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ON SOME APPROXIMATE FUNCTIONAL RELATIONS STEMMING FROM ORTHOGONALITY PRESERVING PROPERTY

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ABSTRACT. Referring to previous papers on orthogonality preserving mappings we deal with some relations, connected with orthogonality, which are preserved exactly or approximately. In particular, we investigate the class of mappings approximately preserving the right-angle. We show some properties similar to those characterizing mappings which exactly preserve the right-angle. Besides, some kind of stability of the considered property is established. We study also the property that a particular value c of the inner product is preserved. We compare the case $c \neq 0$ with $c = 0$, i.e., with orthogonality preserving property. Also here some stability results are given.

Key words and phrases: Orthogonality preserving mappings, Right-angle preserving mappings, Preservation of the inner product, Stability of functional equations.

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1. PREREQUISITES

Let X and Y be real inner product spaces with the standard orthogonality relation $\perp$. For a mapping $f : X \to Y$ it is natural to consider the *orthogonality preserving property*:

(OP) $$\forall x, y \in X : x \perp y \Rightarrow f(x) \perp f(y).$$

The class of solutions of (OP) contains also very irregular mappings (cf. [1, Examples 1 and 2]). On the other hand, a *linear* solution f of (OP) has to be a linear similarity, i.e., it satisfies (cf. [1, Theorem 1])

(1.1) $$\|f(x)\| = \gamma \|x\|, \qquad x \in X$$

or, equivalently,

(1.2) $$\langle f(x) | f(y) \rangle = \gamma^2 \langle x | y \rangle , \qquad x, y \in X$$

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with some $\gamma \geq 0$ ($\gamma > 0$ for $f \neq 0$). (More generally, a linear mapping between real normed spaces which preserves the Birkhoff-James orthogonality has to satisfy (1.1) – see [5].) Therefore, linear orthogonality preserving mappings are not far from inner product preserving mappings (linear isometries), i.e., solutions of the functional equation:

$$\forall x, y \in X : \langle f(x)|f(y)\rangle = \langle x|y\rangle \, . \tag{1.3}$$

A property similar to (OP) was introduced by Kestelman and Tissier (see [6]). One says that f has the *right-angle preserving property* iff:

$$\text{(RAP)} \qquad \forall x, y, z \in X : x - z \perp y - z \Rightarrow f(x) - f(z) \perp f(y) - f(z).$$

For the solutions of (RAP) it is known (see [6]) that they must be affine, continuous similarities (with respect to some point).

It is easily seen that if f satisfies (RAP) then, for an arbitrary $y_0 \in Y$, the mapping $f + y_0$ satisfies (RAP) as well. In particular, $f_0 := f - f(0)$ satisfies (RAP) and $f_0(0) = 0$.

Summing up we have:

Theorem 1.1. *The following conditions are equivalent:*

(i) *f satisfies (RAP) and $f(0) = 0$;*
(ii) *f satisfies (OP) and f is continuous and linear;*
(iii) *f satisfies (OP) and f is linear;*
(iv) *f is linear and satisfies* (1.1) *for some constant $\gamma \geq 0$;*
(v) *f satisfies* (1.2) *for some constant $\gamma \geq 0$;*
(vi) *f satisfies (OP) and f is additive.*

Proof. (i)⇒(ii) follows from [6] (see above); (ii)⇒(iii) is trivial; (iii)⇒(iv) follows from [1] (see above); (iv)⇒(v) by use of the polarization formula and (v)⇒(vi)⇒(i) is trivial. □

In particular, one can consider a real vector space X with two inner products $\langle\cdot|\cdot\rangle_1$ and $\langle\cdot|\cdot\rangle_2$ and $f = \mathrm{id}|_X$ a linear and continuous mapping between $(X, \langle\cdot|\cdot\rangle_1)$ and $(X, \langle\cdot|\cdot\rangle_2)$. Then we obtain from Theorem 1.1:

Corollary 1.2. *Let X be a real vector space equipped with two inner products $\langle\cdot|\cdot\rangle_1$ and $\langle\cdot|\cdot\rangle_2$ generating the norms $\|\cdot\|_1$, $\|\cdot\|_2$ and orthogonality relations $\perp_1$, $\perp_2$, respectively. Then the following conditions are equivalent:*

(i) $\forall x, y, z \in X : x - z \perp_1 y - z \Rightarrow x - z \perp_2 y - z$;
(ii) $\forall x, y \in X : x \perp_1 y \Rightarrow x \perp_2 y$;
(iii) $\|x\|_2 = \gamma \|x\|_1$ *for $x \in X$ with some constant $\gamma > 0$;*
(iv) $\langle x|y\rangle_2 = \gamma^2 \langle x|y\rangle_1$ *for $x, y \in X$ with some constant $\gamma > 0$;*
(v) $\forall x, y, z \in X : x - z \perp_1 y - z \Leftrightarrow x - z \perp_2 y - z$;
(vi) $\forall x, y \in X : x \perp_1 y \Leftrightarrow x \perp_2 y$.

For $\varepsilon \in [0, 1)$ we define an ε-orthogonality by

$$u \perp^\varepsilon v :\Leftrightarrow |\langle u|v\rangle| \leq \varepsilon \|u\| \|v\|.$$

(Some remarks on how to extend this definition to normed or semi-inner product spaces can be found in [2].)

Then, it is natural to consider an approximate orthogonality preserving (a.o.p.) property:

$$(\varepsilon\text{-OP}) \qquad \forall x, y \in X : x \perp y \Rightarrow f(x) \perp^\varepsilon f(y)$$

and the approximate right-angle preserving (a.r.a.p.) property:

$$(\varepsilon\text{-RAP}) \qquad \forall x, y, z \in X : x - z \perp y - z \Rightarrow f(x) - f(z) \perp^\varepsilon f(y) - f(z).$$

The class of linear mappings satisfying (ε-OP) has been considered by the author (cf. [1, 3]). In the present paper we are going to deal with mappings satisfying (ε-RAP) and, in the last section, with mappings which preserve (exactly or approximately) a given value of the inner product. We will deal also with some stability problems. (For basic facts concerning the background and main results in the theory of stability of functional equations we refer to [4].) The following result establishing the stability of equation (1.3) has been proved in [3] and will be used later on.

