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Harmonic Sum and Duality

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We consider an operation on subsets of a topological vector space which is closely related to what has been called the inverse addition by R.T. Rockafellar. Applied to closed convex sets, it appears as the operation corresponding to the addition under polarity. However, our study is not limited to the convex case. Crucial tools for it are the gauges one can associate with a subset. We stress the role played by asymptotic cones in such a context. We present an application to the calculus of conjugate functions for one of the most fruitful dualities for quasiconvex problems. We also present an extension of the well-known rule for the computation of the normal cone to a convex set defined by a convex inequality.

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1. Introduction

The use of gauges for the study of convex sets is well established after the work of Minkowski during the first decade of this century (see classical monographs on functional analysis and convex analysis) and the recent works of Barbara-Crouzeix ([4]), Rubinov and his co-authors ([21]-[24]). It is the purpose of the present paper to show the usefulness of such a tool for the study of an operation on the family of subsets of a topological vector space which is akin to the operation called the inverse sum (see [18], [19]). Here we modify the formula defining the inverse sum in order to get a closed subset when the given sets are closed. To reach this aim one needs to take into account the asymptotic cones or some substitutes. This aim has been achieved by Ioffe-Tikhomirov [6], Rockafellar [18] and Rubinov-Yagubov [24] when the sets are closed, convex and contain 0 (or are star-shaped). Here we consider more general cases by using explicitly asymptotic cones. Moreover, we also modify the classical definitions of the gauges associated with a subset in order to obtain semicontinuity properties. In the particular case of closed shady (or star-shaped at infinity [8], [9], [22]) convex subsets not containing 0 (and even for a larger class of subsets) our proposals coincide with the functions obtained via regularization in [4]. The simultaneous uses of these modifications enable one to get pleasant relationships

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between the set-theoretical operation of inverse sum (or rather its variant considered here, we call the harmonic sum) and sums of gauges. These relationships are especially useful when considering duality questions.

The terminology we use is justified by the following simple case of two closed intervals $A := [\,\underline{a}, \overline{a}\,]$, $B := [\,\underline{b}, \overline{b}\,]$ with $0 \leq \underline{a} \leq \overline{a} < \infty$ and $0 \leq \underline{b} \leq \overline{b} < \infty$; then the inverse sum $A \# B$ and the harmonic sum $A \diamond B$ are given by

$$A \# B = A \diamond B = \left[(\underline{a}^{-1} + \underline{b}^{-1})^{-1}, (\overline{a}^{-1} + \overline{b}^{-1})^{-1}\right].$$

For $A := [\,\underline{a}, \infty[$, $B := [\,\underline{b}, \overline{b}\,]$ one has

$$A \# B = \left[(\underline{a}^{-1} + \underline{b}^{-1})^{-1}, \overline{b}\right[, \quad A \diamond B = \left[(\underline{a}^{-1} + \underline{b}^{-1})^{-1}, \overline{b}\right].$$

This simple example shows one has to modify the definition of the inverse sum in order to get a closed set even when A and B are closed convex subsets.

As an application, we show that the harmonic sum enters in the computation of the conjugate of a quasiconvex function for one of the most important duality schemes of quasiconvex analysis (see [5], [10], [12], [14], [15], [31]...). In general, explicit formulae for such conjugates are not easy to obtain. Here our computation strongly relies on a convex duality result. A by-product of this duality result is a refined form of the representation theorem for the normal cone to a convex set defined by a convex inequality under a Slater condition.

Let us note that the formulae we obtain for this quasiconvex conjugate show that it is useful to consider gauges of sets which are neither shady nor star-shaped. Another such situation appears in the calculus of the classical subdifferentials specific to quasiconvex functions. This question (which motivated the present study) is considered in the paper [16].

Acknowledgements. We are indebted to an anonymous referee for the suggestion of putting apart the contents of the present paper from [16] and for several references related to the subject. We are also indebted to another referee for pointing out the links of the present paper with a previous study by A. Seeger ([25]). There the harmonic sum is included in a general spectrum of operations on convex sets containing the origin; that paper also contains such operations on real-valued functions and applications are given to second order subdifferentials of convex functions and to network connections.

2. Gauges

Henceforth we consider X to be a topological vector space. In the sequel, we say that a subset A of X is star-shaped with respect to 0, or in short is *star-shaped*, if $0 \in A$ and $]0,1] \cdot A := \{ta \mid 0 < t \leq 1, a \in A\} \subset A$. We say that A is *shady* if $[1,\infty[A \subset A$ (see [8], [9], [22], [23]). For a nonempty subset A of X we use the convention that $0 \cdot A = 0A = 0^+A := \limsup_{t \to 0_+} tA$, the *asymptotic cone* of A, i.e. the set of limits of nets of the form $(t_i a_i)_{i \in I}$ with $(t_i)_{i \in I} \to 0$, $t_i \in \mathbb{P} :=]0, \infty[$, $a_i \in A$ for each $i \in I$. We observe that this convention ensures that $[0, \gamma] \cdot A$ is closed for every $\gamma \in \mathbb{R}_+ := [0, \infty[$ when A is closed. If $A = \emptyset$ we consider that $0^+A = \{0\}$, as in [20]. In the sequel, for the empty set in $\mathbb{R}$, we use the familiar conventions $\inf \emptyset = +\infty$ and $\sup \emptyset = -\infty$.

Note that, when $0 \in A$, Ioffe-Tikhomirov [6] and Rubinov [21, p. 296] took $0 \cdot A$ as being $\cap_{\lambda>0} \lambda A$; if A is a closed and star-shaped set, one obtains again the asymptotic cone of A.

Recall that the Minkowski gauge functional of a subset A of X is given by

$$\mu_A : X \to \overline{\mathbb{R}}, \quad \mu_A(x) := \begin{cases} \inf\{t > 0 \mid x \in tA\} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

where $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$. Here we do not suppose $0 \in A$ as it is done usually; we will see in Proposition 5.2 that there are natural situations in which $0 \notin A$.

Another functional associated to A, considered and studied by Barbara-Crouzeix in [4] under the name of concave gauge function, is given by

$$\nu_A : X \to \overline{\mathbb{R}}, \quad \nu_A(x) := \begin{cases} \sup\{t > 0 \mid x \in tA\} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Note that $\mu_B \leq \mu_A$ and $\nu_A \leq \nu_B$ for $A \subset B$, $\mu_{A\cup B} = \mu_A \wedge \mu_B := \min(\mu_A, \mu_B)$, $\nu_{A\cup B} = \nu_A \vee \nu_B := \max(\nu_A, \nu_B)$. Also note that $\mu_A = \mu_{]0,1]\cdot A} = \mu_{A\cup\{0\}}$ and $\nu_A = \nu_{A\setminus\{0\}} = \nu_{[1,\infty[\cdot A}$.

Example 2.1. Note that it may happen that μ_A and $-\nu_A$ are not lower semicontinuous, even if A is closed and convex. For example, if $A = \{(r,s) \in \mathbb{R}^2 \mid r, s > 0,\ rs \geq 1\}$, then

$$\mu_A(r,s) = \begin{cases} 0 & \text{if } (r,s) \in \mathbb{P}^2 \cup \{(0,0)\}, \\ \infty & \text{otherwise}, \end{cases} \qquad \nu_A(r,s) = \begin{cases} \sqrt{rs} & \text{if } (r,s) \in \mathbb{P}^2 \cup \{(0,0)\}, \\ -\infty & \text{otherwise}. \end{cases}$$

The preceding example justifies the introduction of the following functionals (in the definition of which we use the above convention about $0A$):

$$\begin{aligned} \alpha_A : X \to \overline{\mathbb{R}}, \qquad & \alpha_A(x) := \inf\{t \geq 0 \mid x \in tA\}, \\ \beta_A : X \to \overline{\mathbb{R}}, \qquad & \beta_A(x) := \sup\{t \geq 0 \mid x \in tA\}. \end{aligned}$$

Thus, the domains of μ_A and $-\nu_A$ are $\mathbb{P}A \cup \{0\}$, where $\mathbb{P}A =: \operatorname{cone} A$ is the cone generated by A. Let us observe that the functions μ_A and α_A differ only on $0^+A \setminus (0_r^+A \cup \{0\})$, where $0_r^+A := \bigcap_{\gamma>0}]0,\gamma]A$ is the radial asymptotic cone of A and the functions ν_A and β_A are different only on $0^+A \setminus \mathbb{P}A$. Let us also note that in the definition of $\nu_A(x)$ (resp. $\beta_A(x)$) the supremum is attained when it is finite and A is *centrally closed* in the sense that for any $x \in X$, $(t_n) \to 1$, one has $x \in A$ whenever $t_n x \in A$ for each n. A similar observation can be made for $\mu_A(x)$ and $\alpha_A(x)$.

Of course, if $A \subset B$ then $\alpha_B \leq \alpha_A$, $\beta_A \leq \beta_B$. Note that β_A, as well as ν_A, does not take the value $+\infty$ if $0 \notin \operatorname{cl} A$ or if $0 \notin A$ and A is centrally closed.

The following relations, in which $\gamma \in \mathbb{P}$, are obvious:

$$\{x \in X \mid \mu_A(x) < \gamma\} \subset (]0,\gamma] \cdot A) \cup \{0\} \subset \{x \in X \mid \mu_A(x) \leq \gamma\}, \tag{2.1}$$

$$\{x \in X \mid \nu_A(x) > \gamma\} \subset [\gamma, \infty[\cdot A \subset \{x \in X \mid \nu_A(x) \geq \gamma\}. \tag{2.2}$$

The four functions are positively homogeneous and if A is convex, μ_A is sublinear and, when the cone $\mathbb{P}A$ is pointed, $-\nu_A$ is sublinear, too. When A is a closed convex subset, then α_A is convex, hence sublinear, and if moreover $\mathbb{R}_+ A$ is pointed, β_A is concave. Here we use the familiar convention from convex analysis $(+\infty)+(-\infty) = +\infty$. The preceding discussion can be subsumed in the following statement which completes results in [22], [24].

