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OPTIMAL INEQUALITIES CHARACTERISING QUASI-UMBILICAL SUBMANIFOLDS

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ABSTRACT. A family of optimal inequalities is obtained involving the intrinsic scalar curvature and the extrinsic Casorati curvature of submanifolds of real space forms. Equality holds in the inequalities if and only if these submanifolds are invariantly quasi-umbilical. In the particular case of a hypersurface in a real space form, the equality case characterises a special class of rotation hypersurfaces.

Key words and phrases: Chen curvature, Casorati curvature, quasi-umbilical.

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1. INTRODUCTION

B.-Y. Chen obtained many *optimal inequalities between intrinsic and extrinsic quantities* for n-dimensional Riemannian manifolds which are isometrically immersed into $(n+m)$-dimensional real space forms, in particular, in terms of some new intrinsic scalar-valued curvature invariants on these manifolds, the so-called *δ-curvatures of Chen* (see e.g. [4, 5, 6]). The δ-curvatures of Chen originated by considering the minimum or maximum value of the *sectional curvature* of all *two-planes*, or the extremal values of the *scalar curvature* of all *k-planes* ($2 < k < n$), etc., in the tangent space at a point of the manifold. These invariants provide lower bounds for the *squared mean curvature* and equality holds if and only if the second fundamental form assumes some specified expressions with respect to special adapted

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orthonormal frames. For the corresponding immersions, these Riemannian manifolds receive the least amount of "surface-tension" from the surrounding spaces and therefore are called *ideal submanifolds*. Such inequalities have been extended, amongst others, to submanifolds in general Riemannian manifolds [8], to Kaehler submanifolds in Kaehler manifolds [7, 20, 22] and to Lorentzian submanifolds in semi-Euclidean spaces [18].

Instead of balancing *intrinsic scalar valued curvatures*, such as the *scalar curvature* or the more sophisticated *Chen curvatures*, with the *extrinsic squared mean curvature*, in the following, we will obtain optimal inequalities using the *Casorati curvature* of hyperplanes in the tangent space at a point. For a surface in $\mathbb{E}^3$ the Casorati curvature is defined as the normalised sum of the squared *principal curvatures* [2]. This curvature was preferred by Casorati over the traditional Gauss curvature because the Casorati curvature vanishes if and only if both principal curvatures of a surface in $\mathbb{E}^3$ are zero at the same time and thus corresponds better with the *common* intuition of curvature.

In Section 2 we obtain a family of *optimal inequalities* involving the *scalar curvature* and the *Casorati curvature* of a Riemannian submanifold in a real space form. The proof is based on an optimalisation procedure by showing that a quadratic polynomial in the components of the second fundamental form is parabolic. Further we show that *equality* in the inequalities at every point characterises the *invariantly quasi-umbilical submanifolds*. Submanifolds for which the equality holds, will be called *Casorati ideal submanifolds*. It turns out that they are all intrinsically *pseudo-symmetric* and, if the codimension is one, they constitute a special class of *rotation hypersurfaces*.

2. OPTIMAL INEQUALITIES

Let (M^n, g) be an n-dimensional Riemannian manifold and denote by R and τ the *Riemann-Christoffel curvature tensor* and the *scalar curvature* of M, respectively. We assume that (M^n, g) admits an isometric immersion $x : M^n \to \widetilde{M}^{n+m}(\widetilde{c})$ into an $(n+m)$-dimensional Riemannian space form $(\widetilde{M}^{n+m}(\widetilde{c}), \widetilde{g})$ with constant sectional curvature $\widetilde{c}$. The *Levi-Civita connections* on $\widetilde{M}$ and M will be denoted by $\widetilde{\nabla}$ and ∇, respectively. The *second fundamental form* h of M in $\widetilde{M}$ is defined by the *Gauss formula*:

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

whereby X and Y are *tangent* vector fields on M. The *shape operator* A_ξ associated with a normal vector field ξ and the *normal connection* $\nabla^\perp$ of M in $\widetilde{M}$ are defined by the *Weingarten formula*:

$$\widetilde{\nabla}_X \xi = -A_\xi(X) + \nabla^\perp_X \xi.$$

Since $\widetilde{g}(h(X, Y), \xi) = g(A_\xi(X), Y)$, the knowledge of the second fundamental form is equivalent to the knowledge of the shape operators A_ξ (for all ξ's of a normal frame on M in $\widetilde{M}$).