Theorem 1.3 ([3], Theorem 2)**.** *Let X and Y be inner product spaces and let X be finite-dimensional. Then, there exists a continuous mapping $\delta : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{\varepsilon\to 0^+} \delta(\varepsilon) = 0$ which satisfies the following property: For each mapping $f : X \to Y$ (not necessarily linear) satisfying*

$$|\langle f(x)|f(y)\rangle - \langle x|y\rangle| \leq \varepsilon\|x\|\|y\|, \qquad x, y \in X \tag{1.4}$$

there exists a linear isometry $I : X \to Y$ such that

$$\|f(x) - I(x)\| \leq \delta(\varepsilon)\|x\|, \qquad x \in X.$$

2. ADDITIVITY OF APPROXIMATELY RIGHT-ANGLE PRESERVING MAPPINGS

Tissier [6] showed that a mapping f satisfying the (RAP) property has to be additive up to a constant $f(0)$. Following his idea we will show that a.r.a.p. mappings are, in some sense, quasi-additive. We start with the following lemma.

Lemma 2.1. *Let X be a real inner product space. Let a set of points $a, b, c, d, e \in X$ satisfies the following relations, with $\varepsilon \in [0, \frac{1}{8})$,*

$$a - b\perp^\varepsilon c - b, \qquad b - c\perp^\varepsilon d - c, \qquad c - d\perp^\varepsilon a - d, \qquad d - a\perp^\varepsilon b - a; \tag{2.1}$$

$$a - e\perp^\varepsilon b - e, \qquad b - e\perp^\varepsilon c - e, \qquad c - e\perp^\varepsilon d - e, \qquad d - e\perp^\varepsilon a - e. \tag{2.2}$$

Then,

$$\left\| e - \frac{a+c}{2} \right\| \leq \delta\|a - c\|$$

with $\delta := \sqrt{\frac{3\varepsilon}{1-4\varepsilon}}$.

Proof. We have

$$\|c - e\|^2 = \|e - a + a - c\|^2 = \|e - a\|^2 + \|a - c\|^2 + 2\langle e - a|a - c\rangle$$

whence

$$\langle e - a|a - c\rangle = \frac{\|c - e\|^2 - \|a - e\|^2 - \|a - c\|^2}{2}.$$

Thus

$$\begin{aligned}
\left\| e - \frac{a+c}{2} \right\|^2 &= \left\| e - a + \frac{a-c}{2} \right\|^2 \\
&= \|e - a\|^2 + \frac{1}{4}\|a - c\|^2 + \langle e - a|a - c\rangle \\
&= \|e - a\|^2 + \frac{1}{4}\|a - c\|^2 + \frac{1}{2}\|c - e\|^2 - \frac{1}{2}\|a - e\|^2 - \frac{1}{2}\|a - c\|^2 \\
&= \frac{1}{2}\|a - e\|^2 + \frac{1}{2}\|c - e\|^2 - \frac{1}{4}\|a - c\|^2.
\end{aligned}$$

Finally,

$$\left\|e-\frac{a+c}{2}\right\|^2=\frac{1}{4}\left(2\|a-e\|^2+2\|c-e\|^2-\|a-c\|^2\right) \tag{2.3}$$

and, analogously,

$$\left\|e-\frac{b+d}{2}\right\|^2=\frac{1}{4}\left(2\|b-e\|^2+2\|d-e\|^2-\|b-d\|^2\right). \tag{2.4}$$

Adding the equalities:

$$\begin{aligned}
\|a-b\|^2&=\|a-e+e-b\|^2=\|a-e\|^2+\|e-b\|^2+2\langle a-e|e-b\rangle,\\
\|b-c\|^2&=\|b-e+e-c\|^2=\|b-e\|^2+\|e-c\|^2+2\langle b-e|e-c\rangle,\\
\|c-d\|^2&=\|c-e+e-d\|^2=\|c-e\|^2+\|e-d\|^2+2\langle c-e|e-d\rangle,\\
\|d-a\|^2&=\|d-e+e-a\|^2=\|d-e\|^2+\|e-a\|^2+2\langle d-e|e-a\rangle
\end{aligned}$$

one gets

$$\begin{aligned}
&\|a-b\|^2+\|b-c\|^2+\|c-d\|^2+\|d-a\|^2\\
&\quad=2\|a-e\|^2+2\|b-e\|^2+2\|c-e\|^2+2\|d-e\|^2\\
&\qquad+2\langle a-e|e-b\rangle+2\langle b-e|e-c\rangle\\
&\qquad\quad+2\langle c-e|e-d\rangle+2\langle d-e|e-a\rangle.
\end{aligned} \tag{2.5}$$

Similarly, adding

$$\begin{aligned}
\|a-c\|^2&=\|a-b+b-c\|^2=\|a-b\|^2+\|b-c\|^2+2\langle a-b|b-c\rangle,\\
\|a-c\|^2&=\|a-d+d-c\|^2=\|a-d\|^2+\|d-c\|^2+2\langle a-d|d-c\rangle,\\
\|b-d\|^2&=\|b-a+a-d\|^2=\|b-a\|^2+\|a-d\|^2+2\langle b-a|a-d\rangle,\\
\|b-d\|^2&=\|b-c+c-d\|^2=\|b-c\|^2+\|c-d\|^2+2\langle b-c|c-d\rangle
\end{aligned}$$

one gets

$$\begin{aligned}
\|a-c\|^2+\|b-d\|^2&=\|a-b\|^2+\|a-d\|^2+\|c-b\|^2+\|c-d\|^2\\
&\quad+\langle a-b|b-c\rangle+\langle a-d|d-c\rangle\\
&\qquad+\langle b-a|a-d\rangle+\langle b-c|c-d\rangle.
\end{aligned} \tag{2.6}$$

