Journal of Convex Analysis
Volume 5 (1998), No. 2, 329–351

Optimal Control of Problems Governed by Obstacle Type Variational Inequalities: A Dual Regularization-Penalization Approach

Maïtine Bergounioux
Département de Mathématiques, UMR 6628 - Université d'Orléans, B.P. 6759 - 45067 Orléans Cedex 2, France.
e-mail: maitine@labomath.univ-orleans.fr

Helmut Dietrich
Institute of Optimization and Stochastics, Martin-Luther-University of Halle-Wittenberg, 06099 Halle, Germany.
e-mail: dietrich@mathematik.uni-halle.de

Received April 28, 1997
Revised manuscript received October 8, 1997

In this paper we investigate optimal control problems governed by variational inequalities. We give optimality conditions under classical assumptions using a dual regularized functional to interpret the Variational Inequality.

Keywords: Variational Inequalities, Optimal Control, Dual Methods, Penalization Methods

1991 Mathematics Subject Classification: 49J20, 49M29

1. Introduction

In this paper we investigate optimal control problems governed by variational inequalities of obstacle type. This problem has been widely studied during the last years by many authors. It is now known that one cannot obtain classical optimality systems (in the sense of Mathematical Programming) for such problems. This come essentially from the fact that the mapping S which associates the state y solution of a Variational Inequality to the control v, is not differentiable as pointed it out Mignot [9] and one can only define a conical derivative for S. In [10], Mignot and Puel obtain optimality conditions using the results of [9]. Different methods have been used to consider this problem. Barbu [2, 3] studies approximations of the Variational Inequality which lead to optimal control problems governed by variational *equations*. Then he gets existence results and optimality conditions using a passage to the limit in the approximation process. In [5, 6], the first author has obtained classical optimality systems for suitable approximations of the original problem which can be easily used from the numerical point of view.

On the other hand, Rubio and Wenbin [14] obtain results for strongly monotone variational inequalities of obstacle type introducing a dual penalization for the variational inequality on increasing radius balls. We have adopted these last authors point of view to interpret the variational inequality in a dual way. Anyway the motivation is slightly different since we have been thinking of conjugated functions occurring in classical convex

ISSN 0944-6532 / $ 2.50 © Heldermann Verlag

analysis. So we define a regularized dual function which is $\mathcal{C}^1$ to describe the Variational Inequality. We do not claim that our results are completely new (though we are able to weaken some assumptions that are commonly used) but we think that this method may be extended to general Variational Inequalities and may provide some new results and a better understanding of the behavior of solutions.

The paper is organized as follows. We first present the problem and set the basic assumptions. Existence results are well known for this kind of problems and we are only interested in optimality conditions. Then we define the dual functional h, give some properties and reinterpret the original problem as a classical mathematical programming problem with an $\mathcal{C}^1$ equality constraint. A short example will show that it is hopeless to obtain classical optimality systems so that usual results cannot be applied. So we use a penalty method to get first-order conditions with additional assumptions. We end the paper with applications.

2. Setting the problem

Consider the following abstract problem:

$$\min J(y,v), \langle T(y,v), z-y\rangle \geq 0 \quad \forall z \in K,\ y \in K,\ v \in U_{ad} \tag{$\mathcal{P}$}$$

where

- V is an Hilbert space and V' its dual; we denote by $\langle\ ,\ \rangle$ the duality product between V and V', by $(\ ,\ )_V$ the inner scalar product of V and by $\|\ \|_V$ the V-norm. We shall omit the index for the space V most of time, i.e. $\|\ \|$ will mean $\|\ \|_V$. Moreover $\Lambda : V \to V'$ is the duality mapping (i.e. here the canonical isomorphism) and we recall that:
$$\forall (y,z) \in V \times V \quad (y,z)_V = \langle y, \Lambda z\rangle.$$
- U is an Hilbert space and $\Lambda_U : U \to U'$ denotes the duality mapping.
- K and U_{ad} are non empty closed convex subsets of V and U respectively.
- $T : V \times U \to V'$ is an operator (not necessarily linear) which satisfies:
$$T \text{ is } \mathcal{C}^1 \text{ in the Fréchet sense.} \tag{2.1}$$
and
$$\forall y \in K \quad v \mapsto T(y,v) \text{ is strongly continuous from } U \text{ to } V'. \tag{2.2}$$

We recall (see for example Zeidler [15, p.515]) that $T(y,.)$ is strongly continuous if it is weakly-strongly sequentially continuous, i.e.

$$v_n \rightharpoonup v \text{ in } U \Rightarrow T(y,v_n) \to T(y,v) \text{ in } V'.$$

- $J : V \times U \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex and lower semicontinuous functional; moreover it is supposed to be bounded below and Gâteaux-differentiable.

We assume also that

$$\text{Problem } (\mathcal{P}) \text{ has at least one optimal solution denoted } (y^*,v^*) \in K \times U_{ad}. \tag{2.3}$$

All the previous assumptions are supposed to be ensured in the sequel of the paper. We summarize them as assumption $(\mathcal{H})$.

Remark 2.1. We must underline that the Gâteaux-differentiability assumption for the cost functional J can be omitted. Indeed, using Moreau-Yosida approximation of J one can replace the original problem by a similar problem where the cost functional is Gâteaux-differentiable (and even $\mathcal{C}^1$) and the results of this paper can be applied. Then, one can pass to the limit in optimality systems to get the new one which involves sub-gradients instead of Gâteaux-derivatives (see Bergounioux [6] for instance). However, to make the presentation clearer we decided to add this Gâteaux-differentiability assumption for J.

2.1. Examples

Let us give some examples to illustrate such problems. Let A be an operator from V onto V' such that A is

- strongly monotone i.e.

$$\exists \nu > 0 \text{ such that } \forall (y,z) \in V \times V \quad \langle Ay - Az, y - z\rangle \geq \nu \|y - z\|_V^2 .$$

- demi-continuous, i.e. strongly-weakly sequentially continuous:

$$x_n \to x \text{ strongly in } V \Rightarrow Ax_n \rightharpoonup Ax \text{ weakly in } V'.$$

Let $\varphi : V \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous functional such that

$$\exists y_o \in \operatorname{dom} \varphi \;\text{ such that } \lim_{\|y\| \to +\infty} \frac{\langle Ay, y - y_o\rangle + \varphi(y)}{\|y\|} = +\infty,$$

and $f \in V'$.

Then the following variational inequality has a unique solution y (see [3, pp. 125–127]):

$$\forall z \in V \quad \langle Ay, y - z\rangle + \varphi(y) - \varphi(z) \leq \langle f, y - z\rangle. \tag{2.4}$$

In particular we may choose $\langle Ay, z\rangle = a(y,z)$ where a is a bilinear, continuous and V-elliptic (i.e. $a(y,y) \geq \mu \|y\|^2$) form and φ is the indicatrix function 1_K of a nonempty, convex and closed set K. We recall that

$$1_K(y) = \begin{cases} 0 & \text{if } y \in K \\ +\infty & \text{else.} \end{cases}$$

Then, the Variational Inequality (2.4) has a unique solution y and the application $f \mapsto y$ is lipschitz-continuous from V' onto V.