A submanifold M^n in a Riemannian manifold $\widetilde{M}^{n+m}$ is called (properly) *quasi-umbilical* with respect to a normal vector field ξ if the shape operator A_ξ has an eigenvalue with multiplicity $\geq n-1$ $(= n-1)$. In this case, ξ is called a *quasi-umbilical normal section* of M. An n-dimensional submanifold M of an $(n+m)$-dimensional Riemannian manifold $\widetilde{M}$ is called *totally quasi-umbilical* if there exist m *mutually orthogonal* quasi-umbilical normal sections $\xi_1, \ldots, \xi_m$ of M. In the particular case that the *distinguished eigendirections* of the shape operators A_α with respect to ξ_α, i.e. the tangent directions corresponding to the eigenvalues of the matrices A_α with *multiplicity* 1, are the same for all ξ_α, the totally quasi-umbilical submanifold under consideration is called *invariantly quasi-umbilical* [1, 3].

The *squared norm* of the second fundamental form h over the dimension n is called the *Casorati curvature* $\mathcal{C}$ of the submanifold M in $\widetilde{M}$, i.e.,

$$\mathcal{C} = \frac{1}{n}\sum_{\alpha=1}^{m}\left(\sum_{i,j=1}^{n}(h_{ij}^{\alpha})^2\right),$$

where $h_{ij}^{\alpha} = \widetilde{g}(h(e_i, e_j), \xi_\alpha)$ are the components of the second fundamental form with respect to an orthonormal tangent frame $\{e_1, \dots, e_n\}$ and an orthonormal normal frame $\{\xi_1, \dots, \xi_m\}$ of M in $\widetilde{M}$. The *squared mean curvature* of a submanifold M in $\widetilde{M}$ being given by

$$\|\mathbf{H}\|^2 = \frac{1}{n^2}\sum_{\alpha=1}^{m}\left(\sum_{i=1}^{n}h_{ii}^{\alpha}\right)^2,$$

from the *Gauss equation*

$$R_{ijkl} = \sum_{\alpha=1}^{m}\Big(h_{il}^{\alpha}h_{jk}^{\alpha} - h_{ik}^{\alpha}h_{jl}^{\alpha}\Big) + \widetilde{c}\Big(g_{il}g_{jk} - g_{ik}g_{jl}\Big),$$

one readily obtains the following well-known relation between the *scalar curvature*, the *squared mean curvature* and the *Casorati curvature* for any n-dimensional *submanifold* M in any *real space form* $\widetilde{M}$ of curvature $\widetilde{c}$ [3]:

$$\tau = n^2\|\mathbf{H}\|^2 - n\,\mathcal{C} + n(n-1)\widetilde{c}.$$

The *Casorati curvature of a w-plane field W*, spanned by $\{e_{q+1}, \dots, e_{q+w}\}$, $q < n - w$, $w \geq 2$, is defined by

$$\mathcal{C}(W) = \frac{1}{w}\sum_{\alpha=1}^{m}\left(\sum_{i,j=q+1}^{q+w}(h_{ij}^{\alpha})^2\right).$$

At any point p of M^n in a Euclidean ambient space $\mathbb{E}^{n+m}$, $(\mathcal{C}(W))(p)$ is the Casorati curvature at p of the *w-dimensional normal section* Σ_W^w of M^n in $\mathbb{E}^{n+m}$ which is obtained by locally cutting M^n with the normal $(w+m)$-space in $\mathbb{E}^{n+m}$ passing through p and spanned by W and $T_p^{\perp}M$: $(\mathcal{C}(W))(p) = \mathcal{C}_{\Sigma_W^w}(p)$. For any positive real number r, different from $n(n-1)$, set

$$a(r) := \frac{(n-1)(r+n)(n^2-n-r)}{nr},$$

in order to define the *normalized δ-Casorati curvatures* $\delta_{\mathcal{C}}(r; n-1)$ and $\widehat{\delta}_{\mathcal{C}}(r; n-1)$ of M in $\widetilde{M}$ as follows:

$$\delta_{\mathcal{C}}(r; n-1)\mid_p := r\,\mathcal{C}\mid_p + a(r)\cdot\inf\{\mathcal{C}(W) \mid W \text{ a hyperplane of } T_pM\},$$

if $0 < r < n(n-1)$, and:

$$\widehat{\delta}_{\mathcal{C}}(r; n-1)\mid_p := r\,\mathcal{C}\mid_p + a(r)\cdot\sup\{\mathcal{C}(W) \mid W \text{ a hyperplane of } T_pM\},$$

if $r > n(n-1)$.