Using (2.3) – (2.6) we derive

$$\begin{aligned}
&\left\|e-\frac{a+c}{2}\right\|^2+\left\|e-\frac{b+d}{2}\right\|^2\\
&\overset{(2.3),(2.4)}{=}\frac{2\|a-e\|^2+2\|c-e\|^2+2\|b-e\|^2+2\|d-e\|^2-\|a-c\|^2-\|b-d\|^2}{4}\\
&\overset{(2.5)}{=}\frac{1}{4}\Big(\|b-c\|^2+\|c-d\|^2+\|d-a\|^2+\|a-b\|^2\\
&\qquad-2\langle b-e|e-c\rangle-2\langle c-e|e-d\rangle-2\langle d-e|e-a\rangle\\
&\qquad-2\langle a-e|e-b\rangle-\|a-c\|^2-\|b-d\|^2\Big)
\end{aligned}$$

$$\stackrel{(2.6)}{=} -\frac{1}{4}\Big(\langle a-b|b-c\rangle + \langle a-d|d-c\rangle + \langle b-a|a-d\rangle + \langle b-c|c-d\rangle + 2\langle b-e|e-c\rangle + 2\langle c-e|e-d\rangle + 2\langle d-e|e-a\rangle + 2\langle a-e|e-b\rangle \Big).$$

Thus

$$\left\|e-\frac{a+c}{2}\right\|^2 + \left\|e-\frac{b+d}{2}\right\|^2 \le \frac{1}{4}\Big(|\langle a-b|b-c\rangle| + |\langle a-d|d-c\rangle| + |\langle b-a|a-d\rangle| + |\langle b-c|c-d\rangle| + 2|\langle b-e|e-c\rangle| + 2|\langle c-e|e-d\rangle| + 2|\langle d-e|e-a\rangle| + 2|\langle a-e|e-b\rangle|\Big).$$

Using the assumptions (2.1) and (2.2), we obtain

$$\left\|e-\frac{a+c}{2}\right\|^2 + \left\|e-\frac{b+d}{2}\right\|^2 \le \frac{1}{4}\varepsilon\Big(\|a-b\|\|b-c\| + \|a-d\|\|d-c\| + \|b-a\|\|a-d\| + \|b-c\|\|c-d\| + 2\|b-e\|\|e-c\| + 2\|c-e\|\|e-d\| + 2\|d-e\|\|e-a\| + 2\|a-e\|\|e-b\|\Big) = \frac{1}{4}\varepsilon\Big((\|b-c\|+\|a-d\|)(\|a-b\|+\|c-d\|) + 2(\|b-e\|+\|d-e\|)(\|a-e\|+\|c-e\|)\Big). \tag{2.7}$$

Notice, that for $\varepsilon = 0$, (2.7) yields $e = \frac{a+c}{2} = \frac{b+d}{2}$.
Let

$$\varrho := \max\{\|a-b\|, \|b-c\|, \|c-d\|, \|d-a\|, \|a-e\|, \|b-e\|, \|c-e\|, \|d-e\|\}.$$

It follows from (2.7) that

$$\left\|e-\frac{a+c}{2}\right\|^2 + \left\|e-\frac{b+d}{2}\right\|^2 \le \frac{1}{4}\varepsilon(2\varrho\cdot 2\varrho + 2\cdot 2\varrho\cdot 2\varrho) = 3\varepsilon\varrho^2.$$

Then, in particular

$$\left\|e-\frac{a+c}{2}\right\|^2 \le 3\varepsilon\varrho^2. \tag{2.8}$$

Since we do not know for which distance the value ϱ is attained, we are going to consider a few cases.

(1) $\varrho \in \{\|a-b\|, \|b-c\|, \|c-d\|, \|d-a\|\}$.
Suppose that $\varrho = \|a-b\|$ (other possibilities in this case are similar). Then

$$\begin{aligned}\|a-c\|^2 &= \|a-b+b-c\|^2 \\ &= \|a-b\|^2 + \|b-c\|^2 + 2\langle a-b|b-c\rangle \\ &\ge \varrho^2 + 0 - 2\varepsilon\|a-b\|\|b-c\| \\ &\ge \varrho^2 - 2\varepsilon\varrho^2 = (1-2\varepsilon)\varrho^2.\end{aligned}$$

Assuming $\varepsilon < \frac{1}{2}$,

$$\varrho^2 \leq \frac{1}{1-2\varepsilon}\|a-c\|^2.$$

Using (2.8) we have

$$\left\|e - \frac{a+c}{2}\right\|^2 \leq \frac{3\varepsilon}{1-2\varepsilon}\|a-c\|^2$$

whence

$$\left\|e - \frac{a+c}{2}\right\| \leq \sqrt{\frac{3\varepsilon}{1-2\varepsilon}}\|a-c\|. \tag{2.9}$$

(2) $\varrho \in \{\|a-e\|, \|c-e\|\}$.

Suppose that $\varrho = \|a-e\|$ (the other possibility is similar). Then, from (2.3), we have

$$\frac{1}{4}\|a-c\|^2 + \left\|e - \frac{a+c}{2}\right\|^2 = \frac{1}{2}\|a-e\|^2 + \frac{1}{2}\|c-e\|^2 \geq \frac{1}{2}\varrho^2,$$

whence

$$\varrho^2 \leq \frac{1}{2}\|a-c\|^2 + 2\left\|e - \frac{a+c}{2}\right\|^2.$$

From it and (2.8) we get

$$\left\|e - \frac{a+c}{2}\right\|^2 \leq 3\varepsilon\varrho^2 \leq \frac{3\varepsilon}{2}\|a-c\|^2 + 6\varepsilon\left\|e - \frac{a+c}{2}\right\|^2$$

whence (assuming $\varepsilon < \frac{1}{6}$)

$$\left\|e - \frac{a+c}{2}\right\| \leq \sqrt{\frac{3\varepsilon}{2(1-6\varepsilon)}}\|a-c\|. \tag{2.10}$$

(3) $\varrho \in \{\|b-e\|, \|d-e\|\}$.