Proposition 2.2. *The set of centrally closed convex subsets of X not containing 0 is in bijection with the set of pairs (μ,ν), where μ and $-\nu$ are extended real-valued sublinear functions on X, such that $\operatorname{dom}\mu = \operatorname{dom}(-\nu)$ and $0 \le \mu \le \nu$ on $\operatorname{dom}\mu$. The inverse of the mapping $A \mapsto (\mu_A, \nu_A)$ is the mapping which assigns to (μ,ν) the set $A := \{x \in X : \mu(x) \le 1 \le \nu(x)\}$.*

When one restricts one's attention to the family $\mathcal{S}t(X)$ of centrally closed star-shaped subsets of X or to the family $\mathcal{S}h(X)$ of centrally closed convex shady subsets of X not containing 0, it suffices to use $A \mapsto \mu_A$ or $A \mapsto \nu_A$, respectively (see [22], [24]).

For star-shaped subsets and shady subsets nice characterizations of the asymptotic cone in terms of the gauges introduced above are available. For A closed convex containing 0, the relation $0^+A = \mu_A^{-1}(0)$, obtained from part (a) (taking assertions (c) and (d) into account) of the following proposition, is well known (see [20] p. 101 for example).

A relationship between μ_A and α_A is given in the following proposition. For an extended real-valued function f on X we denote by $\overline{f}$ the lower semicontinuous (lsc) hull of f and by $\underline{f}$ the upper semicontinuous (usc) hull of f.

Proposition 2.3. *Let A be a subset of X. Then:*

(a) $\alpha_A = \alpha_{]0,1]A} = \alpha_{[0,1]A}$ *and* $0^+A = \{x \in X \mid \alpha_A(x) = 0\}$;

(b) *for each* $\gamma \in [0,\infty[$ *one has*

$$[0,\gamma]\cdot A \subset \{x \in X \mid \alpha_A(x) \le \gamma\} \subset [0,\gamma]\cdot \operatorname{cl} A. \tag{2.3}$$

Thus, if A is closed one has $[0,\gamma]\cdot A = \{x \in X \mid \alpha_A(x) \le \gamma\}$ and α_A is lsc;

(c) $\alpha_{\operatorname{cl} A} \le \alpha_A \le \mu_A$ *and* $\overline{\mu_A} = \overline{\alpha_A} = \alpha_{\operatorname{cl} A}$.

(d) *If $A =]0,1]A \ne \emptyset$ one has $\overline{\mu_A} = \mu_{\operatorname{cl} A}$. Moreover,*

$$\operatorname{cl} A = \operatorname{cl}\{x \in X \mid \mu_A(x) < 1\} = \operatorname{cl}\{x \in X \mid \mu_A(x) \le 1\}. \tag{2.4}$$

In particular, if A is closed and star-shaped, then $\mu_A = \alpha_A$ and $A = \mu_A^{-1}([0,1])$.

Proof. (a) For the first equalities it is sufficient to observe that $0^+A = 0^+(]0,1]A) = 0^+([0,1]A)$. The inclusion $0^+A \subset \alpha_A^{-1}(0)$ is obvious. Conversely, given $x \in \alpha_A^{-1}(0)$, for each $n \in \mathbb{N}^*$ there exist $t_n \in [0,1/n]$ and $a_n \in A$ such that $x = t_n a_n$. If $t_n = 0$ for some n then $x \in 0^+A$. If $t_n > 0$ for every $n \in \mathbb{N}$, then $x \in 0^+A$.

(b) The first inclusion in relation (2.3) is obvious. Let x be such that $t := \alpha_A(x) \le \gamma$. Then there exists a sequence $(t_n) \in [0,\infty[$ converging to t such that $x \in t_n A$ for each $n \in \mathbb{N}$. If $t = 0$, as above, we have that $x \in 0^+A \subset [0,\gamma]\cdot A$. If $t > 0$ we have $(t_n^{-1}x) \to t^{-1}x$, whence $x \in t \operatorname{cl} A \subset [0,\gamma]\cdot \operatorname{cl} A$. Therefore (2.3) holds. Moreover, when A is closed, equalities hold in (2.3) and since $[0,\gamma]\cdot A$ is closed for every $\gamma \in [0,\infty[$ in view of our convention, we obtain that α_A is l.s.c.

(c) The inequalities $\alpha_{\operatorname{cl} A} \le \alpha_A \le \mu_A$ are obvious. It follows that $\alpha_{\operatorname{cl} A} \le \overline{\alpha_A} \le \overline{\mu_A}$. Let $x \in X$ and $\gamma := \alpha_{\operatorname{cl} A}(x) \in \mathbb{R}_+ \cup \{\infty\}$. If $\gamma = \infty$ there is nothing to prove. In the contrary case, by (2.3), there exists $t \in [0,\gamma]$ such that $x \in t \operatorname{cl} A$. If $t = 0$ then $x \in 0^+(\operatorname{cl} A) = 0^+A$. Therefore there exist nets $(t_i) \to 0_+$ and $(a_i) \subset A$ such that $(t_i a_i) \to x$. Since $\mu_A(t_i a_i) \le t_i$ for every i, we get that $\overline{\mu_A}(x) \le \liminf_{i\in I} \mu_A(t_i a_i) \le \liminf_{i\in I} t_i = 0 \le \gamma$. If $t \in]0,\infty[$, since $t^{-1}x \in \operatorname{cl} A$, there exists a net $(a_i) \subset A$ converging to $t^{-1}x$, i.e. $(ta_i) \to x$. As above, we obtain that $\overline{\mu_A}(x) \le \liminf_{i\in I} \mu_A(ta_i) \le t \le \gamma$. Therefore $\alpha_{\operatorname{cl} A} = \overline{\mu_A}$.

(d) Taking (2.1) into account, (2.4) can be reduced to the inclusion $\mu_A^{-1}([0,1]) \subset \operatorname{cl} \mu_A^{-1}([0,1[)$. So, let $x \in \mu_A^{-1}([0,1])$. Then $\mu_A(\frac{n}{n+1}x) < 1$, whence $x \in \operatorname{cl} \mu_A^{-1}([0,1[)$. Finally, in order to prove that $\overline{\mu_A} = \mu_{\operatorname{cl} A}$ when $A =]0,1]A \neq \emptyset$, since $\overline{\mu_A} = \alpha_{\operatorname{cl} A}$, it suffices to show that when B is closed and star-shaped one has $\alpha_B = \mu_B$. These two functions coincide on $X \setminus 0^+B$. Now, for $x \in 0^+B = \bigcap_{t>0} tB$ one has $\alpha_B(x) = 0$ and $\mu_B(x) = 0$ as $x \in tB$ for every $t > 0$. $\square$

In the following statement we use the *radial tangent cone* at 0 to a subset A of X which is the set $T^r(A,0) := \bigcap_{\gamma>0} [\gamma,\infty[A$, and the usual *tangent cone* (or contingent cone) to A at 0, $T(A,0) := \bigcap_{\gamma>0} \operatorname{cl}([\gamma,\infty[A)$. We note that $\beta_A^{-1}(\infty) = T^r(A,0)$ and that when $0 \notin \operatorname{cl} A$ the cones $T^r(A,0)$ and $T(A,0)$ are empty. They also coincide when A is a polyhedral convex set or when there exists a neighborhood V of 0 such that $A \cap V$ is polyhedral.

Proposition 2.4. *Let $A \subset X$. Then:*

(a) $[0,\infty[\cdot A = \{x \in X \mid \beta_A^{-1}(x) \ge 0\}$, $\mathbb{P}A = \{x \in X \mid \beta_A(x) > 0\}$ *and* $0^+A \setminus \mathbb{P}A = \beta_A^{-1}(\{0\})$;

(b) *for each $\gamma \in [0,\infty[$ one has*

$$[\gamma,\infty[\cdot A \subset \{x \in X \mid \beta_A(x) \ge \gamma\} \subset [\gamma,\infty[\cdot \operatorname{cl} A. \tag{2.5}$$

Moreover, if A is closed and $T(A,0) = T^r(A,0)$, in particular if $0 \notin A$, then $\beta_A = \beta_{[1,\infty[\cdot A}$ and β_A is upper semicontinuous;

(c) $\beta_{\operatorname{cl} A} \ge \beta_A \ge \nu_A$, $\underline{\beta_{\operatorname{cl} A}} = \underline{\beta_A} = \underline{\nu_A}$ *and for each $\gamma \in [0,\infty[$ one has*

$$\begin{aligned} \operatorname{cl}([\gamma,\infty[\cdot A) &= \operatorname{cl}(]\gamma,\infty[\cdot A) = \{x \in X \mid \underline{\beta_A}(x) \ge \gamma\}, \quad (2.6)\\ \operatorname{cl}(\mathbb{P}A) \setminus \mathbb{P}(\operatorname{cl}([1,\infty[A)) &= \{x \in X \mid \underline{\beta_A}(x) = 0\}. \quad (2.7) \end{aligned}$$

Moreover, if $T(A,0) = T^r(\operatorname{cl} A,0)$ then $\underline{\nu_A} = \underline{\beta_A} = \beta_{\operatorname{cl} A}$.

Proof. (a) The first two equalities are obvious; they entail the third one.

(b) The first inclusion of relation (2.5) is obvious. For $\gamma = 0$ the second inclusion of relation (2.5) follows from (a). Let $\gamma > 0$ and let $x \in X$ be such that $t := \beta_A(x) \ge \gamma$. Then there exists a sequence (t_n) in $]0,\infty[$ converging to t and such that $x \in t_n A$ for each $n \in \mathbb{N}$. If $t \neq \infty$ we have $(t_n^{-1}x) \to t^{-1}x$, whence $x \in t \operatorname{cl} A \subset [\gamma,\infty[\cdot \operatorname{cl} A$. If $t = \infty$ then $t_n \ge \gamma$ for some $n \in \mathbb{N}$, and so $x \in [\gamma,\infty[\cdot A \subset [\gamma,\infty[\cdot \operatorname{cl} A$.

Suppose that $A = \operatorname{cl} A$ and $T(A,0) = T^r(A,0)$. Given $\gamma \in \mathbb{R}_+$, using the relation

$$\operatorname{cl}([\gamma,\infty[\cdot B) = ([\gamma,\infty[\cdot \operatorname{cl} B) \cup T(B,0),$$

valid for any subset B of X, we get that $[\gamma,\infty[\cdot A$ is closed. Thus, because β_A takes its values in $\{-\infty\} \cup [0,\infty]$, β_A is upper semicontinuous. Moreover, since $A \subset [1,\infty[\cdot A$, $\beta_A \le \beta_{[1,\infty[\cdot A}$. For the converse inequality, consider $x \in X$ and $\gamma \in \mathbb{R}$ such that $\beta_{[1,\infty[\cdot A}(x) > \gamma$.