Suppose now that we have $V \subset U \subset V'$ continuously and densely and define B a strongly continuous operator from U to V' (B may be, for instance, the canonical injection if U is compactly embedded in V) and assume, in addition that J is coercive. Then ([3, p. 151]) the problem $(\mathcal{P})$ has at least one optimal solution. In this case $T(y,v) = Ay - Bv - f$ and satisfies (2.2).

2.2. Reformulation of the Variational Inequality

We are going to transform the variational inequality via a duality process. For $\alpha > 0$, we define the functional $h_\alpha : V \times U \to \mathbb{R}$ as following:

$$h_\alpha(y,v) = \sup_{z \in K} \left[\langle -\alpha T(y,v), z - y\rangle - \frac{1}{2}\|z - y\|_V^2 \right] . \tag{2.5}$$

Remark 2.2. α is a parameter that will be fixed later greater than ν^{-1}, where ν will be a monotonicity constant for T, as it appears in Example 2.1 for instance. We will explain this choice in the sequel of the paper.

Theorem 2.3. *h_α is well-defined and*

(i) $\forall y \in K,\ \forall v \in U \quad h_\alpha(y,v) \geq 0$.
(ii) $h_\alpha(y,v) = \frac{1}{2}\left[\alpha^2 \|T(y,v)\|^2_{V'} - d^2_K(y - \alpha\Lambda^{-1}T(y,v))\right]$, *where d_K is the distance to the set K.*
(iii) $h_\alpha(y,v) = -\alpha\langle T(y,v), y_K - y\rangle - \frac{1}{2}\|y_K - y\|^2_V$ *where $y_K = P_K(y - \alpha\Lambda^{-1}T(y,v))$ and P_K is the V-projection on K.*

Proof. Recall that

$$d^2_K(y) = \min_{z\in K} \|z - y\|^2_V = \|y - P_K(y)\|^2_V,$$

Assertion (i) is obvious: one takes $z = y$ to compute the supremum in (2.5).

Let (y,v) be in $V \times U$.

$$\begin{aligned} h_\alpha(y,v) &= \alpha\,\langle T(y,v), y\rangle - \frac{1}{2}\|y\|^2 + \sup_{z\in K}\left[\langle -\alpha T(y,v), z\rangle - \frac{1}{2}\|z\|^2 + (y,z)_V\right] \\ &= \alpha\,\langle T(y,v), y\rangle - \frac{1}{2}\|y\|^2 - \inf_{z\in K}\left[\frac{1}{2}\|z\|^2 - \left(y - \alpha\Lambda^{-1}T(y,v), z\right)_V\right], \end{aligned}$$

since $\langle T(y,v), z\rangle = (\Lambda^{-1}T(y,v), z)_V$. So

$$\begin{aligned} &h_\alpha(y,v) \\ &\quad= \alpha\,\langle T(y,v), y\rangle - \frac{1}{2}\|y\|^2 - \frac{1}{2}\inf_{z\in K}\|z - (y - \alpha\Lambda^{-1}T(y,v))\|^2_V + \frac{1}{2}\|y - \alpha\Lambda^{-1}T(y,v))\|^2_V \\ &\quad= \frac{\alpha^2}{2}\|T(y,v)\|^2_{V'} - \frac{1}{2}d^2_K(y - \alpha\Lambda^{-1}T(y,v)) \end{aligned}$$

Setting $y_K = P_K(y - \alpha\Lambda^{-1}T(y,v))$, we have then

$$\begin{aligned} h_\alpha(y,v) &= \frac{\alpha^2}{2}\|T(y,v)\|^2_{V'} - \frac{1}{2}\|y - \alpha\Lambda^{-1}T(y,v) - y_K\|^2_V \\ h_\alpha(y,v) &= -\alpha\,\langle T(y,v), y_K - y\rangle - \frac{1}{2}\|y_K - y\|^2_V. \end{aligned}$$

□

Remark 2.4. We may relate this function h to the classical conjugate functions. Let us call $\tilde{y} = -\alpha T(y,v)$ for a while:

$$\begin{aligned} h_\alpha(y,v) &= \sup_{z\in K}\left[\langle \tilde{y}, z - y\rangle - \frac{1}{2}\|z - y\|^2\right], \\ &= -\frac{1}{2}\|y\|^2 - \langle \tilde{y}, y\rangle + \sup_{z\in K}\left[\langle \tilde{y} + \Lambda y, z\rangle - \frac{1}{2}\|z\|^2\right], \\ &= -\frac{1}{2}\|y\|^2 - \langle \tilde{y}, y\rangle + \sup_{z\in V}\left[\langle \tilde{y} + \Lambda y, z\rangle - \psi(z)\right], \end{aligned}$$

where $\psi(z) = \frac{1}{2}\|z\|^2 + 1_K(z)$ is convex. So we get

$$h_\alpha(y,v) = \psi^*(\Lambda y - \alpha T(y,v)) - \frac{1}{2}\|y\|^2 + \alpha \langle T(y,v), y\rangle$$

that is

$$h_\alpha(y,v) = \psi^*(\Lambda y - \alpha T(y,v)) - \frac{1}{2}\|\Lambda y - \alpha T(y,v)\|^2_{V'} + \frac{\alpha^2}{2}\|T(y,v)\|^2_{V'},$$

where ψ^* is the conjugate function of ψ (see [8] for example).

Theorem 2.5. *If the operator T is Fréchet-$\mathcal{C}^1$ then h_α is Fréchet-$\mathcal{C}^1$ and*

$$\begin{aligned} &h'_\alpha(y,v) \\ &\quad = \alpha^2[T'(y,v)]^*\Lambda^{-1}T(y,v) - \left[\begin{pmatrix} I \\ 0 \end{pmatrix} - \alpha[T'(y,v)]^*\Lambda^{-1}\right]\Lambda(I-P_K)(y-\alpha\Lambda^{-1}T(y,v)), \end{aligned}$$

where $[T'(y,v)]^*$ is the adjoint operator of $T'(y,v)$.

Proof. We know that $\sigma : V \to \mathbb{R}$ such that $\sigma(y) = \frac{1}{2}d_K^2(y) = \frac{1}{2}\|(I-P_K)(y)\|^2_V$ is Fréchet-$\mathcal{C}^1$ and $\sigma' = I - P_K$. (This is a consequence of Moreau-Yosida approximation theory: one can refer to [3, p. 67] for more details). So h_α is Fréchet-$\mathcal{C}^1$ since T is.

Let us fix $(y,v) \in V \times U$ and compute first $\langle \nabla_y h_\alpha(y,v), z\rangle$ for $z \in V$.

