



# SCHRÖDINGER EQUATION FOR A PARTICLE ON A CURVED SPACE AND SUPERINTEGRABILITY

PAUL BRACKEN

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**Abstract.** A formulation of quantum mechanics on spaces of constant curvature is studied by quantizing the Noether momenta and using these to form the quantum Hamiltonian. This approach gives the opportunity of studying a superintegrable quantum system. It is shown there are three different ways of obtaining a Hilbert space of common eigenstates. Three different orthogonal coordinate systems are determined, one for each case. It is shown how the Schrödinger equation can be rendered separable in each of the cases.

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## 1. Introduction

The study of a quantum free particle in Euclidean space leads to the straightforward conclusion that the solutions are plane-wave states that are in fact eigenfunctions of the linear momentum operator. Plane waves are therefore simultaneous eigenfunctions of energy and linear momentum. As soon as the problem is thought of in a space with curvature, the analysis becomes much more complicated [11, 14, 15]. First of all, the canonical momenta do not in general coincide with the Noether momenta. Secondly, the Noether momenta do not Poisson commute classically, so the corresponding self-adjoint quantum operators do not commute. A plane-wave is more of a Euclidean concept, and its meaning needs to be clarified in a curved space. The approach taken here is mainly suited for discussing questions which arise in applications of nonrelativistic quantum mechanics. For example, a two-dimensional application of quantum mechanics arises in condensed matter physics. This is the existence of Landau levels for the motion of a charged particle under perpendicular magnetic fields. This problem and its application to the quantum Hall effect has been studied before in the case of non-Euclidean geometries.