Then there exists $t \geq \max(0,\gamma)$ such that $x \in t\,([1,\infty[\cdot A)$. If $t > 0$ then $x \in [t,\infty[\cdot A$ and thus $\beta_A(x) \geq t \geq \gamma$. If $t = 0$ then $x \in 0^+\,([1,\infty[\cdot A)$. Therefore there exist $(t_i) \to 0_+$, $(s_i) \subset [1,\infty[$ and $(a_i) \subset A$ such that $(t_i s_i a_i) \to x$. Then $x \in \mathrm{cl}\,([0,\infty[\cdot A) = [0,\infty[A$ as observed above. Thus $\beta_A(x) \geq 0 \geq \gamma$. Therefore $\beta_A = \beta_{[1,\infty[\cdot A}$.

(c) The inequalities $\nu_A \leq \beta_A \leq \beta_{\mathrm{cl}\,A}$ are obvious. Therefore $\underline{\nu_A} \leq \underline{\beta_A} \leq \underline{\beta_{\mathrm{cl}\,A}}$. Let us show that $\underline{\beta_{\mathrm{cl}\,A}} \leq \underline{\nu_A}$. Let $x \in X$ and $\gamma \in \mathbb{R}$ be such that $\gamma < \underline{\beta_{\mathrm{cl}\,A}}(x)$. There exists $t \in [\max(\gamma,0),\infty[$ such that $x \in t\,\mathrm{cl}\,A$. If $t = 0$ (and so $\gamma \leq 0$), $x \in 0^+(\mathrm{cl}\,A) = 0^+A$. Therefore there exist nets $(t_i) \to 0_+$ and $(a_i) \subset A$ such that $(t_i a_i) \to x$. Since $\nu_A(t_i a_i) \geq t_i$ for every i, we get that $\underline{\nu_A}(x) \geq \limsup \nu_A(t_i a_i) \geq \limsup t_i = 0 \geq \gamma$. The case $t \in]0,\infty[$ is similar but simpler. Thus $\underline{\beta_{\mathrm{cl}\,A}}(x) \leq \underline{\nu_A}(x)$ for every $x \in X$. Therefore $\underline{\beta_{\mathrm{cl}\,A}} = \underline{\beta_A} = \underline{\nu_A}$.

Given $\gamma \in \mathbb{R}_+$, the definitions and the inequalities $\underline{\beta_A} \geq \beta_A \geq \nu_A$ yield the inclusions

$$\begin{aligned}]\gamma,\infty[\cdot A \subset \{x \in X \mid \nu_A(x) > \gamma\} \\ \subset [\gamma,\infty[\cdot A \subset \{x \in X \mid \beta_A(x) \geq \gamma\} \subset \{x \in X \mid \underline{\beta_A}(x) \geq \gamma\}.\end{aligned}$$

Consider $x \in X$ such that $t := \underline{\beta_A}(x) \geq \gamma$. By definition, there exists a net $(x_i)_{i\in I} \to x$ such that $(\beta_A(x_i))_{i\in I} \to t$. If there exists a cofinal subset J of I such that $\beta_A(x_j) > \gamma$ for each $j \in J$ then $x_j \in]\gamma,\infty[A$, whence $x \in \mathrm{cl}(]\gamma,\infty[\cdot A)$. Thus, we may suppose that $\gamma/2 \leq \beta_A(x_i) \leq \gamma$ for each $i \in I$, and so, $t = \gamma$. If $\gamma = 0$, then $x_i \in 0^+A$ for each $i \in I$ and so $x \in 0^+A$; therefore, there exist nets $(t_k)_{k\in K} \to 0_+$ and $(a_k)_{k\in K}$ in A such that $(t_k a_k)_{k\in K} \to x$ and so $x \in \mathrm{cl}(]\gamma,\infty[\cdot A)$. If $\gamma > 0$, consider $(\varepsilon_n)_{n\in\mathbb{N}} \to 0$ with $\varepsilon_n \in]0,\gamma/2[$. Then, for every $n \in \mathbb{N}$ and $i \in I$, there exists $t_{i,n} \in]\beta_A(x_i) - \varepsilon_n, \gamma]$ such that $x_i \in t_{i,n}A$. It follows that $(t_{i,n}) \to \gamma$ and $x_{i,n} := t_{i,n}^{-1}(\gamma + \varepsilon_n)x_i \in]\gamma,\infty[\cdot A$. Since $(x_{i,n}) \to x$ one has $x \in \mathrm{cl}(]\gamma,\infty[\cdot A)$. The proof of relation (2.6) is complete. Taking into account the fact that $\mathrm{cl}([\gamma,\infty[\cdot A) = \gamma\,\mathrm{cl}([1,\infty[\cdot A)$ for $\gamma > 0$, relation (2.7) follows from (2.6). Taking into account the fact that $\mathrm{cl}([\gamma,\infty[\cdot A) = \gamma\,\mathrm{cl}([1,\infty[\cdot A)$ for $\gamma > 0$, relation (2.7) follows from (2.6).

When $T^r(\mathrm{cl}\,A, 0) = T(A,0)$, from (b) we obtain that $\beta_{\mathrm{cl}\,A}$ is usc and so $\underline{\nu_A} = \underline{\beta_A} = \beta_{\mathrm{cl}\,A}$. □

Note that β_A and $\beta_{[1,\infty[\cdot A}$ may be different if $T(A,0) \neq T^r(\mathrm{cl}\,A,0)$ (f.i. $A = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 2y\}$) or if A is not closed (f.i. $A = \{(x,y) \in \mathbb{R}^2 \mid x,y > 0,\ x + y = 1\}$).

Also note that β_A is usc at 0 if and only if $0 \in A$ or $0 \notin \mathrm{cl}\,A$.

Moreover, if X is a normed space and if $0 \in \mathrm{cl}(A\backslash\{0\})$ then $\underline{\nu_A}(0) = +\infty$. In fact, if (a_n) is a sequence of $A\backslash\{0\}$ with limit 0, for $r_n := \|a_n\|^{-1/2}$ and $x_n := r_n a_n$ we have $\nu_A(x_n) \geq r_n$, hence $\underline{\nu_A}(0) = +\infty$. We will see that if one assumes that $0 \notin \mathrm{cl}\,A$ the situation is much improved. Some duality observations will be useful for such a purpose and for other aims.

We do not intend to give a complete treatment of the operations with gauges and sets. Let us however consider some important cases, at least for μ and ν.

Proposition 2.5. *Let X, Y, Z be vector spaces, $A \subset X$, $B \subset Y$ and $T : X \to Y$ a linear operator.*

(a) *For every $y \in Y$ one has $\mu_{T(A)}(y) = \inf\{\mu_A(x) \mid Tx = y\}$.*
(b) *If $0 \in B$ then for every $x \in X$ one has $\mu_{T^{-1}(B)}(x) = \mu_B(Tx)$.*
(c) *If C and D are two starshaped subsets of Z then $\mu_{C\cap D} = \mu_C \vee \mu_D$.*

(d) *If A and B are star-shaped then for all $x \in X$, $y \in Y$ one has*

$$\mu_{A\times B}(x, y) = \mu_A(x) \vee \mu_B(y). \tag{2.8}$$

Proof. (a) The conclusion is obvious for $y \notin \operatorname{Im} T$ and for $y = 0$. Let $y \in \operatorname{Im} T \setminus \{0\}$. If $x \in X$ is such that $y = Tx$, then $\{t > 0 \mid x \in tA\} \subset \{t > 0 \mid y \in tT(A)\}$, whence $\mu_{T(A)}(y) \le \mu_A(x)$. Hence $\mu_{T(A)}(y) \le \inf\{\mu_A(x) \mid Tx = y\}$. Let now $\gamma \in \mathbb{R}$ be such that $\mu_{T(A)}(y) < \gamma$. Then $y \in tT(A) = T(tA)$ for some $t \in]0, \gamma[$. It follows that there exists $x \in tA$ with $y = Tx$. Since $\mu_A(x) \le t < \gamma$, we obtain that $\inf\{\mu_A(x) \mid Tx = y\} < \gamma$. The conclusion follows.

(b) The proof is simple; just use the definition and the fact that $\ker T \subset T^{-1}(B)$.

(c) The inequality $\mu_{C\cap D} \ge \mu_C \vee \mu_D$ is obvious, while for the reverse one use that C, D are star-shaped.

(d) Apply (c) for $Z = X\times Y$, $C = A\times Y$, $D = X \times B$ and (b) for the two projections. □

As a consequence of the preceding proposition we have the following result about the gauge of a sum which differs from a formula suggested in [7] p. 17. Here we deal with the *sublevel convolution* (or level convolution or level sum) $f\Diamond g$ of two functions f, g, as used in several recent works (see [3], [17], [26], [29], [30] ...).

Corollary 2.6. *Let $A, B \subset X$ be star-shaped sets. Then for every $x \in X$,*

$$\mu_{A+B}(x) = \inf\{\mu_A(u) \vee \mu_B(v) \mid u, v \in X,\ x = u + v\} =: (\mu_A \Diamond \mu_B)(x).$$

Proof. Consider $T : X \times X \to X$ defined by $T(u, v) := u + v$. Using points (a) and (d) of the preceding proposition, for each $x \in X$ we have that

$$\begin{aligned}\mu_{A+B}(x) &= \mu_{T(A\times B)}(x) = \inf\{\mu_{A\times B}(u, v) \mid u, v \in X,\ u + v = x\}\\ &= \inf\{\mu_A(u) \vee \mu_B(v) \mid u, v \in X,\ x = u + v\}.\end{aligned}$$

□

Although the conditions are different for ν, the proofs are similar, so that we skip them.