As

$$h_\alpha(y,v) = \frac{1}{2}\left[\alpha^2\|T(y,v)\|^2_{V'} - d_K^2(y - \alpha\Lambda^{-1}T(y,v))\right].$$

then

$$\begin{aligned} \langle \nabla_y h_\alpha(y,v), z\rangle &= \alpha^2\left(T(y,v), T'_y(y,v)z\right)_{V'} \\ &\quad - \left((I-P_K)(y-\alpha\Lambda^{-1}T(y,v)), z - \alpha\Lambda^{-1}T'_y(y,v)z\right)_V \\ &= \alpha^2\left\langle \Lambda^{-1}T(y,v), T'_y(y,v)z\right\rangle_{V,V'} \\ &\quad - \left\langle \Lambda(I-P_K)(y-\alpha\Lambda^{-1}T(y,v)), z - \alpha\Lambda^{-1}T'_y(y,v)z\right\rangle_{V',V} \\ &= \alpha^2\left\langle T'_y(y,v)^*\Lambda^{-1}T(y,v), z\right\rangle_{V',V} \\ &\quad - \left\langle [I - \alpha\Lambda^{-1}T'_y(y,v)]^*\Lambda(I-P_K)(y-\alpha\Lambda^{-1}T(y,v)), z\right\rangle_{V',V}. \end{aligned}$$

So

$$\begin{aligned} &\nabla_y h_\alpha(y,v) \\ &= \alpha^2 T'_y(y,v)^*\Lambda^{-1}T(y,v) - [I - \alpha(\Lambda^{-1}T'_y(y,v))^*]\Lambda(I-P_K)(y-\alpha\Lambda^{-1}T(y,v)). \end{aligned} \tag{2.6}$$

Similarly one computes $\nabla_v h_\alpha(y,v)$:

$$\begin{aligned}
\langle \nabla_v h_\alpha(y,v), u\rangle_{U',U} &= \alpha^2 \left(T(y,v), T'_v(y,v)u\right)_{V'} \\
&\quad - \left((I-P_K)(y-\alpha\Lambda^{-1}T(y,v)), -\alpha\Lambda^{-1}T'_v(y,v)\ u\right)_V \\
&= \alpha^2 \left\langle \Lambda^{-1}T(y,v), T'_v(y,v)u\right\rangle_{V,V'} \\
&\quad + \alpha \left\langle \Lambda(I-P_K)(y-\alpha\Lambda^{-1}T(y,v)), \Lambda^{-1}T'_v(y,v)\ u\right\rangle_{V',V} \\
&= \alpha^2 \left\langle T'_v(y,v)^*\Lambda^{-1}T(y,v), u\right\rangle_{U',U} \\
&\quad + \alpha \left\langle [\Lambda^{-1}T'_v(y,v)]^*\Lambda(I-P_K)(y-\alpha\Lambda^{-1}T(y,v)), u\right\rangle_{U',U} .
\end{aligned}$$

Finally we get

$$\nabla_v h_\alpha(y,v) = \alpha^2 T'_v(y,v)^*\Lambda^{-1}T(y,v) + \alpha T'_v(y,v)^*(I-P_K)(y-\alpha\Lambda^{-1}T(y,v)) ,$$

i.e.

$$\nabla_v h_\alpha(y,v) = \alpha T'_v(y,v)^*[y - P_K(y-\alpha\Lambda^{-1}T(y,v))]. \tag{2.7}$$

□

Theorem 2.6. *For any $\alpha > 0$, the three following assertions are equivalent:*

(i) $h_\alpha(y,v) = 0\ ,\ y \in K,\ v \in U_{ad}$.
(ii) $\langle T(y,v), z-y\rangle \geq 0 \quad \forall z \in K,\ y \in K,\ v \in U_{ad}$.
(iii) $y = P_K(y-\alpha\Lambda^{-1}T(y,v)),\ v \in U_{ad}$.

Proof. We show (ii) ⇒ (i).
Let be $v \in U_{ad}$ and $y \in K$ be a solution of

$$\langle T(y,v), z-y\rangle \geq 0 \quad \forall z \in K.$$

Then for any $z \in K$ and $\alpha > 0$, we get

$$-\alpha\langle T(y,v), z-y\rangle - \frac{1}{2}\|z-y\|^2 \leq 0,$$

so that $h_\alpha(y,v) \leq 0$. As $y \in K$, we already know that $h_\alpha(y,v) \geq 0$ and we get (i).
Conversely: (i) ⇒ (ii).

$$h_\alpha(y,v) = 0 \rightarrow -\alpha\langle T(y,v), z-y\rangle - \frac{1}{2}\|z-y\|^2 \leq 0, \quad \forall z \in K.$$

Let be $t \in]0,1[$ and set $z = (1-t)y + t\xi$ with $\xi \in K$. we obtain

$$-\alpha t\langle T(y,v), \xi-y\rangle - \frac{t^2}{2}\|\xi-y\|^2 \leq 0.$$

Dividing by t and letting t tend towards 0 implies

$$\forall \xi \in K \quad \alpha\langle T(y,v), \xi-y\rangle \geq 0,$$

i.e. (ii) since $\alpha > 0$.

At last, it is clear that (ii) is equivalent to (iii) since (ii) is equivalent to

$$\left(\alpha\Lambda^{-1}T(y,v), z-y\right)_V \geq 0 \quad \forall z \in K,$$

that is

$$\left(y-(y-\alpha\Lambda^{-1}T(y,v)), z-y\right)_V \geq 0 \quad \forall z \in K.$$

The characterization of the projection yields that

$$y = P_K(y-\alpha\Lambda^{-1}T(y,v)).$$

□

Finally, we have proved that problem $(\mathcal{P})$ is equivalent to

$$\min J(y,v),\ h_\alpha(y,v)=0,\ y \in K,\ v \in U_{ad} \qquad (\mathcal{P}_\alpha)$$

where h_α is $\mathcal{C}^1$ but not convex and $\alpha > 0$.

We end this section with a lower semi-continuity result for h_α. More precisely

Theorem 2.7. *Let y_o be in K and assume the operator $v \to T(y_o,v)$ is strongly continuous at the point $v_o \in U$, then the function $v \mapsto h_\alpha(y_o,v)$ is weakly continuous at v_o.*

Proof. Consider a weakly convergent sequence $v_n \rightharpoonup v_o$ and $\alpha > 0$; Theorem 2.3 gives

$$h_\alpha(y_o,v_n) = -\langle \alpha T(y_o,v_n), \bar{z}_n - y_o\rangle - \frac{1}{2}\|\bar{z}_n - y_o\|^2$$

where

$$\bar{z}_n = P_K(y_o - \alpha\Lambda^{-1}T(y_o,v_n)) \to z_o = P_K(y_o - \alpha\Lambda^{-1}T(y_o,v_o)).$$

(since $T(y_o,v_n) \to T(y_o,v_o)$); so

$$\lim_{n\to+\infty} h_\alpha(y_o,v_n) = \lim_{n\to+\infty}\left[-\alpha\left\langle T(y_o,v_n), \bar{z}_n - y_o\right\rangle - \frac{1}{2}\|\bar{z}_n - y_o\|^2\right],$$

$$\lim_{n\to+\infty} h_\alpha(y_o,v_n) = -\alpha\left\langle T(y_o,v_o), z_o - y_o\right\rangle - \frac{1}{2}\|z_o - y_o\|^2 = h_\alpha(y_o,v_o).$$