Suppose that $\varrho = \|b-e\|$ (the other possibility is similar). We have then

$$\begin{aligned}\|b-a\|^2 &= \|b-e+e-a\|^2 \\ &= \|b-e\|^2 + \|e-a\|^2 + 2\langle b-e|e-a\rangle \\ &\geq \varrho^2 + 0 - 2\varepsilon\|b-e\|\|e-a\| \\ &\geq \varrho^2 - 2\varepsilon\varrho^2 = (1-2\varepsilon)\varrho^2,\end{aligned}$$

whence

$$\|b-a\|^2 \geq (1-2\varepsilon)\varrho^2.$$

Using this estimation we have

$$\begin{aligned}\|a-c\|^2 &= \|a-b+b-c\|^2 \\ &= \|a-b\|^2 + \|b-c\|^2 + 2\langle a-b|b-c\rangle \\ &\geq (1-2\varepsilon)\varrho^2 + 0 - 2\varepsilon\|a-b\|\|b-c\| \\ &\geq (1-2\varepsilon)\varrho^2 - 2\varepsilon\varrho^2 \\ &= (1-4\varepsilon)\varrho^2,\end{aligned}$$

whence (for $\varepsilon < \frac{1}{4}$)

$$\varrho^2 \leq \frac{1}{1-4\varepsilon}\|a-c\|^2.$$

Using (2.8) we get

$$\left\|e - \frac{a+c}{2}\right\|^2 \le 3\varepsilon\varrho^2 \le \frac{3\varepsilon}{1-4\varepsilon}\|a-c\|^2$$

and

$$\left\|e - \frac{a+c}{2}\right\| \le \sqrt{\frac{3\varepsilon}{1-4\varepsilon}}\|a-c\|. \tag{2.11}$$

Finally, assuming $\varepsilon < \frac{1}{8}$, we have

$$\max\left\{\sqrt{\frac{3\varepsilon}{1-2\varepsilon}}, \sqrt{\frac{3\varepsilon}{2(1-6\varepsilon)}}, \sqrt{\frac{3\varepsilon}{1-4\varepsilon}}\right\} = \sqrt{\frac{3\varepsilon}{1-4\varepsilon}}$$

and it follows from (2.9) – (2.11)

$$\left\|e - \frac{a+c}{2}\right\| \le \sqrt{\frac{3\varepsilon}{1-4\varepsilon}}\|a-c\|,$$

which completes the proof.

□

Theorem 2.2. *Let X and Y be real inner product spaces and let $f : X \to Y$ satisfy (ε-RAP) with $\varepsilon < \frac{1}{8}$. Then f satisfies*

$$\left\|f\left(\frac{x+y}{2}\right) - \frac{f(x)+f(y)}{2}\right\| \le \delta\|f(x)-f(y)\| \quad \textit{for} \quad x,y \in X \tag{2.12}$$

with $\delta = \sqrt{\frac{3\varepsilon}{1-4\varepsilon}}$.

Moreover, if additionally $f(0) = 0$, then f satisfies (ε-OP) and

$$\|f(x+y) - f(x) - f(y)\| \le 2\delta(\|f(x+y)\| + \|f(x)-f(y)\|), \quad \textit{for} \quad x,y \in X. \tag{2.13}$$

Proof. Fix arbitrarily $x, y \in X$. The case $x = y$ is obvious. Assume $x \neq y$. Choose $u, v \in X$ such that x, u, y, v are consecutive vertices of a square with the center at $\frac{x+y}{2}$. Denote

$$a := f(x), \qquad b := f(u), \qquad c := f(y), \qquad d := f(v), \qquad e := f\left(\frac{x+y}{2}\right).$$

Since $x-u \perp y-u$, $u-y \perp v-y$, $y-v \perp x-v$, $v-x \perp u-x$ and $x-\frac{x+y}{2} \perp u-\frac{x+y}{2}$, $u-\frac{x+y}{2} \perp y-\frac{x+y}{2}$, $y-\frac{x+y}{2} \perp v-\frac{x+y}{2}$, $v-\frac{x+y}{2} \perp x-\frac{x+y}{2}$, it follows from ($\varepsilon$-RAP) that the conditions (2.1) and (2.2) are satisfied. The assertion of Lemma 2.1 yields (2.12).

For the second assertion, it is obvious that f satisfies (ε-OP). Inequality (2.13) follows from (2.12). Indeed, putting $y = 0$ we get

$$\left\|f\left(\frac{x}{2}\right) - \frac{f(x)}{2}\right\| \le \delta\|f(x)\|, \qquad x \in X.$$

Now, for $x, y \in X$

$$\begin{aligned}
\|f(x+y) &- f(x) - f(y)\| \\
&= 2\left\|\frac{f(x+y)}{2} - f\left(\frac{x+y}{2}\right) + f\left(\frac{x+y}{2}\right) - \frac{f(x)+f(y)}{2}\right\| \\
&\le 2\left\|\frac{f(x+y)}{2} - f\left(\frac{x+y}{2}\right)\right\| + 2\left\|f\left(\frac{x+y}{2}\right) - \frac{f(x)+f(y)}{2}\right\| \\
&\le 2\delta\|f(x+y)\| + 2\delta\|f(x)-f(y)\|.
\end{aligned}$$

□

For $\varepsilon = 0$ we obtain that if f satisfies (RAP) and $f(0) = 0$, then f is additive.
The following, reverse in a sense, statement is easily seen.

Lemma 2.3. *If f satisfies (ε-OP) and f is additive, then f satisfies (ε-RAP) and $f(0) = 0$.*

Example 2 in [1] shows that it is not possible to omit completely the additivity assumption in the above lemma. However, the problem arises if additivity can be replaced by a weaker condition (e.g. by (2.13)). This problem remains open.