Proposition 2.7. *Let X, Y, Z be vector spaces, $A \subset X$, $B \subset Y$ and $T : X \to Y$ a linear operator.*

(a) *For every $y \in Y \setminus \{0\}$ one has $\nu_{T(A)}(y) = \sup\{\nu_A(x) \mid Tx = y\}$. If $A \cap \ker T \subset \{0\}$, the preceding formula holds for $y = 0$, too.*
(b) *For every $x \in (X \setminus \ker T) \cup \{0\}$ one has $\nu_{T^{-1}(B)}(x) = \nu_B(Tx)$.*
(c) *If C and D are two shady subsets of Z then $\nu_{C\cap D} = \nu_C \wedge \nu_D$.*
(d) *If A and B are shady then for all $x \in X \setminus \{0\}$, $y \in Y \setminus \{0\}$ one has*

$$\nu_{A\times B}(x, y) = \nu_A(x) \wedge \nu_B(y). \tag{2.9}$$

Note that, in the framework of the preceding proposition, if $x \in \ker T\setminus\{0\}$ then $\nu_{T^{-1}(B)}(x) = \infty$ when $0 \in B$ and $\nu_{T^{-1}(B)}(x) = -\infty$ when $0 \notin B$, while $\sup\{\nu_A(x) \mid Tx = 0\} \ge 1 > 0 = \nu_{T(A)}(0)$ if $A \cap \ker T \not\subset \{0\}$.

The next result is a counterpart to Corollary 2.6.

Corollary 2.8. *Let $A, B \subset X$ be shady sets. Suppose that $0 \notin A \cup B \cup (A+B)$. Then for every $x \in \mathbb{P}(A+B) \cup (X \setminus (\mathbb{P}A \cup \mathbb{P}B))$ one has*

$$\nu_{A+B}(x) = \sup\{\nu_A(u) \wedge \nu_B(v) \mid u, v \in X,\ x = u + v\}. \tag{2.10}$$

In particular, if $\mathbb{P}A \cup \mathbb{P}B \subset \mathbb{P}(A+B)$, the above formula holds for every $x \in X$.

Proof. Note first that (2.10) holds for $x = 0$ since $0 \notin \mathbb{P}(A+B) = \mathbb{P}A + \mathbb{P}B$. So, we may suppose $x \in X \setminus \{0\}$. Consider $T : X \times X \to X$ defined by $T(u,v) := u + v$. In our hypotheses $(A \times B) \cap \ker T = \emptyset$. So, from Proposition 2.7 (a) and (d) we have that

$$\begin{aligned}
\nu_{A+B}(x) &= \nu_{T(A\times B)}(x) = \sup\{\nu_{A\times B}(u,v) \mid u, v \in X,\ u+v = x\} \\
&= \max\left(\sup\{\nu_A(u) \wedge \nu_B(v) \mid u, v \in X \setminus \{0\},\ u+v = x\}, \nu_{A\times B}(x,0), \nu_{A\times B}(0,x)\right) \\
&= \sup\{\nu_A(u) \wedge \nu_B(v) \mid u, v \in X \setminus \{0\},\ u+v = x\},
\end{aligned} \tag{2.11}$$

since $\nu_{A\times B}(x,0) = \nu_{A\times B}(0,x) = -\infty$, as $0 \notin A \cup B$.

If $x \in \mathbb{P}(A+B) = \mathbb{P}A + \mathbb{P}B$, then $x = u + v$ for some $u \in \mathbb{P}A$, $v \in \mathbb{P}B$, whence $\nu_A(u) \wedge \nu_B(v) > 0 \geq \nu_A(x) \wedge \nu_B(0)$, $\nu_A(0) \wedge \nu_B(x)$. Using relation (2.11) we obtain that (2.10) holds in this case. Suppose now that $x \notin \mathbb{P}A \cup \mathbb{P}B$. Then $\nu_A(x) \wedge \nu_B(0) = \nu_A(0) \wedge \nu_B(x) = -\infty$, and so the conclusion follows again using (2.11). □

Note that for $x \in (\mathbb{P}A \cup \mathbb{P}B) \setminus \mathbb{P}(A+B)$ one has $\nu_{A+B}(x) = -\infty$ and $\sup\{\nu_A(u) \wedge \nu_B(v) \mid u, v \in X,\ x = u+v\} = 0$.

Example 2.9. Consider the following subsets of $\mathbb{R}^2$: $A_1 = \{(x,y) \mid y \geq \max(|x|, 1)\}$, $A_2 = \{(x,y) \mid y \geq \sqrt{x^2+1}\}$ and $B = \{(x,y) \mid x, y > 0,\ xy \geq 1\}$. Then $\mathbb{P}A_1 \not\subset \mathbb{P}(A_1 + B)$ and $\mathbb{P}A_2 \cup \mathbb{P}B \subset \mathbb{P}(A_2 + B)$.

Similar results could be given for the gauges α and β but would involve conditions on the asymptotic cones of images, sums and intersections which are outside the scope of this paper.

3. Harmonic sum and gauges

It is the purpose of the present section to study a variant of the operation of inverse addition of two subsets A, B of a vector space introduced by Rockafellar ([18], [19, p. 21]) as being

$$A\#B := \left(\bigcup_{\lambda \in\,]0,1[} \lambda A \cap (1-\lambda)B\right) \cup (\{0\} \cap B) \cup (A \cap \{0\}).$$

This operation plays an important role in the study of the Plastria's subdifferential of the sublevel-convolution of two functions which will be undertaken in the paper [16]. We give a direct definition; alternatively, for certain cases, the gauges described above may be used in a way similar to what is done in [24] (in the case of continuous gauges).

For two subsets A, B of X we introduce their *harmonic sum*, denoted by $A\Diamond B$, as being $A\#B$, but with the convention made above:

$$A\Diamond B := \left(\bigcup_{\lambda\in\,]0,1[} \lambda A \cap (1-\lambda)B \right) \cup \left(0^+A \cap B\right) \cup \left(A \cap 0^+B\right).$$

This definition has been introduced by A. Seeger for convex sets containing 0. It also appears as a special case of an operation studied in [21] in the general framework of c^2-lattices, at least for closed star-shaped subsets. Note that, if A is a closed cone, then $A\Diamond B = A \cap [0,1]B$. Other examples are described and drawn in [24] or are given below.

Let us note that $A\Diamond B$ is convex when A and B are closed convex subsets. The following result has been obtained by A. Seeger [25], Prop. 4.2 for convex sets containing 0, in which case our assumption $\mathbb{P}A \cap \mathbb{P}B \neq \emptyset$ is automatically satisfied.

Lemma 3.1. *Let A, B be two closed subsets of X.*

(a) *Then the set $A\Diamond B$ is closed and contains $A\#B$.*

(b) *Moreover, if A, B are convex and $\mathbb{P}A \cap \mathbb{P}B \neq \emptyset$, or if A, B are star-shaped, then*

$$A\Diamond B = \mathrm{cl}\left(\bigcup_{\lambda\in\,]0,1[} \lambda A \cap (1-\lambda)B \right) = \mathrm{cl}\,(A\#B). \tag{3.1}$$

Proof. (a) The first assertion follows from the inclusion, valid for any subsets E, F of X,

$$\mathrm{cl}\,(E\#F) \subset \left(\bigcup_{\lambda\in\,]0,1[} \lambda\,\mathrm{cl}\,E \cap (1-\lambda)\,\mathrm{cl}\,F \right) \cup \left(0^+E \cap \mathrm{cl}\,F\right) \cup \left(\mathrm{cl}\,E \cap 0^+F\right),$$

so that the closure of $C := \bigcup_{\lambda\in\,]0,1[} \lambda A \cap (1-\lambda)B$ is contained in $(0^+A \cap B) \cup (A \cap 0^+B) \cup C = A\Diamond B$. As $0 \in 0^+D$ for any subset D of X, the inclusion $A\#B \subset A\Diamond B$ is immediate.

(b) It is sufficient to show that $A\Diamond B \subset \mathrm{cl}\,C$, or even that $0^+A \cap B \subset \mathrm{cl}\,C$.

Suppose first that A and B are convex and $\mathbb{P}A \cap \mathbb{P}B \neq \emptyset$, or equivalently, $C \neq \emptyset$, i.e. there exist $a \in A$, $b \in B$ and $r \in]0,1[$ such that $ra = (1-r)b$. Let $x \in 0^+A \cap B$. Then for each $n > 1$, for $t_n := nr(nr - r + 1)^{-1} \in]0,1[$, we have $a_n := a + nx \in A$, $b_n := (1-t_n)b + t_n x \in B$ and $\lambda_n a_n = (1-\lambda_n)b_n$ for $\lambda_n := r(nr+1)^{-1} \in]0,1[$. Since $(\lambda_n a_n) \to x$ as $n \to \infty$, we get $x \in \mathrm{cl}\,C$.

Suppose now that A, B are star-shaped. Again, let $x \in 0^+A \cap B$. Since $0^+A = \bigcap_{t>0} tA$, it follows that $x_n := (1-\frac{1}{n})x = \frac{1}{n}[(n-1)x] \in \frac{1}{n}A \cap (1-\frac{1}{n})B \subset C$ for every $n > 1$. Since $(x_n) \to x$, $x \in \mathrm{cl}\,C$. $\square$

Example 3.2. Observe that the relations (3.1) are not valid for non closed sets (even if they are convex and star-shaped); consider the subsets of $\mathbb{R}^2$, $A = [0,1] \times [0,1]$ and $B =]-\infty, 0[\times]0, \infty[\cup \{(0,0)\}$. Then $A\#B = \{(0,0)\}$ and $A\Diamond B = \{0\} \times [0,1]$.

They may fail if A and B are closed non star-shaped subsets; take for example the subsets of $\mathbb{R}^2$, $A = \{(x, x^2) \mid x \in \mathbb{R}\}$, $B = \{0\} \times [0, \infty[$. They may also fail if A and B are closed and convex but $\mathbb{P}A \cap \mathbb{P}B$ is empty. Take for example $A = \{(x, y) \mid x > 0, y > 0, xy \geq 1\}$, $B = [0, \infty[\times \{0\}$; then $C = \emptyset$, $A\#B = \{0\}$ and $A\Diamond B = B$.

Lemma 3.3.