□

3. Classical Mathematical Programming Approach

In this section,we suppose that J is differentiable (in the Fréchet sense). Problem $(\mathcal{P}_\alpha)$ appears to be a classical mathematical programming problem where the functions are smooth and there are no inequality constraints. So we are going to see how general mathematical programming methods in Banach spaces can be adapted here. Let us recall a result mainly due to J. Zowe and S. Kurcyusz [16]. We consider real Banach spaces

$\mathcal{X}, \mathcal{Y}$; let C be a convex closed subset of $\mathcal{X}$ and M a closed cone of $\mathcal{Y}$ with vertex at 0. We deal also with:

$$f : \mathcal{X} \to \mathbb{R}, \text{ Fréchet-differentiable functional and}$$
$$g : \mathcal{X} \to \mathcal{Y} \text{ continuously Fréchet-differentiable.}$$

Now, let be the mathematical programming problem defined by:

$$\min\{f(x) \mid g(x) \in M,\ x \in C\}. \tag{3.1}$$

We suppose that the problem (3.1) has an optimal solution that we call $\bar{x}$, and we introduce the conical hulls of $C - \{\bar{x}\}$ and $M - \{y\}$:

$$C(\bar{x}) = \{x \in \mathcal{X} \mid \exists\, \lambda \geq 0, \exists\, c \in C,\ x = \lambda\,(c - \bar{x})\},$$
$$M(y) = \{z \in \mathcal{Y} \mid \exists\, \lambda \geq 0, \exists\, \zeta \in M,\ z = \zeta - \lambda y\}.$$

One may now enounce the main result about the existence of Lagrange multipliers for such a problem.

Theorem 3.1. *Let $\bar{x}$ be an optimal solution for problem* (3.1) *and suppose that the following regularity condition is fulfilled:*

$$g'(\bar{x}) \cdot C(\bar{x}) - M(g(\bar{x})) = \mathcal{Y}\ . \tag{3.2}$$

Then a Lagrange multiplier $\mu^* \in \mathcal{Y}'$ exists such that

$$\forall z \in M \quad \langle \mu^*, z\rangle_{\mathcal{Y}',\mathcal{Y}} \geq 0, \tag{3.3}$$

$$\langle \mu^*, g(\bar{x})\rangle_{\mathcal{Y}',\mathcal{Y}} = 0, \tag{3.4}$$

$$f'(\bar{x}) - \mu^* \circ g'(\bar{x}) \in C(\bar{x})^+, \tag{3.5}$$

where $A^+ = \{x^* \in \mathcal{X}^* \mid \langle x^*, a\rangle_{\mathcal{X}',\mathcal{X}} \geq 0,\ \forall a \in A\}$.

Remark 3.2. A classical regularity condition to ensure the existence of Lagrange multipliers for such problems is one of the following:

$$\text{either } \bar{x} \in \operatorname{Int} C \text{ and } g'(\bar{x}) \text{ is surjective}$$
$$\text{or, there is some } x \in C(\bar{x}) \text{ such that } g'(\bar{x})x \in \operatorname{Int} M(g(\bar{x})).$$

These conditions are not fulfilled when the considered interiors are empty (which happens quite often). The so-called Zowe and Kurcyusz regularity condition (3.2) is a weak variant and allows to get (classical) Lagrange multipliers even if the previous regularity conditions are not satisfied. Indeed, if (3.2) is fulfilled then the linearizing cone of the feasible set at $\bar{x}$ is included in the sequential tangent cone at $\bar{x}$ and therefore Lagrange multipliers exist.

Let us apply this formalism to our case with $\mathcal{X} = V \times U,\ \mathcal{Y} = \mathbb{R},\ f = J,\ g = h_\alpha,\ C = K \times U_{ad}$ and $M = \{0\}$. We set $x = (y, v)$ and $\bar{x} = (y^*, v^*)$. In our setting condition (3.2) becomes

$$h'_\alpha(y^*, v^*) \cdot C(y^*, v^*) = \mathbb{R}. \tag{3.6}$$

Applying the previous general result leads to

Theorem 3.3. *Let (y^*, v^*) be an optimal solution for problem $(\mathcal{P})$ and suppose that* (3.6) *is fulfilled.*

Then, there exists $\lambda^* \in \mathbb{R}$ such that

$$J'(y^*, v^*) - \lambda^* h'_\alpha(y^*, v^*) \in C(y^*, v^*)^+,$$

i.e.

$$\forall z \in K \quad \langle \nabla_y J(y^*, v^*) - \lambda^* \nabla_y h_\alpha(y^*, v^*), z - y^* \rangle_{V',V} \geq 0, \tag{3.7}$$

$$\forall v \in U_{ad} \quad \langle \nabla_v J(y^*, v^*) - \lambda^* \nabla_v h_\alpha(y^*, v^*), v - v^* \rangle_{U',U} \geq 0. \tag{3.8}$$

The regularity condition involves the derivative of h_α at (y^*, v^*) so we compute it:

Lemma 3.4.

$$h'_\alpha(y^*, v^*) = \alpha\, T(y^*, v^*) \in V'.$$

Proof. We know that $y^* = P_K(y^* - \alpha\Lambda^{-1}T(y^*, v^*))$ because of Theorem 2.6. So

$$(I - P_K)(y^* - \alpha\Lambda^{-1}T(y^*, v^*)) = -\alpha\Lambda^{-1}T(y^*, v^*).$$

Using Theorem 2.5 we get: $\forall (y, v) \in V \times U$

$$\begin{aligned}
\langle h'_\alpha(y^*, v^*), (y, v) \rangle &= \alpha^2 \left(T(y^*, v^*), T'(y^*, v^*)(y, v)\right)_{V'} \\
&\quad + \alpha \left(\Lambda^{-1}T(y^*, v^*), y - \alpha\Lambda^{-1}T'(y^*, v^*)(y, v)\right)_V \\
&= \alpha^2 \left(T(y^*, v^*), T'(y^*, v^*)(y, v)\right)_{V'} \\
&\quad - \alpha^2 \left(T(y^*, v^*), T'(y^*, v^*)(y, v)\right)_{V'} + \alpha \left(\Lambda^{-1}T(y^*, v^*), y\right)_V .
\end{aligned}$$

□

Let us write the regularity condition (3.6). It precisely means:

$$\forall t \in \mathbb{R}, \ \exists \lambda \geq 0, \ \exists y \in K \ \text{ such that } \lambda\alpha \langle T(y^*, v^*), y \rangle = t. \tag{3.9}$$

We remark that the above condition does not depend on the set U_{ad}. Nevertheless it is quite difficult to ensure and we think that most of time it not useful since it cannot be fulfilled. Let us give a simple example which is the case of obstacle problem.