Theorem 2.1. *For any Riemannian submanifold M^n of any real space form $\widetilde{M}^{n+m}(\widetilde{c})$, for any real number r such that $0 < r < n(n-1)$:*

$$\tau \leq \delta_{\mathcal{C}}(r; n-1) + n(n-1)\,\widetilde{c}, \tag{2.1}$$

and for any real number r such that $n(n-1) < r$:

$$\tau \leq \widehat{\delta}_{\mathcal{C}}(r; n-1) + n(n-1)\,\widetilde{c}. \tag{2.2}$$

Proof. Consider the following function $\mathcal{P}$ which is a quadratic polynomial in the components of the second fundamental form:

$$\mathcal{P} = r\,\mathcal{C} + a(r)\,\mathcal{C}(W) - \tau + n(n-1)\,\widetilde{c}.$$

Assuming, without loss of generality, that the hyperplane W involved is spanned by the tangent vectors $e_1, e_2, \ldots$ and e_{n-1}, it follows that

$$\begin{aligned}\mathcal{P} = \sum_{\alpha=1}^{m} \Bigg\{ & \left(\frac{r}{n} + \frac{a(r)}{n-1}\right) \sum_{i=1}^{n-1} (h_{ii}^{\alpha})^2 + \frac{r}{n}(h_{nn}^{\alpha})^2 \\ & + 2\left(\frac{r}{n} + \frac{a(r)}{n-1} + 1\right) \sum_{i,j=1(i\neq j)}^{n-1} (h_{ij}^{\alpha})^2 + 2\left(\frac{r}{n}+1\right)\sum_{i=1}^{n-1}(h_{in}^{\alpha})^2 - 2\sum_{i,j=1(i\neq j)}^{n} h_{ii}^{\alpha} h_{jj}^{\alpha} \Bigg\}. \end{aligned} \tag{2.3}$$

The *critical points* $h^c = (h_{11}^1, h_{12}^1, \ldots, h_{nn}^1, \ldots, h_{11}^m, \ldots, h_{nn}^m)$ of $\mathcal{P}$ are the solutions of the following system of linear homogeneous equations:

$$\begin{aligned} \frac{\partial \mathcal{P}}{\partial h_{ii}^{\alpha}} &= 2\left(\frac{r}{n} + \frac{a(r)}{n-1}\right) h_{ii}^{\alpha} - 2\sum_{k\neq i, k=1}^{n} h_{kk}^{\alpha} = 0, \\ \frac{\partial \mathcal{P}}{\partial h_{nn}^{\alpha}} &= 2\frac{r}{n} h_{nn}^{\alpha} - 2\sum_{k=1}^{n-1} h_{kk}^{\alpha} = 0, \\ \frac{\partial \mathcal{P}}{\partial h_{ij}^{\alpha}} &= 4\left(\frac{r}{n} + \frac{a(r)}{n-1} + 1\right) h_{ij}^{\alpha} = 0, \quad (i\neq j), \\ \frac{\partial \mathcal{P}}{\partial h_{in}^{\alpha}} &= 4\left(\frac{r}{n}+1\right) h_{in}^{\alpha} = 0, \end{aligned} \tag{2.4}$$

with $i, j \in \{1, \ldots, n-1\}$ and $\alpha \in \{1, \ldots, m\}$. Thus, every solution h^c of (2.4) has $h_{ij}^{\alpha} = 0$ for $i \neq j$ (which corresponds to submanifolds with *trivial normal connection*) and the determinant of the first two sets of equations of (2.4) is zero (implying that there exist solutions which do not correspond to *totally geodesic submanifolds*). Moreover, the eigenvalues of the Hessian matrix of $\mathcal{P}$ are

$$\lambda_{11}^{\alpha} = 0; \quad \lambda_{22}^{\alpha} = \frac{2}{nr}\Big(r^2 + n^2(n-1)\Big); \quad \lambda_{33}^{\alpha} = \cdots = \lambda_{nn}^{\alpha} = \frac{2(n-1)}{r}(r+n);$$

$$\lambda_{ij}^{\alpha} = 4\left(\frac{r}{n} + \frac{a(r)}{n-1} + 1\right), \quad (i\neq j); \qquad \lambda_{in}^{\alpha} = 4\left(\frac{r}{n}+1\right), \quad (i,j \in \{1,\ldots,n-1\}).$$