(a) *For any subsets A, B of X one has $0^+(A\Diamond B) \subset 0^+A \cap 0^+B$.*

(b) *If A and B are closed convex subsets and $A\Diamond B \neq \emptyset$, or if A and B are star-shaped and one of them is closed, then*

$$0^+A \cap 0^+B = 0^+(A\Diamond B). \tag{3.2}$$

Proof. (a) Let $x \in 0^+(A\Diamond B)$; there exist nets $(t_i)_{i\in I} \to 0_+$, $(w_i)_{i\in I}$ in $A\Diamond B$ and $(\lambda_i)_{i\in I}$ in $[0, 1]$ such that $(t_i w_i)_{i\in I} \to x$ and $w_i \in \lambda_i A \cap (1 - \lambda_i)B$. If $\lambda_j = 0$ for j in a cofinal subset of I, then $t_j w_j \in 0^+A \cap t_j B$ so that $x \in 0^+A \cap 0^+B$. The case $\lambda_j = 1$ for j in a cofinal subset of I is similar. So, we may suppose that $\lambda_i \in]0, 1[$ for every i. Then $w_i = \lambda_i a_i = (1 - \lambda_i)b_i$ for some $a_i \in A$, $b_i \in B$ for each $i \in I$; it follows that $x = \lim_i t_i\lambda_i a_i = \lim_i t_i(1 - \lambda_i)b_i \in 0^+A \cap 0^+B$.

(b) Suppose A and B are closed convex subsets and $A\Diamond B \neq \emptyset$. Let $z_0 \in A\Diamond B$ and $z \in 0^+A \cap 0^+B$. There exists $\lambda \in [0, 1]$ such that $z_0 \in \lambda A \cap (1 - \lambda)B$. Consider first the case $\lambda \in]0, 1[$. Then $z_0 = \lambda x_0 = (1 - \lambda)y_0$ for some $x_0 \in A$ and $y_0 \in B$. It follows that $z_0 + \gamma z = \lambda(x_0 + \lambda^{-1}\gamma z) = (1 - \lambda)[y_0 + (1 - \lambda)^{-1}\gamma z]$, with $x_0 + \lambda^{-1}\gamma z \in A$ and $y_0 + (1 - \lambda)^{-1}\gamma z \in B$ for every $\gamma \geq 0$, whence $z_0 + \gamma z \in A\Diamond B$ for every $\gamma \geq 0$. Therefore $z \in 0^+(A\Diamond B)$. Suppose now that $\lambda = 0$; then $z_0 \in 0^+A \cap B$. It follows that $z_0 + \gamma z \in 0^+A \cap B \subset A\Diamond B$ for every $\gamma \geq 0$, whence, once again, $z \in 0^+(A\Diamond B)$.

Suppose now that A and B are star-shaped sets, A being closed. In this case $0^+A = \bigcap_{t>0} tA$ and $0^+B = 0^+(\operatorname{cl} B) = \bigcap_{t>0} t \operatorname{cl} B$. So, if $x \in 0^+A \cap 0^+B$ then $tx \in A \cap 0^+B \subset A\Diamond B$ for every $t > 0$, whence $x \in 0^+(A\Diamond B)$. □

Example 3.4. Counter-examples for (3.2):

The sets A, B are convex and closed but $A\Diamond B = \emptyset : A = \{(x, y) \in \mathbb{R}^2 \mid x, y > 0,\ xy \geq 1\}$, $B = \{(x, y) \in \mathbb{R}^2 \mid x < 0,\ y > 0,\ xy \leq -1\}$; here $0^+A \cap 0^+B = \{0\} \times \mathbb{R}_+$.

The sets A, B are closed but are not star-shaped and $A\Diamond B \neq \emptyset$: $A = \{(x, x^2) \in \mathbb{R}^2 \mid x \geq 0\}$, $B = \{(x, x^2) \in \mathbb{R}^2 \mid x \leq 0\}$; here $A\Diamond B = \{(0, 0)\}$ and $0^+A = 0^+B = \{0\} \times \mathbb{R}_+$.

The sets A, B are star-shaped (even convex cones), but none of them is closed: $A = \{(x, y) \in \mathbb{R}^2 \mid x, y > 0\} \cup \{(0, 0)\}$, $B = \{(x, y) \in \mathbb{R}^2 \mid x < 0,\ y > 0\} \cup \{(0, 0)\}$.

Let us note that for $A, B \subset X$,

$$(]0, 1] \cdot A) \# (]0, 1] \cdot B) =]0, 1] \cdot (A\#B), \quad ([1, \infty[\cdot A) \# ([1, \infty[\cdot B) = [1, \infty[\cdot (A\#B).$$

In particular, if A, B are star-shaped (shady) then $A\#B$ is star-shaped (shady). However, similar relations for the harmonic sum do not hold without additional assumptions. More precisely,

$$([0, 1] \cdot A) \Diamond ([0, 1] \cdot B) = [0, 1] \cdot (A\Diamond B) \;\Leftrightarrow\; 0^+(A\Diamond B) = 0^+A \cap 0^+B.$$

Since for a shady set B we have that $0^+B = \overline{\operatorname{cone}} B := \operatorname{cl}(\operatorname{cone} B)$, we have that $0^+ ([1, \infty[\cdot A) = \overline{\operatorname{cone}} A$. Using this relation we obtain easily that

$$[1, \infty[\cdot (A\Diamond B) \subset ([1, \infty[\cdot A) \Diamond ([1, \infty[\cdot B),$$

the reverse inclusion being valid if $0^+A = \overline{\text{cone}}A$ and $0^+B = \overline{\text{cone}}B$ or if A, B are closed and $0 \notin A \cup B$.

Proposition 3.5. *Let $A, B \subset X$. Then $\mu_A + \mu_B = \mu_{A\#B}$ and $\alpha_A + \alpha_B \leq \alpha_{A\diamond B}$. Moreover,*

$$\alpha_A + \alpha_B = \alpha_{A\diamond B} \;\Leftrightarrow\; 0^+(A\diamond B) = 0^+A \cap 0^+B. \tag{3.3}$$

Proof. We give the proof for α only; the proof for μ is similar and even simpler.

Let $x \in X$ and $\gamma \in]0, \infty[$ be such that $\alpha_{A\diamond B}(x) < \gamma$. Then there exists $t \in [0, \gamma]$ such that $x \in t \cdot (A\diamond B)$. If $t = 0$ then $x \in 0^+(A\diamond B)$. Using Lemma 3.3 we see that $\alpha_A(x) + \alpha_B(x) = 0 \leq \gamma$. If $t > 0$ then $t^{-1}x \in A\diamond B$; therefore $t^{-1}x \in \lambda A \cap (1-\lambda)B$ with $\lambda \in [0, 1]$. So $x \in \lambda t A \cap (1-\lambda)tB$, whence $\alpha_A(x) + \alpha_B(x) \leq \lambda t + (1-\lambda)t = t \leq \gamma$. It follows that $\alpha_A(x) + \alpha_B(x) \leq \alpha_{A\diamond B}(x)$.

The implication "$\Rightarrow$" of (3.3) follows easily from Proposition 2.3 (a). So suppose that $0^+(A\diamond B) = 0^+A \cap 0^+B$ and take $x \in X$ and $\gamma \in]0, \infty[$ such that $\alpha_A(x) + \alpha_B(x) < \gamma$. There exist $\gamma_1, \gamma_2 \in \mathbb{R}$ such that $\gamma = \gamma_1 + \gamma_2$ and $\alpha_A(x) < \gamma_1$, $\alpha_B(x) < \gamma_2$. So there exist $t_1 \in [0, \gamma_1]$, $t_2 \in [0, \gamma_2]$ such that $x \in t_1 A \cap t_2 B$. If $t_1 + t_2 = 0$ then $x \in 0^+A \cap 0^+B$, whence, by hypothesis, $x \in 0^+(A\diamond B)$; thus $\alpha_{A\diamond B}(x) = 0 \leq \gamma$. If $t_1 + t_2 > 0$, then $x \in (t_1 + t_2)\left(\frac{t_1}{t_1+t_2}A \cap \frac{t_2}{t_1+t_2}B\right) \subset (t_1 + t_2)(A\diamond B)$. Therefore $\alpha_{A\diamond B}(x) \leq t_1 + t_2 \leq \gamma$. It follows that $\alpha_{A\diamond B}(x) \leq \alpha_A(x) + \alpha_B(x)$. As the converse inequality is true always, we obtain that $\alpha_A + \alpha_B = \alpha_{A\diamond B}$. $\square$

Examples illustrating the way the harmonic sum and gauges operate are given in [24] in the case the gauge μ is continuous; next, we give an example in which this assumption is not satisfied.

Example 3.6. Let $A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ and $B = \{(x, y) \in \mathbb{R}^2 \mid y \geq x^2\}$. Then

$$A\diamond B = \left\{(x, y) \in \mathbb{R}^2 \mid y \geq 0,\; y\sqrt{x^2 + y^2} + x^2 \leq y\right\}.$$

Indeed, $\mu_A(x, y) = \alpha_A(x, y) = \sqrt{x^2 + y^2}$ and

$$\mu_B(x, y) = \alpha_B(x, y) = \begin{cases} \frac{x^2}{y} & \text{if } y > 0, \\ 0 & \text{if } x = y = 0, \\ +\infty & \text{if } y < 0 \text{ or } y = 0,\; x \neq 0. \end{cases}$$

Since $0^+A \cap 0^+B \subset 0^+A = \{0\}$, it follows from Propositions 2.3 and 3.5 that

$$\begin{aligned} A\diamond B &= \operatorname{cl}(A\#B) = \{(x, y) \in \mathbb{R}^2 \mid \alpha_A(x, y) + \alpha_B(x, y) \leq 1\} \\ &= \left\{(x, y) \in \mathbb{R}^2 \mid y \geq 0,\; y\sqrt{x^2 + y^2} + x^2 \leq y\right\}. \end{aligned}$$

We observe that, since 0 is in the interior $\operatorname{int} A$ of A, $A\#B \subset \operatorname{int} A$, but the point $(0, 1)$ of $A\diamond B$ belongs to the boundary of A.

A similar result holds for ν and β.

Proposition 3.7. *Let $A, B \subset X$. Then, with the convention that $(-\infty)+(+\infty)=-\infty$, one has $\nu_A+\nu_B=\nu_{A\#B}$ and $\beta_{A\diamond B}\le\beta_A+\beta_B$. Moreover,*

$$\beta_A+\beta_B=\beta_{A\diamond B} \;\Leftrightarrow\; 0^+A\cap 0^+B\subset [0,\infty[\cdot(A\Diamond B). \tag{3.4}$$

In particular $\beta_A+\beta_B=\beta_{A\diamond B}$ whenever $0^+A\cap 0^+B=0^+(A\Diamond B)$.

