n)) \Bigg] + \overline{\omega_{2,i}} \\
&= \overline{y_2}(t_i) + \overline{\omega_{2,i}}, \qquad \text{when } i = \overline{0,n}.
\end{aligned} \tag{3.15}
$$

By induction, for $m \geq 3$, we obtain for $i = \overline{0,n}$ that

$$
\begin{aligned}
x_m(t_i) &= g(t_i) + \frac{(b-a)}{2n} \cdot \Bigg[f\left(t_i, t_0, \overline{x_{m-1}}(t_0) + \overline{R}_{m-1,0}, \overline{y_{m-1}}(t_0) + \overline{\omega}_{m-1,0}\right) \\
&\qquad + 2\sum_{j=1}^{n-1} f\left(t_i; t_j, \overline{x_{m-1}}(t_j) + \overline{R}_{m-1,j}, \overline{y_{m-1}}(t_j) + \overline{\omega}_{m-1,j}\right) \\
&\qquad + f\left(t_i, t_n, \overline{x_{m-1}}(t_n) + \overline{R}_{m-1,n}, \overline{y_{m-1}}(t_n) + \overline{\omega}_{m-1,n}\right) \Bigg] + R_{m,i} \\
&= g(t_i) + \frac{(b-a)}{2n} \cdot \Bigg[f(t_i, t_0, \overline{x_{m-1}}(t_0), \overline{y_{m-1}}(t_0)) \\
&\qquad + 2\sum_{j=1}^{n-1} f(t_i; t_j, \overline{x_{m-1}}(t_j), \overline{y_{m-1}}(t_j)) + f(t_i, t_n, \overline{x_{m-1}}(t_n), \overline{y_{m-1}}(t_n)) \Bigg] + \overline{R_{m,i}} \\
&= \overline{x_m}(t_i) + \overline{R_{m,i}}
\end{aligned} \tag{3.16}
$$

and

$$
y_m(t_i) = g'(t_i) + \frac{(b-a)}{2n} \cdot \Bigg[\frac{\partial f}{\partial t}\left(t_i, t_0, \overline{x_{m-1}}(t_0) + \overline{R}_{m-1,0}, \overline{y_{m-1}}(t_0) + \overline{\omega}_{m-1,0}\right) \tag{3.17}
$$

$$+ 2\sum_{j=1}^{n-1} \frac{\partial f}{\partial t}\left(t_i; t_j, \overline{x_{m-1}}(t_j) + \overline{R}_{m-1,j}, \overline{y_{m-1}}(t_j) + \overline{\omega}_{m-1,j}\right)$$

$$\left. + \frac{\partial f}{\partial t}\left(t_i, t_n, \overline{x_{m-1}}(t_n) + \overline{R}_{m-1,n}, \overline{y_{m-1}}(t_n) + \overline{\omega}_{m-1,n}\right)\right] + \omega_{m,i}$$

$$= g'(t_i) + \frac{(b-a)}{2n} \cdot \left[\frac{\partial f}{\partial t}\left(t_i, t_0, \overline{x_{m-1}}(t_0), \overline{y_{m-1}}(t_0)\right)\right.$$

$$+ 2\sum_{j=1}^{n-1} \frac{\partial f}{\partial t}\left(t_i; t_j, \overline{x_{m-1}}(t_j), \overline{y_{m-1}}(t_j)\right)$$

$$\left. + \frac{\partial f}{\partial t}\left(t_i, t_n, \overline{x_{m-1}}(t_n), \overline{y_{m-1}}(t_n)\right)\right] + \overline{\omega_{m,i}}$$

$$= \overline{y_m}(t_i) + \overline{\omega_{m,i}}.$$

At the remainder estimation we have for any $i = \overline{0,n}$

$$\|R_{1,i}\|_X \le \frac{L_1 (b-a)^2}{4n}, \qquad \|\omega_{1,i}\|_X \le \frac{L_2 (b-a)^2}{4n}.$$

Using in (3.14) the Lipschitz property (2.1) we obtain:

$$\left\|\overline{R}_{2,i}\right\| \le [1 + (b-a)(\alpha_1 + \beta_1)] \cdot \frac{L_1 (b-a)^2}{4n}, \qquad \text{for any } i = \overline{0,n}.$$