3. APPROXIMATE RIGHT-ANGLE PRESERVING MAPPINGS ARE APPROXIMATE SIMILARITIES

As we know from [6], a right-angle preserving mappings are similarities. Our aim is to show that a.r.a.p. mappings behave similarly. We start with a technical lemma.

Lemma 3.1. *Let $a, x \in X$ and $\varepsilon \in [0, 1)$. Then*

$$(a - x)\perp^{\varepsilon}(-a - x) \tag{3.1}$$

if and only if

$$\left|\|x\|^2 - \|a\|^2\right| \le \frac{2\varepsilon}{\sqrt{1-\varepsilon^2}}\sqrt{\|a\|^2\|x\|^2 - \langle a|x\rangle^2}. \tag{3.2}$$

Moreover, it follows from (3.1) *that*

$$\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\|a\| \le \|x\| \le \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\|a\|. \tag{3.3}$$

Proof. The condition (3.1) is equivalent to:

$$|\langle a + x|a - x\rangle| \le \varepsilon\|a + x\|\|a - x\|,$$

$$\left|\|a\|^2 - \|x\|^2\right| \le \varepsilon\sqrt{\|a\|^2 + 2\langle a|x\rangle + \|x\|^2}\sqrt{\|a\|^2 - 2\langle a|x\rangle + \|x\|^2},$$

$$\begin{aligned}\left(\|a\|^2 - \|x\|^2\right)^2 &\le \varepsilon^2(\|a\|^2 + \|x\|^2 + 2\langle a|x\rangle)(\|a\|^2 + \|x\|^2 - 2\langle a|x\rangle)\\ &= \varepsilon^2\Big((\|a\|^2 + \|x\|^2)^2 - 4\langle a|x\rangle^2\Big)\\ &= \varepsilon^2\Big((\|a\|^2 - \|x\|^2)^2 + 4\|a\|^2\|x\|^2 - 4\langle a|x\rangle^2\Big),\end{aligned}$$

and finally

$$(1-\varepsilon^2)(\|a\|^2 - \|x\|^2)^2 \le 4\varepsilon^2\left(\|a\|^2\|x\|^2 - \langle a|x\rangle^2\right)$$

which is equivalent to (3.2).

Inequality (3.2) implies

$$\left|\|x\|^2 - \|a\|^2\right| \le \frac{2\varepsilon}{\sqrt{1-\varepsilon^2}}\|a\|\|x\|,$$

which yields

$$\left|\frac{\|x\|}{\|a\|} - \frac{\|a\|}{\|x\|}\right| \le \frac{2\varepsilon}{\sqrt{1-\varepsilon^2}} \tag{3.4}$$

(we assume $x \neq 0$ and $a \neq 0$, otherwise the assertion of the lemma is trivial). Denoting $t := \frac{\|x\|}{\|a\|} > 0$ and $\alpha := \frac{2\varepsilon}{\sqrt{1-\varepsilon^2}}$, the inequality (3.4) can be written in the form

$$|t - t^{-1}| \le \alpha$$

with a solution

$$\frac{-\alpha+\sqrt{\alpha^2+4}}{2} \leq t \leq \frac{\alpha+\sqrt{\alpha^2+4}}{2}.$$

Therefore,

$$\sqrt{\frac{1-\varepsilon}{1+\varepsilon}} \leq \frac{\|x\|}{\|a\|} \leq \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}$$

whence (3.3) is satisfied. □

Theorem 3.2. *If $f: X \to Y$ is homogeneous and satisfies (ε-RAP), then with some $k \geq 0$:*

$$k \cdot \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{\frac{3}{2}} \|x\| \leq \|f(x)\| \leq k \cdot \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{\frac{3}{2}} \|x\|, \qquad x \in X. \tag{3.5}$$

Proof. 1. For arbitrary $x, y \in X$ we have

$$\|x\| = \|y\| \Leftrightarrow x - y \perp -x - y.$$

It follows from (ε-RAP) and the oddness of f that

$$\|x\| = \|y\| \Rightarrow f(x) - f(y) \perp^{\varepsilon} -f(x) - f(y).$$

Lemma 3.1 yields

$$\|x\| = \|y\| \Rightarrow \sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\|f(x)\| \leq \|f(y)\| \leq \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\|f(x)\|. \tag{3.6}$$

2. Fix arbitrarily $x_0 \neq 0$ and define for $r \geq 0$, $\varphi(r) := \left\|f\left(\frac{r}{\|x_0\|}x_0\right)\right\|$. Using (3.6) we have

$$\|x\| = r \Rightarrow \sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\varphi(r) \leq \|f(x)\| \leq \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\varphi(r),$$

whence

$$\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\varphi(\|x\|) \leq \|f(x)\| \leq \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\varphi(\|x\|), \qquad x \in X. \tag{3.7}$$

3. For $t \geq 0$ and $\|x\| = r$ we have

$$\|f(tx)\| \in \left[\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\varphi(tr), \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\varphi(tr)\right]$$

and

$$t\|f(x)\| \in \left[\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}t\varphi(r), \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}t\varphi(r)\right].$$

Since $\|f(tx)\| = t\|f(x)\|$ (homogeneity of f),

$$\left[\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\varphi(tr), \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\varphi(tr)\right] \cap \left[\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}t\varphi(r), \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}t\varphi(r)\right] \neq \emptyset.$$

Thus there exist $\lambda, \mu \in \left[\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}, \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\right]$ such that $\lambda\varphi(tr) = \mu t\varphi(r)$, whence

$$\frac{1-\varepsilon}{1+\varepsilon}t\varphi(r) \leq \varphi(tr) \leq \frac{1+\varepsilon}{1-\varepsilon}t\varphi(r).$$

In particular, for $r = 1$ and $k := \varphi(1)$ we get

$$\frac{1-\varepsilon}{1+\varepsilon}kt \leq \varphi(t) \leq \frac{1+\varepsilon}{1-\varepsilon}kt, \qquad t \geq 0. \tag{3.8}$$

4. Using (3.7) and (3.8), we get

$$\|f(x)\| \le \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\varphi(\|x\|) \le \sqrt{\frac{1+\varepsilon}{1-\varepsilon}} \cdot \frac{1+\varepsilon}{1-\varepsilon}k\|x\|$$

and

$$\|f(x)\| \ge \sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\varphi(\|x\|) \ge \sqrt{\frac{1-\varepsilon}{1+\varepsilon}} \cdot \frac{1-\varepsilon}{1+\varepsilon}k\|x\|$$

whence (3.5) is proved. □

For $\varepsilon = 0$ we get $\|f(x)\| = k\|x\|$ for $x \in X$ and, from (2.13), f is additive hence linear. Thus f is a similarity.