Let Ω be a bounded open subset of $\mathbb{R}^n$ $(n \leq 3)$ with a smooth boundary. We set $V = H^1_o(\Omega)$ and $U = L^2(\Omega)$.The isomorphism Λ is equal to $-\Delta + I$ (where Δ is the laplacian operator. We define T as: $T(y, v) = -\Delta y - v - f$ where $f \in H^{-1}(\Omega)$. T is of course a $\mathcal{C}^1$-operator from $H^1_o(\Omega) \times L^2(\Omega)$ to $H^{-1}(\Omega)$.

We set $K = \{y \in V \mid y \geq 0 \text{ a.e. in } \Omega\}$ and $\xi^* = T(y^*, v^*) = -\Delta y^* - v^* - f$. It is a classical result (see [10, 5] for instance) that $T(y^*, v^*) \geq 0$ a.e. and $\langle T(y^*, v^*), y^* \rangle_{V',V} = 0$. Condition (3.9) becomes

$$\forall t \in \mathbb{R}, \ \exists \lambda \geq 0, \ \exists y \geq 0 \ \text{ such that } \lambda\alpha \langle T(y^*, v^*), y \rangle = t.$$

This is obviously impossible for $t < 0$.

Remark 3.5. Lemma 3.4 gives the relation between h_α and T when the pair (y,v) is solution of the variational inequality i.e. $h_\alpha(y,v)=0$. We get precisely $T(y,v)=\frac{1}{\alpha}h'_\alpha(y,v)$.

The previous analysis shows that is hopeless to get "classical" Lagrange multipliers that satisfy an optimality system as (3.7)–(3.8). Indeed, even a quite weak regularity condition cannot be satisfied. One cannot use the general results of mathematical programming theory. So we are going to use other techniques to get what we still will call Lagrange multipliers; however these multipliers will satisfy more restrictive optimality systems.

4. The penalty method

4.1. Penalization of $(\mathcal{P}_\alpha)$

We have underlined, it is hopeless to expect a "classical" optimality system as the previous example shows it; however we are going to prove there exists some "weaker" optimality conditions via a penalization method.

Let be $(y^*,v^*)\in K\times U_{ad}$ a solution of $(\mathcal{P})$, and consider the following problem $(\mathcal{P}_\alpha^n)$ which is a penalization of problem $(\mathcal{P}_\alpha)$.

$$\inf J_n(y,v),\quad \forall y\in K,\ \forall v\in U_{ad} \qquad (\mathcal{P}_\alpha^n)$$

with

$$J_n(y,v) := J(y,v)+nh_\alpha(y,v)+\frac{1}{2}\|y-y^*\|_V^2+\frac{1}{2}\|v-v^*\|_U^2.$$

For a sequence $\varepsilon_n\to+0$ there exists by Ekeland's variational principle an element $(y_n,v_n)\in K\times U_{ad}$ such that

$$J(y_n,v_n)+nh_\alpha(y_n,v_n)+\frac{1}{2}\|y_n-y^*\|_V^2+\frac{1}{2}\|v_n-v^*\|_U^2 \quad\leq\quad \inf\ (\mathcal{P}_\alpha^n)+\varepsilon_n$$

and

$$\begin{aligned}&J(y_n,v_n)+nh_\alpha(y_n,v_n)+\frac{1}{2}\|y_n-y^*\|_V^2+\frac{1}{2}\|v_n-v^*\|_U^2-\varepsilon_n\left\|\begin{pmatrix}y-y_n\\ v-v_n\end{pmatrix}\right\|_{V\times U}\\ &\leq J(y,v)+nh_\alpha(y,v)+\frac{1}{2}\|y-y^*\|_V^2+\frac{1}{2}\|v-v^*\|_U^2\quad\forall(y,v)\in K\times U_{ad}.\end{aligned}\tag{4.1}$$

Theorem 4.1. *Assume $(\mathcal{H})$ and one of the following:*

(1) *Either, $y\mapsto T(y,v)$ is strongly monotone uniformly with respect to $v\in U$, i.e*

$$\exists\nu>0\ \langle T(y,v)-T(z,v),y-z\rangle\geq\frac{\nu}{2}\|z-y\|_V^2\quad\forall y,z\in K,\ \forall v\in U_{ad}\tag{4.2}$$

and $\alpha>\nu^{-1}$.

(2) *Or T is weakly-strongly continuous with respect to both variables y and v i.e.*

$$y_n\rightharpoonup y \text{ and } v_n\rightharpoonup v\Rightarrow T(y_n,v_n)\to T(y,v)\ .\tag{4.3}$$

Then, the sequence (y_n,v_n) strongly converges to (y^*,v^*) in $V\times U$.

Moreover $\lim_{n\to+\infty} n\cdot h_\alpha(y_n,v_n)=0$.

Proof. With $(y,v) = (y^*, v^*) \in K \times U_{ad}$ it follows from (4.1)

$$\begin{aligned} 0 &\leq nh_\alpha(y_n, v_n) + \frac{1}{2}\|y_n - y^*\|_V^2 + \frac{1}{2}\|v_n - v^*\|_U^2 \\ &\leq J(y^*, v^*) - J(y_n, v_n) + \varepsilon_n \left\| \begin{pmatrix} y_n - y^* \\ v_n - v^* \end{pmatrix} \right\|_{V\times U}. \end{aligned} \tag{4.4}$$

If the cost function $J : Y \times U \to \mathbb{R}$ is convex, continuous and Gâteaux-differentiable we obtain

$$\begin{aligned} 0 &\leq nh_\alpha(y_n, v_n) + \frac{1}{2}\|y_n - y^*\|_V^2 + \frac{1}{2}\|v_n - v^*\|_U^2 \\ &\leq \|J'(y^*, v^*)\| \left\| \begin{pmatrix} y_n - y^* \\ v_n - v^* \end{pmatrix} \right\|_{V\times U} + \varepsilon_n \left\| \begin{pmatrix} y_n - y^* \\ v_n - v^* \end{pmatrix} \right\|_{V\times U}. \end{aligned} \tag{4.5}$$

It follows that the sequences $\{v_n\}$ and $\{y_n\}$ are bounded. Therefore there exist a weakly convergent subsequences of $\{v_n\}$ and $\{y_n\}$ (still denoted $\{v_n\}$ and $\{y_n\}$) such that

$$v_n \rightharpoonup \bar{v} \in U_{ad} \quad \text{and} \quad y_n \rightharpoonup \bar{y} \in K. \tag{4.6}$$

Moreover we see that

$$h_\alpha(y_n, v_n) \to 0 \quad \text{as} \quad n \to +\infty. \tag{4.7}$$

Let us detail the two different cases occurring in the theorem.