If A and B are closed, shady and do not contain 0, then $0^+(A\Diamond B)=[0,\infty[\cdot(A\Diamond B)$, and

$$\beta_A+\beta_B=\beta_{A\diamond B} \;\Leftrightarrow\; 0^+(A\Diamond B)=0^+A\cap 0^+B.$$

Proof. Again, the proof for ν is similar to the proof for β. Let $x\in X$ and $\gamma\in\mathbb{R}$ be such that $\beta_{A\diamond B}(x)>\gamma$. Then there exists $t\ge\max(0,\gamma)$ such that $x\in t\cdot(A\Diamond B)$. If $t=0$ then $x\in 0^+(A\Diamond B)$. Since $0^+(A\Diamond B)\subset 0^+A\cap 0^+B$, we obtain that $\beta_A(x),\ \beta_B(x)\ge 0$, and so $\beta_A(x)+\beta_B(x)\ge 0\ge\gamma$. Consider now the case $t>0$. Then $t^{-1}x\in A\Diamond B$; therefore $t^{-1}x\in\lambda A\cap(1-\lambda)B$ with $\lambda\in[0,1]$. So $x\in\lambda tA\cap(1-\lambda)tB$, whence $\beta_A(x)+\beta_B(x)\ge\lambda t+(1-\lambda)t=t\ge\gamma$. It follows that $\beta_{A\diamond B}(x)\le\beta_A(x)+\beta_B(x)$.

Suppose that $\beta_A+\beta_B=\beta_{A\diamond B}$ and take $x\in 0^+A\cap 0^+B$. Then $\beta_{A\diamond B}(x)=\beta_A(x)+\beta_B(x)\ge 0$, whence, by Proposition 2.4, $x\in[0,\infty[\cdot(A\Diamond B)$.

The proof of the implication "$\Leftarrow$" of (3.4) is similar to the corresponding one in (3.3), and so is omitted.

Assume now that A and B are shady; then $A\Diamond B$ is also shady. As for a closed shady set C not containing 0 one has $0^+C=\mathrm{cl}(\mathbb{P}C)=\mathbb{R}_+C$, the conclusion follows. $\square$

4. Harmonic sum and polarities

In the sequel we assume that the space X is an Hausdorff locally convex space and X^* is its topological dual. Throughout, for a function f, ∂f and f^* denote the subdifferential and the conjugate of f in the sense of convex analysis.

Let us consider the *polar* and *anti-polar* of $A\subset X$ defined by

$$\begin{aligned} A^\circ &:= \{x^*\in X^* \mid \langle x,x^*\rangle\le 1\ \forall x\in A\},\\ A^\nabla &:= \{x^*\in X^* \mid \langle x,x^*\rangle\ge 1\ \forall x\in A\},\end{aligned}$$

with $A^\circ=A^\nabla=X^*$ if $A=\emptyset$. Here we use the notation of [1] and [2]; for other uses of the anti-polar set, see [4], [9], [11], [27], [28]. We will use the following obvious relations:

$$A^\circ=(\overline{\mathrm{co}}(A\cup\{0\})^\circ=(\overline{\mathrm{co}}(]0,1]A))^\circ,\quad A^\nabla=(\overline{\mathrm{co}}([1,\infty[A))^\nabla.$$

Let us note that since A° is closed, convex and contains 0, one has

$$0^+A^\circ=A^-:=\{x^*\in X^*\mid\langle x,x^*\rangle\le 0\ \forall x\in A\},$$

while if $A^\nabla\ne\emptyset$

$$0^+A^\nabla=A^+:=\{x^*\in X^*\mid\langle x,x^*\rangle\ge 0\ \forall x\in A\}.$$

Taking the last relation into consideration, in this section we consider that $0\cdot A^\nabla=A^+$, even if $A^\nabla=\emptyset$. Note that if A is non-empty and convex, then by an easy separation argument one has $A^\nabla=\emptyset$ if and only if $0\in\mathrm{cl}\,A$ (see also [4]).

In the sequel we denote by ι_C the *indicator function* of a subset C of X given by $\iota_C(x) = 0$ if $x \in C$, $+\infty$ else. In the following lemma we use the well-known fact that the conjugate h^* of a positively homogeneous function $h : X \to \mathbb{R} \cup \{+\infty\}$ is the indicator function of $H := \{x^* \in X^* \mid x^* \leq h\}$ and, when $h(0) = 0$, $H = \partial h(0)$, the subdifferential of h at 0. The relation $(\mu_A)^* = \iota_{A^\circ}$ of the next lemma is given in [6] Proposition 1.6 and in [19]. It implies that the support function of A° is $(\mu_A)^{**}$; when A is convex and contains 0, this last function is $\overline{\mu_A}$ and one gets connections with section 6 of [25] which is devoted to support functions of direct and inverse additions of convex sets.

Lemma 4.1. *Let $A \subset X$ be a non-empty set. Then $(\mu_A)^* = (\alpha_A)^* = \iota_{A^\circ}$, $(-\nu_A)^* = (-\beta_A)^* = \iota_{-A^\nabla}$ and*

$$A^\circ = \partial\mu_A(0) = \partial\alpha_A(0), \tag{4.1}$$

$$A^\nabla = -\partial(-\nu_A)(0) = \{x^* \in X^* \mid x^* \geq \beta_A\} = \{x^* \in X^* \mid x^* \geq \underline{\beta_A}\}. \tag{4.2}$$

In particular, if A is convex, then $\underline{\beta_A}(0) = 0$ if and only if $0 \notin \operatorname{cl} A$; in such a case one has $\partial(-\underline{\beta_A})(0) = -A^\nabla$.

Proof. We prove the assertions dealing with β_A and ν_A, the proof of those concerning α_A and μ_A being similar. Moreover, the equalities $(-\nu_A)^* = (-\beta_A)^* = \iota_{-A^\nabla}$ are consequences of (4.2) and of the observation preceding the statement. Given $x^* \in A^\nabla$ let us prove that $\langle x, x^*\rangle \geq \nu_A(x)$ for every x. We may suppose $x \neq 0$. Let $\lambda < \nu_A(x)$; there exists $t > \max(\lambda, 0)$ such that $t^{-1}x \in A$, whence $\langle t^{-1}x, x^*\rangle \geq 1$, i.e. $\langle x, x^*\rangle \geq t \geq \lambda$. Hence $\langle x, x^*\rangle \geq \nu_A(x)$. Conversely, suppose that $\langle x, x^*\rangle \geq \nu_A(x)$ for every $x \in X$. If $x \in A$, $\langle x, x^*\rangle \geq \nu_A(x) \geq 1$. Hence $x^* \in A^\nabla$. Thus the first equality in (4.2) holds. The other ones follow from $\underline{\nu_A} = \underline{\beta_A}$ (by Proposition 2.4) and a passage to the limit superior.

Suppose now that A is convex. If $0 \notin \operatorname{cl} A$, then A^∇ is nonempty, whence, from (4.2) $\underline{\beta_A}(0) \leq 0$. Therefore, $\underline{\beta_A}(0) = 0$. Conversely, if $\underline{\beta_A}(0) = 0$ then $0 \notin \operatorname{cl} A$ and β_A is usc at 0, with $\beta_A(0) = 0$. The relation $\partial(-\underline{\beta_A})(0) = -A^\nabla$ then follows from (4.2). □

Lemma 4.2. *Let $A \subset X$ be a non-empty convex set. Then $A^{\circ\circ} = \overline{\text{co}}(A \cup \{0\})$. If $0 \notin \operatorname{cl} A$ then $(A^\nabla)^+ = \overline{\text{cone}}A = (0^+A^\nabla)^+$ and $A^{\nabla\nabla} = \operatorname{cl}([1, \infty[\cdot A)$; if $0 \in \operatorname{cl} A$ then $A^{\nabla\nabla} = X$.*

Proof. The first relation is the well-known bipolar theorem.

Suppose that $0 \notin \operatorname{cl} A$; as mentioned above, $A^\nabla \neq \emptyset$. Of course, using also Proposition 2.4, we have that

$$A \subset A^{\nabla\nabla} \subset (A^\nabla)^+ = \left(\overline{\text{cone}}A^\nabla\right)^+ = \left(0^+A^\nabla\right)^+.$$

Therefore $\overline{\text{cone}}A \subset (A^\nabla)^+$. Let $x_0 \notin \overline{\text{cone}}A$. By a separation theorem, there exists $x_0^* \in X^*$ such that $\langle x_0, x_0^*\rangle < 0 \leq \langle x, x_0^*\rangle$ for every $x \in A$. Take $x_1^* \in A^\nabla$; then for every $\lambda \geq 0$ we have that $\langle x, x_1^* + \lambda x_0^*\rangle \geq 1$ for every $x \in A$, whence $x_1^* + \lambda x_0^* \in A^\nabla$ for every $\lambda \geq 0$. As $\langle x_0, x_1^* + \lambda x_0^*\rangle < 0$ for sufficiently great λ, we obtain that $x_0 \notin (A^\nabla)^+$.

The last part can be found in [4]. □

We are ready to deal with the harmonic sum of two sets. The formula given below eases the proof of the following well-known result (which can be deduced from Proposition 2 of [32] or from a separation theorem).

Recall ([32]) that if $f, g : X \to \overline{\mathbb{R}}$ are proper lower semi-continuous sublinear functions, then

$$\partial(f+g)(0) = \mathrm{cl}\left(\partial f(0) + \partial g(0)\right), \tag{4.3}$$

the closure being taken with respect to the weak* topology.

Note that the above formula may not be valid if one of the functions is not lsc; just take $f = \iota_P$ and $g = \iota_Q$, where $P = \mathbb{R}^2_+$ and $Q = \{(0,0)\} \cup \{(x,y) \in \mathbb{R}^2 \mid x < 0,\ y > 0\}$.