4. SOME STABILITY PROBLEMS

The stability of the orthogonality preserving property has been studied in [3]. We present the main result from that paper which will be used in this section.

Theorem 4.1 ([3, Theorem 4])**.** *Let X, Y be inner product spaces and let X be finite-dimensional. Then, there exists a continuous function $\delta : [0, 1) \to [0, +\infty)$ with the property $\lim_{\varepsilon\to 0^+} \delta(\varepsilon) = 0$ such that for each linear mapping $f : X \to Y$ satisfying (ε-OP) one finds a linear, orthogonality preserving one $T : X \to Y$ such that*

$$\|f - T\| \le \delta(\varepsilon) \min\{\|f\|, \|T\|\}.$$

The mapping δ depends, actually, only on the dimension of X. Immediately, we have from the above theorem:

Corollary 4.2. *Let X, Y be inner product spaces and let X be finite-dimensional. Then, for each $\delta > 0$ there exists $\varepsilon > 0$ such that for each linear mapping $f : X \to Y$ satisfying (ε-OP) one finds a linear, orthogonality preserving one $T : X \to Y$ such that*

$$\|f - T\| \le \delta \min\{\|f\|, \|T\|\}.$$

We start our considerations with the following observation.

Proposition 4.3. *Let $f : X \to Y$ satisfy (RAP) and $f(0) = 0$. Suppose that $g : X \to Y$ satisfies*

$$\|f(x) - g(x)\| \le M\|f(x)\|, \qquad x \in X$$

with some constant $M < \frac{1}{4}$. Then g satisfies (ε-OP) with $\varepsilon := \frac{M(M+2)}{(1-M)^2}$ and

$$\|g(x + y) - g(x) - g(y)\| \le 2\sqrt{\varepsilon}(\|g(x)\| + \|g(y)\|), \qquad x, y \in X; \tag{4.1}$$

$$\|g(\lambda x) - \lambda g(x)\| \le 2\sqrt{\varepsilon}|\lambda|\|g(x)\|, \qquad x \in X,\ \lambda \in \mathbb{R}. \tag{4.2}$$

Proof. It follows from Theorem 1.1 that for some $\gamma \ge 0$, f satisfies (1.2) and (1.1). Thus we have for arbitrary $x, y \in X$:

$$\begin{aligned}
&| \langle g(x)|g(y)\rangle - \gamma^2 \langle x|y\rangle | \\
&\quad = | \langle g(x) - f(x)|g(y) - f(y)\rangle + \langle g(x) - f(x)|f(y)\rangle \\
&\qquad\quad + \langle f(x)|g(y) - f(y)\rangle | \\
&\quad \le \|g(x) - f(x)\|\|g(y) - f(y)\| + \|g(x) - f(x)\|\|f(y)\| \\
&\qquad\quad + \|f(x)\|\|g(y) - f(y)\| \\
&\quad \le M^2\|f(x)\|\|f(y)\| + M\|f(x)\|\|f(y)\| + M\|f(x)\|\|f(y)\| \\
&\quad = M(M + 2)\gamma^2\|x\|\|y\|.
\end{aligned}$$

Using [1, Lemma 2], we get, for arbitrary $x, y \in X$, $\lambda \in \mathbb{R}$:

$$\begin{aligned}\|g(x+y) - g(x) - g(y)\| &\le 2\sqrt{M(M+2)}\gamma(\|x\| + \|y\|) \\ &= 2\sqrt{M(M+2)}(\|f(x)\| + \|f(y)\|); \\ \|g(\lambda x) - \lambda g(x)\| &\le 2\sqrt{M(M+2)}\gamma|\lambda|\|x\| \\ &= 2\sqrt{M(M+2)}|\lambda|\|f(x)\|.\end{aligned}$$

Since $\|f(x)\| - \|g(x)\| \le M\|f(x)\|$,

$$\|f(x)\| \le \frac{\|g(x)\|}{1-M}.$$

Therefore

$$| \langle g(x)|g(y)\rangle - \gamma^2 \langle x|y\rangle | \le \frac{M(M+2)}{(1-M)^2}\|g(x)\|\|g(y)\|.$$

Putting $\varepsilon := \frac{M(M+2)}{(1-M)^2}$ we have $x \perp y \Rightarrow g(x) \perp^{\varepsilon} g(y)$ and

$$\begin{aligned}\|g(x+y) - g(x) - g(y)\| &\le 2\sqrt{\varepsilon}(\|g(x)\| + \|g(y)\|) \\ \|g(\lambda x) - \lambda g(x)\| &\le 2\sqrt{\varepsilon}|\lambda|\|g(x)\|.\end{aligned}$$

□

Corollary 4.4. *If f satisfies (RAP), $f(0) = 0$ (whence f is linear) and $g : X \to Y$ is a linear mapping satisfying, with $M < \frac{1}{4}$,*

$$\|f - g\| \le M\|f\|, \tag{4.3}$$

then g is (ε-OP) (and linear) whence (ε-RAP).

The above result yields a natural question if the reverse statement is true. Namely, we may ask if for a linear mapping $g : X \to Y$ satisfying (ε-RAP) (with some $\varepsilon > 0$) there exists a (linear) mapping f satisfying (RAP) such that an estimation of the (4.3) type holds.