(1) We use assumption (4.2) (i.e the strong monotonicity of T with respect to y for any v) to prove that y_n strongly converges to $\bar{y}$; we get

$$\begin{aligned} 0 \;\leq\; \frac{\nu}{2}\|y_n - \bar{y}\|^2 &\leq \langle T(y_n, v_n) - T(\bar{y}, v_n), y_n - \bar{y}\rangle \\ 0 \;\leq\; \frac{\alpha\nu}{2}\|y_n - \bar{y}\|^2 &\leq -\alpha\langle T(y_n, v_n), \bar{y} - y_n\rangle - \alpha\langle T(\bar{y}, v_n), y_n - \bar{y}\rangle \\ \frac{(\alpha\nu - 1)}{2}\|y_n - \bar{y}\|^2 &\leq h_\alpha(y_n, v_n) - \alpha\langle T(\bar{y}, v_n), y_n - \bar{y}\rangle. \end{aligned}$$

Moreover we assume that $\alpha > \nu^{-1}$. As the mapping $v \to T(y, v)$ is strongly continuous for any $y \in K$ (assumption (2.3)) $T(\bar{y}, v_n)$ strongly converges to $T(\bar{y}, \bar{v})$. As $h_\alpha(y_n, v_n) \to 0$ as well, we finally get the strong convergence of y_n toward $\bar{y}$.

As the operator T satisfies (2.1) and (2.2), h_α is (strongly) continuous with respect to y and weakly continuous with respect to v because of Theorem 2.7. Then using relation (4.7) we get

$$0 = \lim_{n\to+\infty} h_\alpha(y_n, v_n) \geq \liminf_{n\to+\infty} h_\alpha(y_n, v_n) \geq h_\alpha(\bar{y}, \bar{v}) \geq 0.$$

Thus, the pair $(\bar{y}, \bar{v})$ is feasible for $(\mathcal{P}_\alpha)$.

(2) We assume (4.3) now. As $z_n = P_K(y_n - \alpha\Lambda^{-1}T(y_n, v_n))$ we know that

$$\begin{aligned} \|z_n - y_n\| &= \|P_K(y_n - \alpha\Lambda^{-1}T(y_n, v_n)) - P_K(y_n)\| \\ &\leq \alpha\|T(y_n, v_n)\| \;\leq\; M. \end{aligned}$$

since (y_n, v_n) weakly converges to $(\bar{y}, \bar{v})$ and T is weakly-strongly continuous. Therefore there exists a subsequence of z_n (still denoted in the same way) weakly convergent to some $\bar{z} \in K$. In addition the characterization of the projection operator yields that

$$\big(y_n - \alpha\Lambda^{-1}T(y_n, v_n) - z_n, z - z_n\big) \;\leq\; 0 \quad \forall z \in K.$$

Setting $z = y_n \in K$ we get

$$\frac{1}{2}\|z_n - y_n\|^2 \;\leq\; -\alpha\langle T(y_n, v_n), z_n - y_n\rangle - \frac{1}{2}\|z_n - y_n\|^2 \;=\; h_\alpha(y_n, v_n).$$

So (4.7) gives the strong convergence of $z_n - y_n$ to 0 and $\bar{z} = \bar{y}$. Moreover as

$$(y_n - z_n, z - z_n) \;\leq\; \alpha\langle T(y_n, v_n), z - z_n\rangle \quad \forall z \in K$$

and T is weakly-strongly continuous we finally obtain

$$0 \leq \langle T(\bar{y}, \bar{v}), z - \bar{y}\rangle \quad \forall z \in K.$$

Therefore $(\bar{y}, \bar{v})$ satisfies $h_\alpha(\bar{y}, \bar{v}) = 0$.

So in both cases $(\bar{y}, \bar{v})$ is a feasible element of $(\mathcal{P})$. Then from (4.4) we obtain

$$0 \leq \lim_{n\to+\infty}\left[n \cdot h_\alpha(y_n, v_n) + \frac{1}{2}\|y_n - y^*\|_V^2 + \frac{1}{2}\|v_n - v^*\|_U^2\right] \leq J(y^*, v^*) - J(\bar{y}, \bar{v}) \leq 0$$

Finally

$$y_n \to y^* = \bar{y}\ , \quad v_n \to v^* = \bar{v} \text{ strongly }, \text{ and } n \cdot h_\alpha(y_n, v_n) \to 0. \tag{4.8}$$

□

Now we set first order optimality conditions for problem $(\mathcal{P}_\alpha^n)$ (though the pair (y_n, v_n) is only an "ε_n-minimizer" for this problem). Then we shall pass to the limit as $n \to +\infty$.

By (4.1) it follows with $y = y_n + ts$, $s \in K - y_n$ and $v = v_n + tr$, $r \in U_{ad} - v_n$ and $t > 0$

$$\begin{aligned} J(y_n + ts, v_n + tr) - J(y_n, v_n) + n\,[h_\alpha(y_n + ts, v_n + tr) - h_\alpha(y_n, v_n)] \\ + \frac{1}{2}t^2\|s\|^2 + t\langle\Lambda(y_n - y^*), s\rangle \\ + \frac{1}{2}t^2\|r\|^2 + t\langle\Lambda_U(v_n - v^*), r\rangle + \varepsilon_n t\left\|\begin{pmatrix} s \\ r\end{pmatrix}\right\|_{V\times U} \;\geq\; 0. \end{aligned}$$

Dividing by t and letting tend $t \to 0$, this implies with $s = z - y_n,\ z \in K$, and $r = u - v_n,\ u \in U_{ad}$, the inequality

$$\begin{aligned} &\left\langle J'(y_n, v_n), \begin{pmatrix} z - y_n \\ u - v_n\end{pmatrix}\right\rangle + n\left\langle h'_\alpha(y_n, v_n), \begin{pmatrix} z - y_n \\ u - v_n\end{pmatrix}\right\rangle \\ &+ \left\langle\begin{pmatrix}\Lambda(y_n - y^*) \\ \Lambda_U(v_n - v^*)\end{pmatrix}, \begin{pmatrix} z - y_n \\ u - v_n\end{pmatrix}\right\rangle + \varepsilon_n\left\|\begin{pmatrix} z - y_n \\ u - v_n\end{pmatrix}\right\|_{V\times U} \;\geq\; 0 \end{aligned} \tag{4.9}$$

for all $(z,u) \in K \times U_{ad}$.

As we assumed that the operator $T : V \times U \to V'$ is continuously Fréchet-differentiable, Theorem 2.5 gives, with

$$z_n = P_K(y_n - \alpha\Lambda^{-1}T(y_n,v_n)) \tag{4.10}$$

$$\begin{aligned} \left\langle h'_\alpha(y_n,v_n), \begin{pmatrix} z-y_n \\ u-v_n \end{pmatrix} \right\rangle &= \left\langle \alpha T'(y_n,v_n)\begin{pmatrix} z-y_n \\ u-v_n \end{pmatrix}, y_n - z_n \right\rangle \\ &+ \langle \alpha T(y_n,v_n), z-y_n\rangle + \langle \Lambda(z_n-y_n), z-y_n\rangle. \end{aligned} \tag{4.11}$$