Taking into account that $0^+A^\circ = A^-$, $A^\circ \subset (0^+A)^\circ$ and $0^+A^\nabla = A^+$, it is easy to obtain the following inclusions for any non-empty subsets:

$$A^\circ \diamond B^\circ \subset (A+B)^\circ, \qquad \mathrm{cl}(A^\circ + B^\circ) \subset (A \diamond B)^\circ, \tag{4.4}$$

$$A^\nabla \diamond B^\nabla \subset (A+B)^\nabla, \qquad \mathrm{cl}(A^\nabla + B^\nabla) \subset (A \diamond B)^\nabla. \tag{4.5}$$

The following proposition completes [25] Theorem 6.1; the second part of its assertion (a) is given in [6] Theorem 1.2^0. There the harmonic sum is even considered for an arbitrary finite number of convex subsets containing 0. Here we only consider two sets; however, given a family $A_1, ..., A_n$ of subsets of X one can set

$$A_1 \diamond \cdots \diamond A_n := \bigcup_{s \in S} (s_1 A_1 \cap \cdots \cap s_n A_n)$$

where S is the canonical simplex of $\mathbb{R}^n$, i.e. the set of $s := (s_1, ..., s_n) \in \mathbb{R}^n_+$ such that $s_1 + \cdots + s_n = 1$. When $A_1, ..., A_n$ are closed convex subsets of X containing 0 (or, more generally, if $\mathbb{P}A_1 \cap \cdots \cap \mathbb{P}A_n \neq \emptyset$), this set coincides with $((A_1 \diamond A_2) \diamond \cdots) \diamond A_n$.

Proposition 4.3. *Let A, B be two non-empty closed convex subsets of X.*

(a) *If $A \diamond B \neq \emptyset$ then $(A \diamond B)^\circ = \mathrm{cl}(A^\circ + B^\circ)$. If $0 \in A \cap B$ then $A^\circ \diamond B^\circ = (A+B)^\circ$.*
(b) *If $A \diamond B \neq \emptyset$ then $(A \diamond B)^\nabla = \mathrm{cl}(A^\nabla + B^\nabla)$.*
(c) *If A and B are shady then $A^\nabla \diamond B^\nabla = (A+B)^\nabla$.*

Proof. (a) A direct proof can be given along the lines of the proof of (c) presented below. As a short variant, use Lemma 4.1, Propositions 2.3, 3.5 and relations (4.3), (4.4) to get

$$(A \diamond B)^\circ = \partial \alpha_{A \diamond B}(0) = \partial\left(\alpha_A + \alpha_B\right)(0) = \mathrm{cl}\left(\partial \alpha_A(0) + \partial \alpha_B(0)\right) = \mathrm{cl}\left(A^\circ + B^\circ\right).$$

The second formula follows from the above one applied for A° and B°, and the bipolar theorem.

(b) If $0 \in A$ then $0 \in 1A \cap 0B \subset A \diamond B$ and $A^\nabla = \emptyset$. It follows that $(A \diamond B)^\nabla = \emptyset = \mathrm{cl}(A^\nabla + B^\nabla)$. Thus we may assume that $0 \notin A \cup B$; then the sublinear functions $-\beta_A$ and $-\beta_B$ are lsc and take the value 0 at 0 by Proposition 2.4; moreover, $(-\beta_A) + (-\beta_B) = -\beta_{A \diamond B}$ by Proposition 3.7 and Lemma 3.3. Thus $-\beta_{A \diamond B}(0) = 0$ and by Lemma 4.1

$$-(A \diamond B)^\circ = \partial\left(-\beta_{A \diamond B}\right)(0) = \mathrm{cl}\left(\partial\left(-\beta_A\right)(0) + \partial\left(-\beta_B\right)(0)\right) = \mathrm{cl}\left(-A^\nabla - B^\nabla\right).$$

(c) Let $z^* \in (A+B)^\nabla$; thus $\langle x, z^* \rangle + \langle y, z^* \rangle \geq 1$ for all $x \in A$, $y \in B$. Since $A = [1, \infty[\cdot A$ and $B = [1, \infty[\cdot B$, we have that $r := \inf_{x \in A} \langle x, z^* \rangle \geq 0$, $s := \inf_{y \in B} \langle y, z^* \rangle \geq 0$ and $r + s \geq 1$. Suppose first that $r, s > 0$; there exist $r', s' > 0$ such that $r \geq r'$, $s \geq s'$ and $r' + s' = 1$. It follows that $z^* \in r'A^\nabla \cap s'B^\nabla \subset A^\nabla \diamond B^\nabla$. Suppose now that $r = 0$. Then $z^* \in A^+ \cap B^\nabla = 0 \cdot A^\nabla \cap 1 \cdot B^\nabla \subset A^\nabla \diamond B^\nabla$. Using relation (4.5), we obtain that $A^\nabla \diamond B^\nabla = (A+B)^\nabla$. □

Example 4.4. Note that the formulae $(A\Diamond B)^\circ = \mathrm{cl}(A^\circ + B^\circ)$, $(A\Diamond B)^\nabla = \mathrm{cl}(A^\nabla + B^\nabla)$ may not hold when $A\Diamond B = \emptyset$. Take for instance $A = -B = [1,\infty[\times\mathbb{R}$. Then $A\Diamond B = \emptyset$, $A^\circ = -B^\circ =]-\infty, 0]\times\{0\}$ and $A^\nabla = -B^\nabla = [1,\infty[\times\{0\}$ so that $A^\circ + B^\circ = A^\nabla + B^\nabla = \mathbb{R}\times\{0\}$.

5. Computation of the flat and sharp conjugates of a function

A number of conjugacies adapted to quasiconvex functions are known. In most cases the computation of the conjugate of a function is not obvious. It is the purpose of this section to present some cases for which an explicit expression of the conjugate can be given.

We start with the flat and the sharp conjugacies which have proved to be useful, especially for Hamilton-Jacobi equations (see [5], [31], [10]). They are defined on $X^* \times \mathbb{R}$ by

$$\begin{aligned} f^\sharp(x^*, s) &:= \sup\{\langle x, x^*\rangle \mid x \in X,\ f(x) \le s\}, \\ f^\flat(x^*, s) &:= \sup\{\langle x, x^*\rangle \mid x \in X,\ f(x) < s\}. \end{aligned}$$

Note that $f^\sharp(x^*, s) = f^\flat(x^*, s)$ if $s > \inf f$ and f is convex because in this case $\{x \in X \mid f(x) \le s\} \subset \mathrm{cl}\{x \in X \mid f(x) < s\}$; this equality also holds for many quasiconvex functions.

Let us first give a slight extension of [31] Proposition 15 which deals with the flat conjugate of a function of the form $f := \varphi \circ \|\cdot\|$, where $\varphi : \mathbb{R}_+ \to \overline{\mathbb{R}}$ is nondecreasing. Here we consider the case of a function of the form $f := \varphi \circ g$, where $\varphi : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is nondecreasing and $g : X \to \overline{\mathbb{R}}$ is sublinear.

As in [31] we use the lowest quasi-inverse φ^e and the greatest quasi-inverse φ^h of φ introduced in [13] by

$$\begin{aligned} \varphi^e(s) &:= \sup\left\{r \in \overline{\mathbb{R}} \mid \varphi(r) < s\right\} = \inf\left\{t \in \overline{\mathbb{R}} \mid s \le \varphi(t)\right\}, \\ \varphi^h(s) &:= \sup\left\{r \in \overline{\mathbb{R}} \mid \varphi(r) \le s\right\} = \inf\left\{t \in \overline{\mathbb{R}} \mid s < \varphi(t)\right\}, \end{aligned}$$

which are characterized, respectively, by the implications

$$\begin{aligned} &\varphi(r) < s \Rightarrow r \le \varphi^e(s) \quad \text{and} \quad t < \varphi^e(s) \Rightarrow \varphi(t) < s, \\ &\varphi(r) \le s \Rightarrow r \le \varphi^h(s) \quad \text{and} \quad t < \varphi^h(s) \Rightarrow \varphi(t) \le s. \end{aligned}$$

In order to obtain formulae for $(\varphi\circ g)^\flat$ and $(\varphi\circ g)^\sharp$ with φ nondecreasing and g sublinear we need the following result which completes Theorem 13.5 of [19] in asserting that the infimum is attained when finite in the formula (5.1) below.

Lemma 5.1. *Let $g : X \to \overline{\mathbb{R}}$ be a proper convex function, $\gamma > \inf g$ and $x^* \in X^*$. Then,*

$$g^\flat(x^*, \gamma) = g^\sharp(x^*, \gamma) = \min\{\lambda\gamma + h(x^*, \lambda) \mid \lambda \ge 0\}, \tag{5.1}$$

where

$$h(x^*, \lambda) = (\lambda g)^*(x^*) = \begin{cases} \lambda g^*(\lambda^{-1}x^*) & \text{if } \lambda > 0, \\ \sup\{\langle x, x^*\rangle \mid x \in \mathrm{dom}\, g\} & \text{if } \lambda = 0. \end{cases}$$

Proof. Consider

$$F : X \times \mathbb{R} \to \overline{\mathbb{R}}, \quad F(x,t) = \begin{cases} -\langle x, x^* \rangle & \text{if } g(x) \le \gamma + t, \\ +\infty & \text{otherwise.} \end{cases}$$

It is obvious that F is a convex function and $P_{\mathbb{R}}(\operatorname{dom} F) \supset]\inf g - \gamma, \infty[$, where $P_{\mathbb{R}}$ is the projection from $X \times \mathbb{R}$ onto $\mathbb{R}$. Therefore $0 \in \operatorname{int} P_{\mathbb{R}}(\operatorname{dom} F)$. Applying the fundamental duality theorem in the case of a finite dimensional parameter space (see for example [33, Th. 2.6.5]) we have that $\inf_{x \in X} F(x,0) = \max_{\lambda \in \mathbb{R}} (-F^*(0,\lambda))$, in the sense that the supremum is attained when it is finite, or equivalently,

$$\sup\{\langle x, x^* \rangle \mid g(x) \le \gamma\} = \min_{\lambda \in \mathbb{R}} F^*(0, -\lambda).$$

But

$$\begin{aligned} F^*(0,-\lambda) &= \sup\{-\lambda t + \langle x, x^* \rangle \mid x \in X,\ t \in \mathbb{R},\ g(x) \le t + \gamma\} \\ &= \lambda\gamma + \sup\{-\lambda r + \langle x, x^* \rangle \mid (x,r) \in \operatorname{epi} g\}. \end{aligned}$$

It is obvious that $F^*(0,-\lambda) = \infty$ if $\lambda < 0$ and $F^*(0,-\lambda) = \sup\{\langle x, x^* \rangle \mid x \in \operatorname{dom} g\}$ if $\lambda = 0$. Let $\lambda > 0$; then

$$F^*(0,-\lambda) = \lambda\gamma + \sup\{\langle x, x^* \rangle - \lambda g(x) \mid x \in \operatorname{dom} g\} = \lambda\gamma + \lambda g^*(\lambda^{-1} x^*).$$

The conclusion follows. □

When g is sublinear, we obtain formulae involving the gauges studied above which will be used in our paper [16].