A particular solution to this problem follows easily from Theorem 4.1.

Theorem 4.5. *Let X be a finite-dimensional inner product space and Y an arbitrary one. There exists a mapping $\delta : [0, 1) \to \mathbb{R}_+$ satisfying $\lim_{\varepsilon \to 0^+} \delta(\varepsilon) = 0$ and such that for each linear mapping satisfying (ε-RAP) $g : X \to Y$ there exists $f : X \to Y$ satisfying (RAP) and such that*

$$\|f - g\| \le \delta(\varepsilon) \min\{\|f\|, \|g\|\}. \tag{4.4}$$

Proof. If g is linear and satisfies (ε-RAP), then g is linear and satisfies (ε-OP). It follows then from Theorem 4.1 that there exists f linear and satisfying (OP), whence (RAP), such that (4.4) holds. □

Corollary 4.6. *Let X be a finite-dimensional inner product space and Y an arbitrary one. Then, for each $\delta > 0$ there exists $\varepsilon > 0$ such that for each linear and satisfying (ε-RAP) mapping $g : X \to Y$ there exists $f : X \to Y$ satisfying (RAP) and such that*

$$\|f - g\| \le \delta \min\{\|f\|, \|g\|\}.$$

It is an open problem to verify if the above result remains true in the infinite dimensional case or without the linearity assumption.

5. On Mappings Preserving a Particular Value of the Inner Product

The following considerations have been inspired by a question of L. Reich during the 43rd ISFE. In this section X and Y are inner product spaces over the field $\mathbb{K}$ of real or complex numbers. Let $f : X \to Y$ and suppose that, for a fixed number $c \in \mathbb{K}$, f preserves this particular value of the inner product, i.e.,

$$\forall x, y \in X : \langle x|y\rangle = c \Rightarrow \langle f(x)|f(y)\rangle = c. \tag{5.1}$$

If $c = 0$, the condition (5.1) simply means that f preserves orthogonality, i.e., that f satisfies (OP).

We will show that the solutions of (5.1) behave differently for $c = 0$ and for $c \neq 0$.

Obviously, if f satisfies (1.3) then f also satisfies (5.1), with an arbitrary c. The converse is not true, neither with $c = 0$ (cf. examples mentioned in the Introduction) nor with $c \neq 0$. Indeed, if $c > 0$, then fixing $x_0 \in X$ such that $\|x_0\|^2 = c$, a constant mapping $f(x) = x_0$, $x \in X$ satisfies (5.1) but not (1.3). Another example: let $X = Y = \mathbb{C}$ and let $0 \neq c \in \mathbb{C}$. Define

$$f(z) := \frac{c}{\bar{z}}, \quad z \in \mathbb{C} \setminus \{0\}; \quad f(0) := 0.$$

Then, for $z, w \in \mathbb{C} \setminus \{0\}$

$$\langle f(z)|f(w)\rangle = \frac{|c|^2}{\overline{\langle z|w\rangle}}$$

and, in particular, if $\langle z|w\rangle = c$, then $\langle f(z)|f(w)\rangle = c$. Thus f satisfies (5.1) but not (1.3).

Let us restrict our investigations to the class of linear mappings. As we will see below (Corollary 5.2), a linear solution of (5.1), with $c \neq 0$, satisfies (1.3).

Let us discuss a stability problem. For fixed $0 \neq c \in \mathbb{K}$ and $\varepsilon \geq 0$ we consider the condition

$$\forall x, y \in X : \langle x|y\rangle = c \Rightarrow |\langle f(x)|f(y)\rangle - c| \leq \varepsilon. \tag{5.2}$$

Theorem 5.1. *For a finite-dimensional inner product space X and an arbitrary inner product space Y there exists a continuous mapping $\delta : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\lim_{\varepsilon \to 0^+} \delta(\varepsilon) = 0$ and such that for each linear mapping $f : X \to Y$ satisfying* (5.2) *there exists a linear isometry $I : X \to Y$ such that*

$$\|f - I\| \leq \delta(\varepsilon).$$

Proof. Let $0 \neq d \in \mathbb{K}$. If $\langle x|y\rangle = d$, then $\left\langle \frac{c}{d}x|y \right\rangle = c$ and hence, using (5.2) and homogeneity of f, $\frac{|c|}{|d|}|\langle f(x)|f(y)\rangle - d| \leq \varepsilon$. Therefore we have (for $d \neq 0$)

$$\langle x|y\rangle = d \Rightarrow |\langle f(x)|f(y)\rangle - d| \leq \frac{|d|}{|c|}\varepsilon. \tag{5.3}$$

Now, let $d = 0$. Let $0 \neq d_n \in \mathbb{K}$ and $\lim_{n\to\infty} d_n = 0$. Suppose that $\langle x|y\rangle = d = 0$ and $y \neq 0$. Then $\left\langle x + \frac{d_n y}{\|y\|^2}|y \right\rangle = d_n$ and thus, from (5.3) and linearity of f,

$$\left|\left\langle f(x) + \frac{d_n}{\|y\|^2} f(y)|f(y) \right\rangle - d_n\right| \leq \frac{|d_n|}{|c|}\varepsilon.$$

Letting $n \to \infty$ we obtain $\langle f(x)|f(y)\rangle = 0$. For $y = 0$ the latter equality is obvious.