So with (4.9), we get

$$\begin{aligned} &\left\langle J'(y_n,v_n) - n\alpha T'^*(y_n,v_n)(z_n-y_n), \begin{pmatrix} z-y_n \\ u-v_n \end{pmatrix} \right\rangle \\ &+ \langle n\Lambda(z_n-y_n), z-y_n\rangle + n\,\langle \alpha T(y_n,v_n), z-y_n\rangle \\ &+ \left\langle \begin{pmatrix} \Lambda(y_n-y^*) \\ \Lambda_U(v_n-v^*) \end{pmatrix}, \begin{pmatrix} z-y_n \\ u-v_n \end{pmatrix} \right\rangle + \varepsilon_n \left\| \begin{pmatrix} z-y_n \\ u-v_n \end{pmatrix} \right\|_{V\times U} \ \geq\ 0 \end{aligned} \tag{4.12}$$

and for $u = v_n \in U_{ad}$ with $p_n := n(z_n - y_n)$ from (4.12)

$$\begin{aligned} \forall z \in K \quad &\left\langle J'_y(y_n,v_n) - \alpha T'^*_y(y_n,v_n)p_n, z-y_n \right\rangle + n\,\langle \alpha T(y_n,v_n), z-y_n\rangle \\ &+ \langle \Lambda(y_n-y^*), z-y_n\rangle + n\,\langle \Lambda(z_n-y_n), z-y_n\rangle + \varepsilon_n\|z-y_n\| \ \geq\ 0. \end{aligned} \tag{4.13}$$

For $z = y_n \in K$ it follows from (4.12)

$$\begin{aligned} \forall u \in U_{ad} \quad &\left\langle J'_v(y_n,v_n) - \alpha T'^*_v(y_n,v_n)p_n, u-v_n \right\rangle + \langle \Lambda_U(v_n-v^*), u-v_n\rangle \\ &+ \varepsilon_n\|u-v_n\| \ \geq\ 0. \end{aligned} \tag{4.14}$$

4.2. Passage to the limit

We would like to pass to the limit in the previous inequalities. Beginning with relation (4.14) we realize that all the n-quantities are either bounded or convergent except p_n. So we first give an estimation on p_n.

Theorem 4.2. *Assume $(\mathcal{H})$ and one of the following:*

(1) *Either* (4.2)

(2) *Or T'_y is coercive in the following sense:*

$$\begin{aligned} &\exists(\mu,\nu) > 0,\ \forall(y,v) \in K \times U_{ad},\ \forall z \in V \\ &s.t.\ \|z\|_V \geq \mu \ \ ,\ \ \left\langle T'_y(y,v)z, z \right\rangle \geq \frac{\nu}{2}\|z\|^2_V \end{aligned} \tag{4.15}$$

and $\alpha > \nu^{-1}$.

Then, there exists $\kappa > 0$ such that

$$\forall n \in \mathbb{N} \quad \|p_n\|_V \;\leq\; \kappa.$$

Proof. Let us take $z = z_n = P_K(y_n - \alpha\Lambda^{-1}T(y_n, v_n))$ in (4.13); this gives

$$\left\langle J_y'(y_n, v_n) - n\alpha T_y'^{*}(y_n, v_n)(z_n - y_n), z_n - y_n \right\rangle + n\,\langle \alpha T(y_n, v_n), z_n - y_n\rangle$$
$$+\, n\,\langle \Lambda(z_n - y_n), z_n - y_n\rangle + \langle \Lambda(y_n - y^*), z_n - y_n\rangle + \varepsilon_n\|z_n - y_n\| \quad \geq \quad 0.$$

As $h_\alpha(y_n, v_n) = -\alpha\langle T(y_n, v_n), z_n - y_n\rangle - \frac{1}{2}\|z_n - y_n\|^2$ (Theorem 2.3(iii)) we get

$$\begin{aligned} &\langle J_y'(y_n, v_n), z_n - y_n\rangle - nh_\alpha(y_n, v_n) \\ &- \frac{n}{2}\|z_n - y_n\|^2 + n\langle \Lambda(z_n - y_n), z_n - y_n\rangle \\ &+ \langle \Lambda(y_n - y^*), z_n - y_n\rangle + \varepsilon_n\|z_n - y_n\| \quad \geq \quad n\alpha\langle T_y'^{*}(y_n, v_n)(z_n - y_n), z_n - y_n\rangle. \end{aligned} \tag{4.16}$$

(1) The operator T is strongly monotone with respect to y so (see Shi [12])

$$\left\langle T_y'^{*}(y, v)z, z\right\rangle \geq \frac{\nu}{2}\|z\|_V^2 \quad \forall z \in V\ , \forall y \in K,\ \forall v \in U_{ad}.$$

With $h_\alpha(y_n, v_n) \geq 0$ relation (4.16) gives

$$\left\langle J_y'(y_n, v_n), z_n - y_n\right\rangle + \frac{n}{2}\|z_n - y_n\|^2 + \langle \Lambda(y_n - y^*), z_n - y_n\rangle + \varepsilon_n\|z_n - y_n\| \geq n\alpha\frac{\nu}{2}\|z_n - y_n\|^2.$$

So

$$\|J_y'(y_n, v_n)\| + \|y_n - y^*\| + \varepsilon_n \geq \frac{n}{2}(\nu\alpha - 1)\|z_n - y_n\|.$$

As the left-hand side quantities are uniformly bounded with respect to n we get the desired result (analog to Shi [12] and Wenbin and Rubio [14]).

(2) We now assume (4.15). Relation (4.16) gives

$$\|J_y'(y_n, v_n)\| + \|y_n - y^*\| + \varepsilon_n \geq \frac{\alpha\left\langle T_y'(y_n, v_n)(p_n), p_n\right\rangle}{\|p_n\|} - \frac{1}{2}\|p_n\|. \tag{4.17}$$

If p_n is not bounded, one can suppose (up to a subsequence) that $\|p_n\| \to +\infty$ and relation (4.15) is true for n large enough. Relation (4.17) gives

$$\|J_y'(y_n, v_n)\| + \|y_n - y^*\| + \varepsilon_n \geq \frac{\alpha\nu - 1}{2}\;\|p_n\|,$$

and we get a contradiction. $\square$

Then one can extract a subsequence of $\{p_n\}$ (still denoted $\{p_n\}$) weakly convergent towards p^* as $n \to +\infty$, and by (4.14) and $\alpha > \nu^{-1}$ we obtain

$$\left\langle J_v'(y^*, v^*) - \alpha T_v'^{*}(y^*, v^*)\; p^*, u - v^*\right\rangle \quad \geq \quad 0 \quad \forall u \in U_{ad} \tag{4.18}$$

From (4.8) we know that $n \cdot h_\alpha(y_n, v_n) \to 0$, so that (with (2.5))

$$- \langle \alpha T(y_n, v_n), p_n \rangle - \frac{1}{2}\|p_n\|\|z_n - y_n\| \to 0 \quad as \quad n \to +\infty.$$

Since $p_n \rightharpoonup p^*$ and $z_n \to y^*$, it follows that the following equation for the adjoint state p^*

$$\langle T(y^*, v^*), p^* \rangle = 0 \tag{4.19}$$

is fulfilled (one may compare to the results of Shi [12]).

It remains to pass to the limit in relation (4.13). Let us precise this point.