Proposition 5.2. *Let $g : X \to \overline{\mathbb{R}}$ be a proper sublinear functional, lsc at 0, and let $x^* \in X^*$. Then for every $\gamma > \inf g$ one has*

$$g^\flat(x^*, \gamma) = g^\sharp(x^*, \gamma) = \begin{cases} \gamma \cdot \alpha_{\partial g(0)}(x^*) & \text{if } \gamma > 0, \\ \iota_{[0,\infty[\partial g(0)}(x^*) & \text{if } \gamma = 0, \\ \gamma \cdot \beta_{\partial g(0)}(x^*) & \text{if } \gamma < 0, \end{cases}$$

with $\alpha_{\partial g(0)}(x^*)$ and $\beta_{\partial g(0)}(x^*)$ attained when finite.

Note that here $\partial g(0)$ is nonempty, as g is lsc at 0, and $0^+(\partial g(0)) = (\operatorname{dom} g)^-$.

Proof. Recall that, since g is sublinear, $\operatorname{dom} g$ is a convex cone and $g^* = \iota_{\partial g(0)}$. In our conditions the function h introduced in Lemma 5.1 is given by

$$h(x^*, \lambda) = \begin{cases} 0 & \text{if } \lambda > 0,\ x^* \in \lambda\partial g(0) \text{ or } \lambda = 0,\ x^* \in (\operatorname{dom} g)^-, \\ +\infty & \text{otherwise,} \end{cases}$$

and so

$$\{x^* \in X^* \mid \exists \lambda \ge 0,\ h(x^*, \lambda) < \infty\} = [0, \infty[\cdot \partial g(0). \tag{5.2}$$

Let $\gamma > 0$. It is obvious that $g^\sharp(x^*, \gamma) \geq 0$ in this case. From relations (5.1) and (5.2), we have that $g^\sharp(x^*, \gamma) = \alpha_{\partial g(0)}(x^*) = \infty$ if $x^* \notin [0, \infty[\cdot\partial g(0)$. If $x^* \in (\operatorname{dom} g)^-$ then $h(x^*, 0) = 0$, and so, from (5.1), $g^\sharp(x^*, \gamma) = 0 = \gamma\alpha_{\partial g(0)}(x^*)$. Suppose now that $x^* \in]0, \infty[\cdot\partial g(0) \setminus (\operatorname{dom} g)^-$. Applying again the preceding lemma we have that $g^\sharp(x^*, \gamma) < \infty$ so that

$$g^\sharp(x^*, \gamma) = \min\{\lambda\gamma + h(x^*, \lambda) \mid \lambda \geq 0\} = \overline{\lambda}\gamma + h(x^*, \overline{\lambda})$$

for some $\overline{\lambda} \geq 0$. Since $x^* \notin (\operatorname{dom} g)^-$, $\overline{\lambda} \neq 0$, and so $\overline{\lambda} > 0$ and $x^* \in \overline{\lambda} \cdot \partial g(0)$. It follows that

$$g^\sharp(x^*, \gamma) = \min\{\lambda\gamma \mid \lambda \geq 0,\ \ x^* \in \lambda\partial g(0)\} = \overline{\lambda}\gamma = \gamma\alpha_{\partial g(0)}(x^*).$$

Suppose now that $\inf g < 0$.

For $\gamma = 0$, from (5.1) and (5.2), we have that $g^\sharp(x^*, 0) = \iota_{[0,\infty[\partial g(0)}(x^*)$.

Let $\gamma < 0$. As above, if $x^* \notin [0, \infty[\cdot\partial g(0)$ then $g^\sharp(x^*, \gamma) = \gamma \cdot \nu_{\partial g(0)}(x^*) = \infty$. If $x^* \in]0, \infty[\cdot\partial g(0)$ then, from (5.1), $g^\sharp(x^*, \gamma) < 0$, and so

$$\begin{aligned} g^\sharp(x^*, \gamma) &= \min\{\lambda\gamma \mid \lambda \geq 0,\ \ x^* \in \lambda\partial g(0)\} = \gamma \max\{\lambda \geq 0 \mid x^* \in \lambda\partial g(0)\} \\ &= \gamma \cdot \beta_{\partial g(0)}(x^*) < 0 \end{aligned}$$

with $\beta_{\partial g(0)}(x^*)$ attained. If $x^* \in (\operatorname{dom} g)^- \setminus]0, \infty[\cdot\partial g(0)$ from (5.1) and (5.2) we get that $g^\sharp(x^*, \gamma) = 0$. The proof is complete. $\square$

The previous result can be extended to functions of the form $\varphi \circ g$ with φ nondecreasing and g sublinear. It has been obtained by Volle [31] when g is a norm on X.

Proposition 5.3. *Let $\varphi : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ be nondecreasing and $g : X \to \overline{\mathbb{R}}$ be a proper convex function. Consider $f := \varphi \circ g$ and $(x^*, r) \in X^* \times \mathbb{R}$. If $\varphi^e(r) \in]\inf g, \infty[$ then*

$$f^\flat(x^*, r) = g^\flat\left(x^*, \varphi^e(r)\right) = g^\sharp\left(x^*, \varphi^e(r)\right),$$

and for $\varphi^h(r) \in]\inf g, \infty[$ one has

$$f^\sharp(x^*, r) = g^\sharp\left(x^*, \varphi^h(r)\right) = g^\flat\left(x^*, \varphi^h(r)\right).$$

Moreover, when g is sublinear and lsc at 0, and $\varphi^e(r) \in]\inf g, \infty[$,

$$f^\flat(x^*, r) = \begin{cases} \varphi^e(r) \cdot \alpha_{\partial g(0)}(x^*) & \text{if } \varphi^e(r) > 0, \\ \iota_{[0,\infty[\partial g(0)}(x^*) & \text{if } \varphi^e(r) = 0, \\ \varphi^e(r) \cdot \beta_{\partial g(0)}(x^*) & \text{if } \varphi^e(r) < 0, \end{cases}$$

similar expressions being valid for $f^\sharp(x^*, r)$ with $\varphi^e(r)$ replaced by $\varphi^h(r) \in]\inf g, \infty[$.

Proof. For the first assertions it is sufficient to observe that

$$\begin{aligned} \{x \mid g(x) < \varphi^e(r)\} \ &\subset\ \{x \mid f(x) < r\} \ \subset\ \{x \mid g(x) \leq \varphi^e(r)\} \subset \operatorname{cl}\{x \mid g(x) < \varphi^e(r)\}, \\ \{x \mid g(x) < \varphi^h(r)\} \ &\subset\ \{x \mid f(x) \leq r\} \ \subset\ \{x \mid g(x) \leq \varphi^h(r)\} \subset \operatorname{cl}\{x \mid g(x) < \varphi^h(r)\}, \end{aligned}$$

while for the last ones it is enough to apply the preceding proposition. $\square$

Using Lemma 5.1 we also obtain the next formula for the normal cone to a sublevel set of a convex function; note that such a formula is usually stated for finite convex functions. Recall that the *normal cone* to a subset C of X at $x_0 \in X$ is given by

$$N(C, x_0) := \{x^* \in X^* \mid \langle x^*, x - x_0 \rangle \leq 0 \; \forall\, x \in C\}.$$

Proposition 5.4. *Let $g : X \to \overline{\mathbb{R}}$ be a proper convex function and $x_0 \in \operatorname{dom} g$. If $g(x_0) > \inf g$ then*

$$N([g \leq g(x_0)], x_0) = [0, \infty[\cdot \partial g(x_0),$$

with the convention that $0 \cdot \partial g(x_0) = 0^+ \partial g(x_0) = N(\operatorname{dom} g, x_0)$.

Proof. The inclusion $\supset$ is obvious. Let $x^* \in N([g \leq g(x_0)], x_0)$ be fixed. Using the definition of the normal cone and Lemma 5.1 with $\gamma = g(x_0)$ we have

$$\langle x_0, x^* \rangle = \sup\{\langle x, x^* \rangle \mid x \in [g \leq \gamma]\} = \lambda g(x_0) + h(x^*, \lambda)$$

for some $\lambda \geq 0$. If $\lambda = 0$ then $\langle x_0, x^* \rangle = \sup\{\langle x, x^* \rangle \mid x \in \operatorname{dom} g\}$, and so $x^* \in N(\operatorname{dom} g, x_0) = 0 \cdot \partial g(x_0)$. If $\lambda > 0$ then $\langle x_0, x^* \rangle = \lambda g(x_0) + \lambda g^*(\lambda^{-1} x^*)$, whence $\lambda^{-1} x^* \in \partial g(x_0)$. The proof is complete. $\square$

Remark 5.5. The relation $0 \cdot \partial g(x_0) = N(\operatorname{dom} g, x_0)$ is justified by the fact that $0^+ \partial g(x_0) = N(\operatorname{dom} g, x_0)$ when $\partial g(x_0) \neq \emptyset$. Moreover, when $\mathbb{P}(\operatorname{dom} g - x_0) = X$ one has $N(\operatorname{dom} g, x_0) = \{0\}$.

Example 5.6. Let $g : \mathbb{R} \to \overline{\mathbb{R}}$ be given by $g(x) = -\sqrt{1 - x^2}$ for $|x| \leq 1$, $+\infty$ otherwise. Taking $x_0 = 1$, we observe that $\partial g(x_0) = \emptyset$ but $N([g \leq g(x_0)], x_0) = N(\operatorname{dom} g, x_0) = 0 \cdot \partial g(x_0)$.

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