Summing up, we obtain that

$$|\langle f(x)|f(y)\rangle - \langle x|y\rangle| \leq \frac{|\langle x|y\rangle|}{|c|}\varepsilon$$

whence also

$$|\langle f(x)|f(y)\rangle - \langle x|y\rangle| \le \frac{\varepsilon}{|c|}\|x\|\|y\|. \tag{5.4}$$

Let $\delta' : \mathbb{R}_+ \to \mathbb{R}_+$ be a mapping from the assertion of Theorem 1.3. Define $\delta(\varepsilon) := \delta'(\varepsilon/|c|)$ and notice that $\lim_{\varepsilon\to 0^+} \delta(\varepsilon) = 0$. Then it follows from (5.4) that there exists a linear isometry $I : X \to Y$ such that

$$\|f - I\| \le \delta'\left(\frac{\varepsilon}{|c|}\right) = \delta(\varepsilon).$$

□

For $\varepsilon = 0$ we obtain from the above result (we can omit the assumption concerning the dimension of X in this case, considering a subspace spanned on given vectors $x, y \in X$):

Corollary 5.2. *Let $f : X \to Y$ be linear and satisfy* (5.1)*, with some $0 \ne c \in \mathbb{K}$. Then f satisfies* (1.3)*.*

Notice that for $c = 0$, the condition (5.2) has the form

$$\langle x|y\rangle = 0 \Rightarrow |\langle f(x)|f(y)\rangle| \le \varepsilon.$$

If $\langle x|y\rangle = 0$, then also $\langle nx|y\rangle = 0$ for all $n \in \mathbb{N}$. Thus $|\langle f(nx)|f(y)\rangle| \le \varepsilon$, whence $|\langle f(x)|f(y)\rangle| \le \frac{\varepsilon}{n}$ for $n \in \mathbb{N}$. Letting $n \to \infty$ one gets

$$\forall x, y \in X : \langle x|y\rangle = 0 \Rightarrow \langle f(x)|f(y)\rangle = 0,$$

i.e, f is a linear, orthogonality preserving mapping.

Now, let us replace the condition (5.2) by

$$\forall x, y \in X : \langle x|y\rangle = c \Rightarrow |\langle f(x)|f(y)\rangle - c| \le \varepsilon\|f(x)\|\|f(y)\|. \tag{5.5}$$

For $c = 0$, (5.5) states that f satisfies (ε-OP).

Let us consider the class of linear mappings satisfying (5.5) with $c \ne 0$.

We proceed similarly as in the proof of Theorem 5.1. Let $0 \ne d \in \mathbb{K}$. If $\langle x|y\rangle = d$, then $\left\langle \frac{c}{d}x|y\right\rangle = c$ and hence, using (5.5) and homogeneity of f, we obtain

$$\frac{|c|}{|d|}|\langle f(x)|f(y)\rangle - d| \le \varepsilon\frac{|c|}{|d|}\|f(x)\|\|f(y)\|.$$

Therefore we have (for $d \ne 0$)

$$\langle x|y\rangle = d \Rightarrow |\langle f(x)|f(y)\rangle - d| \le \varepsilon\|f(x)\|\|f(y)\|. \tag{5.6}$$

Now, suppose that $\langle x|y\rangle = d = 0$. Let $0 \ne d_n \in \mathbb{K}$ and $\lim_{n\to\infty} d_n = 0$. Then $\left\langle x + \frac{d_n y}{\|y\|^2}|y\right\rangle = d_n$ and thus, from (5.6) and linearity of f,

$$\left|\left\langle f(x) + \frac{d_n}{\|y\|^2}f(y)|f(y)\right\rangle - d_n\right| \le \varepsilon\left\|f(x) + \frac{d_n}{\|y\|^2}f(y)\right\|\|f(y)\|.$$

Letting $n \to \infty$ we obtain $|\langle f(x)|f(y)\rangle| \le \varepsilon\|f(x)\|\|f(y)\|$. So we have

$$\langle x|y\rangle = 0 \Rightarrow |\langle f(x)|f(y)\rangle| \le \varepsilon\|f(x)\|\|f(y)\|.$$

Summing up, we obtain that

$$|\langle f(x)|f(y)\rangle - \langle x|y\rangle| \le \varepsilon\|f(x)\|\|f(y)\|, \qquad x, y \in X. \tag{5.7}$$

Putting in the above inequality $x = y$ we get $\|f(x)\| \le \frac{\|x\|}{\sqrt{1-\varepsilon}}$ for $x \in X$, which gives

$$|\langle f(x)|f(y)\rangle - \langle x|y\rangle| \le \frac{\varepsilon}{1-\varepsilon}\|x\|\|y\|, \qquad x, y \in X,$$

i.e., f satisfies (1.4) with the constant $\frac{\varepsilon}{1-\varepsilon}$.

Therefore, applying Theorem 1.3, we get

Theorem 5.3. *If X is a finite-dimensional inner product space and Y an arbitrary inner product space, then there exists a continuous mapping $\delta : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{\varepsilon\to 0^+} \delta(\varepsilon) = 0$ such that for a linear mapping $f : X \to Y$ satisfying* (5.5)*, with $c \neq 0$, there exists a linear isometry $I : X \to Y$ such that*

$$\|f - I\| \leq \delta(\varepsilon).$$

A converse theorem is also true, even with no restrictions concerning the dimension of X. Let $I : X \to Y$ be a linear isometry and $f : X \to Y$ a mapping, not necessarily linear, such that

$$\|f(x) - I(x)\| \leq \delta\|x\|, \qquad x \in X$$

with $\delta = \sqrt{\frac{1+2\varepsilon}{1+\varepsilon}} - 1$ (for a given $\varepsilon \geq 0$). Reasoning similarly as in the proof of Proposition 4.3 one can show that

$$|\langle f(x)|f(y)\rangle - \langle x|y\rangle| \leq \delta(\delta+2)\|x\|\|y\|,$$

which implies

$$\|x\| \leq \frac{1}{\sqrt{1-\delta(\delta+2)}}\|f(x)\|$$

and finally

$$\begin{aligned} |\langle f(x)|f(y)\rangle - \langle x|y\rangle| &\leq \frac{\delta(\delta+2)}{1-\delta(\delta+2)}\|f(x)\|\|f(y)\| \\ &= \varepsilon\|f(x)\|\|f(y)\|. \end{aligned}$$

Thus f satisfies (5.5) with an arbitrary c.

Remark 5.4. From Theorems 5.1 and 5.3 one can derive immediately the stability results formulated as in Corollaries 4.2 and 4.6.

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