Hence, $\mathcal{P}$ is parabolic and reaches a minimum $\mathcal{P}(h^c) = 0$ for each solution h^c of (2.4), as follows from inserting (2.4) in (2.3). Thus, $\mathcal{P} \geq 0$, i.e.,

$$\tau \leq r\,\mathcal{C} + a(r)\,\mathcal{C}(W) + n(n-1)\,\widetilde{c}\,.$$

And because this holds for every tangent hyperplane W of M, (2.1) and (2.2) trivially follow. □

3. CHARACTERISATIONS OF THE EQUALITY CASES

Equality holds in the inequalities (2.1) and (2.2) if and only if

$$h_{ij}^{\alpha} = 0, \quad (i \neq j \in \{1, \ldots, n\}), \tag{3.1}$$

and

$$h_{11}^{\alpha} = \cdots = h_{n-1,n-1}^{\alpha} = \frac{r}{n(n-1)} h_{nn}^{\alpha}, \quad (\alpha \in \{1, \ldots, m\}). \tag{3.2}$$

Equation (3.1) means that the shape operators with respect to all normal directions ξ_α commute, or equivalently, that the *normal connection* $\nabla^\perp$ is flat, or still, that the *normal curvature tensor* $R^\perp$, i.e., the curvature tensor of the normal connection, is *zero*. Furthermore, (3.2) means that there exist m mutually orthogonal unit normal vector fields $\xi_1, \ldots, \xi_m$ such that the shape operators with respect to all directions ξ_α have an eigenvalue of multiplicity $n-1$ and that for each ξ_α the distinguished eigendirection is the same (namely e_n), i.e., that the submanifold is *invariantly quasi-umbilical*. Thus, we have proved the following.

Corollary 3.1. *Let M^n be a Riemannian submanifold of a real space form $\widetilde{M}^{n+m}(\widetilde{c})$. Equality holds in (2.1) or (2.2) if and only if M is invariantly quasi-umbilical with trivial normal connection in $\widetilde{M}$ and, with respect to suitable tangent and normal orthonormal frames, the shape operators are given by*

$$A^1 = \begin{pmatrix} \lambda & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \lambda & 0 \\ 0 & \cdots & 0 & \frac{n(n-1)}{r}\lambda \end{pmatrix}, \qquad A^2 = \cdots = A^m = 0. \tag{3.3}$$

From a result in [11] it follows that every totally quasi-umbilical submanifold of dimension ≥ 4 in a real space form is *conformally flat*. In [10] it is shown that every $n(\geq 4)$-dimensional *conformally flat submanifold with trivial normal connection* in a conformally flat space of dimension $n+m$ is totally quasi-umbilical if $m < n-2$, and in [21] it is shown that every $n(\geq 4)$-dimensional *submanifold in* $\mathbb{E}^{n+m}$ with $m \leq \min\{4, n-3\}$ is totally quasi-umbilical if and only if it is conformally flat. Thus, in particular, we also have the following.

Corollary 3.2. *The Casorati ideal submanifolds for (2.1) and (2.2) with $n \geq 4$ are conformally flat submanifolds with trivial normal connection.*

We remark that an *obstruction* for a manifold to be conformally flat in terms of the δ-curvatures of Chen was given in [9].

The *pseudo-symmetric spaces* were introduced by Deszcz (see e.g. [13, 15]) in the study of totally umbilical submanifolds with parallel mean curvature vector, i.e. of *extrinsic spheres*, in semi-symmetric spaces. A pseudo-symmetric manifold has the property that $R \cdot R = L\,(\wedge_g \cdot R)$, whereby $R \cdot R$ is the $(0,6)$-tensor obtained by the action of the curvature operator $R(X,Y)$ as a derivation on the $(0,4)$ curvature tensor, $\wedge_g \cdot R$ is the $(0,6)$ Tachibana tensor, obtained by the action of the metrical endomorphism $X \wedge_g Y$ as a derivation on the $(0,4)$ curvature tensor, and L is a scalar valued function on the manifold, called the *sectional curvature of Deszcz* (see [19] for a *geometrical interpretation* of this curvature). It follows from (3.3), by a straightforward calculation, that the Casorati ideal submanifolds M in $\widetilde{M}$ are pseudo-symmetric spaces (see also [14]) whose sectional curvature of Deszcz is given by $L = \frac{\tau}{n(n+1)}$ [12]. Thus, we also have the following.