Using in (3.15) the Lipschitz property (2.2) we obtain:

$$\|\overline{\omega}_{2,i}\| \le [1 + (b-a)(\alpha_2 + \beta_2)] \cdot \frac{L_2 (b-a)^2}{4n}, \qquad \text{for each } i = \overline{0,n}.$$

By induction, we obtain for $m \ge 2$ and $i = \overline{0,n}$,

$$\left\|\overline{R}_{m,i}\right\| \le \left[1 + (b-a)(\alpha_1 + \beta_1) + \cdots + (b-a)^{m-1}(\alpha_1 + \beta_1)^{m-1}\right] \cdot \frac{L_1 (b-a)^2}{4n} = \frac{1 - (b-a)^m (\alpha_1 + \beta_1)^m}{1 - (b-a)(\alpha_1 + \beta_1)} \cdot \frac{L_1 (b-a)^2}{4n} \tag{3.18}$$

and

$$\|\overline{\omega}_{m,i}\| \le \left[1 + (b-a)(\alpha_2 + \beta_2) + \cdots + (b-a)^{m-1}(\alpha_2 + \beta_2)^{m-1}\right] \cdot \frac{L_2 (b-a)^2}{4n} = \frac{1 - (b-a)^m (\alpha_2 + \beta_2)^m}{1 - (b-a)(\alpha_2 + \beta_2)} \cdot \frac{L_2 (b-a)^2}{4n}. \tag{3.19}$$

Finally, we can state the following result:

Theorem 3.2. *With the conditions (i), (2.1) – (2.7), (2.10), and if* $(b-a)(\alpha_1 + \beta_1) < 1$ *and* $(b-a)(\alpha_2 + \beta_2) < 1$*, then the solution of the system (1.2) is approximated on the knots of the uniform partition* Δ *given in (3.3), by the sequence*

$$\{(\overline{x_m}(t_i), \overline{y_m}(t_i))\}_{m \in \mathbb{N}}, \qquad i = \overline{0,n}$$

obtained in (3.12) – (3.17) and the following error estimation holds:

$$\begin{pmatrix} \|x^*(t_i) - \overline{x_m}(t_i)\|_X \\ \|y^*(t_i) - \overline{y_m}(t_i)\|_X \end{pmatrix} \leq Q^m (I_2 - Q)^{-1} d(x_0, x_1) + \begin{pmatrix} \frac{L_1(b-a)^2}{4n[1-(b-a)(\alpha_1+\beta_1)]} \\ \frac{L_2(b-a)^2}{4n[1-(b-a)(\alpha_2+\beta_2)]} \end{pmatrix}, \quad \text{for each } m \in \mathbb{N}^* \text{ and } i = \overline{0,n}. \tag{3.20}$$

Proof. Follows from (2.14), (3.18) and (3.19) since

$$\|x^*(t_i) - \overline{x_m}(t_i)\|_X \leq \|x^*(t_i) - x_m(t_i)\|_X + \|x_m(t_i) - \overline{x_m}(t_i)\|_X$$

for each $m \in \mathbb{N}^*$ and $i = \overline{0,n}$ and,

$$\|y^*(t_i) - \overline{y_m}(t_i)\|_X \leq \|y^*(t_i) - y_m(t_i)\|_X + \|y_m(t_i) - \overline{y_m}(t_i)\|_X$$

for each $m \in \mathbb{N}^*$ and $\forall i = \overline{0,n}$. □

Remark 3.3. When $f(t,s,u,v) = H(t,s) \cdot f(s,u,v)$ with $H^1 \in C\left([a,b]^2, X\right)$ we obtain the existence, uniqueness and approximation of the solution for Hammerstein-Fredholm integro-differential equations in Banach spaces. Moreover, in the particular case $H(t,s) = G(t,s)$, the Green function, we obtain a new approach for two point boundary value problems associated to second order differential equations in Banach spaces.

Remark 3.4. For $X = \mathbb{R}^n$ we obtain a new method in analysing systems of Fredholm integro-differential equations and for $X = \mathbb{R}$ we obtain similar results for the approximation of the solution of a scalar Fredholm integro-differential equation.

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