Corollary 3.3. *The Casorati ideal submanifolds of (2.1) and (2.2) are pseudo-symmetric manifolds whose sectional curvature L of Deszcz can be expressed in terms of the Casorati curvature as*

$$L = \frac{nr}{(n-1)(n+1)(r+n)}\Big[r(n-2) + 2n(n-1)\Big]\mathcal{C}^2 + \frac{(n-1)}{n+1}\widetilde{c}.$$

A *rotation hypersurface* of a real space form $\widetilde{M}^{n+1}$ is generated by moving an $(n-1)$-dimensional totally umbilical submanifold along a curve in $\widetilde{M}$ [17]. If M^n is a Casorati ideal *hypersurface* in $\widetilde{M}^{n+1}(\widetilde{c})$, it follows from [16, 17] that M^n is a rotation hypersurface whose *profile curve* is the graph of a function f of one real variable x which satisfies the differential equation

$$f(f'' + \widetilde{c}\, f) + \frac{n(n-1)}{r}(\varepsilon - \widetilde{c}\, f^2 - f'^2) = 0, \tag{3.4}$$

whereby $\varepsilon = 0$, 1 or -1 if $\widetilde{c} < 0$ (the rotation hypersurface M^n is *parabolical*, *spherical* or *hyperbolical*, respectively), and $\varepsilon = 1$ if $\widetilde{c} \geq 0$.

Corollary 3.4. *The Casorati ideal hypersurfaces of real space forms are rotation hypersurfaces whose profile curves are given by the solutions of (3.4).*

By way of examples, we finally list a few solutions of (3.4) for some special values of $\widetilde{c}$, ε and r.
If $\widetilde{c} = 0$, $\varepsilon = 1$ and $r = 2n(n-1)$:

$$f(x) = \frac{c_1^2(x + c_2)^2 - 4}{4c_1};$$

if $\widetilde{c} = -1$, $\varepsilon = 1$ and $r = 2n(n-1)$:

$$f(x) = \frac{4e^x - c_1^2(1 + c_2 e^x)^2 e^{-x}}{4c_1};$$

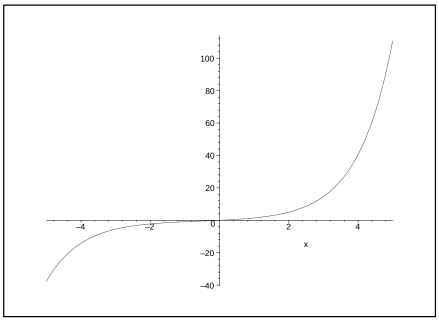
if $\widetilde{c} = -1$, $\varepsilon = 0$ and $r = 2n(n-1)$:

$$f(x) = \frac{1}{4}(c_1 - c_2 e^x)^2 e^{-x};$$

if $\widetilde{c} = -1$, $\varepsilon = -1$ and $r = 2n(n-1)$:

$$f(x) = \frac{4e^x + c_1^2(1 + c_2 e^x)^2 e^{-x}}{4c_1};$$

whereby c_1 and c_2 are integration constants.



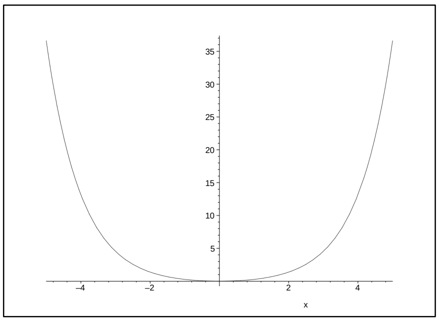


Figure 3.1: The profile curve on the left is $f(x) = \frac{4e^x - e^{-x}(1+e^x)^2}{4}$ *and on the right is* $f(x) = \frac{e^{-x}(1-e^x)^2}{4}$.

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