Let $\mathcal{A} : V \times U \to 2^V$ be the following set-valued mapping

$$\mathcal{A}(y,v) = \begin{cases} \{z \in K | \langle \alpha T(y,v) + \Lambda(y_K - y), z - y \rangle \leq 0\} & \text{if } (y,v) \in K \times U_{ad} \\ \emptyset & \text{else.} \end{cases} \tag{4.20}$$

where $y_K = P_K(y - \alpha\Lambda^{-1}T(y,v))$.

For $(y,v) \in K \times U_{ad}$, $y \in \mathcal{A}(y,v)$ so that $\mathcal{A}(y,v) \neq \emptyset$ and $\text{dom}(\mathcal{A}) = K \times U_{ad}$; moreover the set $\mathcal{A}(y,v)$ is convex and closed. If $(y^*, v^*) \in K \times U_{ad}$ is the solution of $(\mathcal{P})$, then $y^* = P_K(y^* - \alpha\Lambda^{-1}T(y^*, v^*))$ yields

$$\mathcal{A}(y^*, v^*) = \{z \in K | \langle T(y^*, v^*), z - y^* \rangle \leq 0, \}.$$

On the other hand we have seen (Theorem 2.6(ii)) that

$$\forall z \in K \quad \langle T(y^*, v^*), z - y^* \rangle \geq 0,$$

so that we get

$$\mathcal{A}(y^*, v^*) = \{z \in K | \, \langle T(y^*, v^*), z - y^* \rangle = 0\}. \tag{4.21}$$

Moreover, one can easily see that relation (4.13) implies that

$$\begin{aligned} \forall z \in \mathcal{A}(y_n, v_n) \quad & \left\langle J_y'(y_n, v_n) - \alpha T_y'^*(y_n, v_n)p_n, z - y_n \right\rangle \\ & + \langle \Lambda(y_n - y^*), z - y_n \rangle + \varepsilon_n \|z - y_n\| \;\geq\; 0. \end{aligned} \tag{4.22}$$

To study the asymptotic behavior of this inequality we need some continuity properties for $\mathcal{A}$.

We recall (see Berge [4] or Aubin-Frankowska [1] for example) that the *inf-limit* of a sequence of non empty subsets $\{A_n\}$ of X (where X is a Banach space) is defined as following:

$$\liminf_{n \to +\infty} A_n = \{x \in X| \quad \exists \{x_n\} : x_n \in A_n \text{ and } x_n \to x\}$$

(Note that it is also called the Kuratowski-Painlevé set convergence). The set-valued mapping $\mathcal{A} : V \times U \to 2^K$ is said to be *lower semi-continuous (l.s.c.)* at $(y^*, v^*) \in K \times U_{ad}$ if

$$\mathcal{A}(y^*, v^*) \subset \liminf_{n \to +\infty} \mathcal{A}(y_n, v_n)$$

holds for every sequence $(y_n, v_n) \in K \times U_{ad}$ with $y_n \to y^*$ and $v_n \to v^*$. Since the set $\mathcal{A}(y^*, v^*)$ is closed, it follows in addition that

$$\mathcal{A}(y^*, v^*) = \liminf_{n\to+\infty} \mathcal{A}(y_n, v_n).$$

Consider now (4.22) and let $n \to +\infty$; then we obtain

$$\forall z \in \liminf_{n\to+\infty} \mathcal{A}(y_n, v_n), \quad \left\langle J'_y(y^*, v^*) - \alpha T'^*_y(y^*, v^*)p^*, z - y^* \right\rangle \ \geq\ 0. \tag{4.23}$$

Finally we get

Theorem 4.3 (Weak Variant). *Suppose that assumptions of* Theorems 4.1 *and* 4.2 *are fulfilled; let (y^*, v^*) be an optimal solution of $(\mathcal{P})$. Then, there exists $q^* = \alpha\, p^* \ \in V$ such that*

$$\forall u \in U_{ad} \quad \left\langle J'_v(y^*, v^*) - T'^*_v(y^*, v^*)\ q^*,\ u - v^* \right\rangle \ \geq\ 0, \tag{4.24}$$

$$\forall z \in \liminf_{(y,v)\to(y^*,v^*)} \mathcal{A}(y, v) \quad \left\langle J'_y(y^*, v^*) - T'^*_y(y^*, v^*)\ q^*,\ z - y^* \right\rangle \ \geq\ 0, \tag{4.25}$$

$$\langle T(y^*, v^*), q^* \rangle = 0. \tag{4.26}$$

If we suppose that $\mathcal{A}$ is l.s.c. at (y^*, v^*) we obtain a strong variant of the previous result.

Corollary 4.4 (Strong Variant). *Suppose that assumptions of* Theorem 4.3 *are fulfilled; let (y^*, v^*) be an optimal solution of $(\mathcal{P})$ and assume that $\mathcal{A}$ is l.s.c. at (y^*, v^*). Then, there exists $q^* \in V$ such that relations* (4.24) *and* (4.26) *are satisfied with*

$$\begin{aligned} \forall z \in K \quad s.t. \quad & \langle T(y^*, v^*), z - y^* \rangle = 0 \\ & \langle J'_y(y^*, v^*) - T'^*_y(y^*, v^*)\ q^*,\ z - y^* \rangle \geq 0. \end{aligned} \tag{4.27}$$

These results are to be compared with those obtained by Barbu [3] where the operator T is defined as in Example 2.1 by

$$\forall z \in V \quad \langle Ay, y - z \rangle + \varphi(y) - \varphi(z) \leq \langle f + Bv, y - z \rangle\,,$$

and A is a linear continuous operator from V to V' and $U_{ad} = U$. The cost functional J is defined as

$$J(y, v) = g(y) + h(v),$$

with appropriate continuity, convexity and coercivity assumptions on g and h (we refer to [3, p.150]). Then, we get:

Theorem 4.5. *Let (y^*, v^*) be an optimal pair for problem $(\mathcal{P})$. Then there exists $p^* \in V$ such that*

$$-A^*p^* - \eta \in \partial g(y^*), \tag{4.28}$$

$$u^* \in \partial h(B^*p^*). \tag{4.29}$$

where η is the weak limit of a sequence $\nabla^2\varphi^\varepsilon(y_\varepsilon)p_\varepsilon$ in V' with $p_\varepsilon \to p^*$ and $y_\varepsilon \to y^*$.

Thus, the multiplier p^* is not completely "known" since it depends on the choice of an approximation φ_ε of φ and on the approximated quantities y_ε and p_ε. This comes from the approximation method to get the system. Instead of a penalization of the whole state (in)equation as we have done (via the h_α function), this author had rather used an approximation φ_ε of the φ function to get an equation approximating the inequation.

In our setting, $\varphi = \mathbf{1}_K$ and g and h are differentiable, but A is not necessarily linear. Anyway, relations (4.28) and (4.29) become

$$-A^* p^* - \eta = \nabla g(y^*) \ , \ u^* = \nabla h(B^* p^*) \ .$$

We can see (and it is a general remark) that informations are less precise with Barbu's approach. Indeed, taking into account general problems do not allow to use properties of particular cases. We shall meet this problem again when we shall investigate the linear obstacle problem in the next section.

5. Applications and Examples

5.1. Comparison with Rubio-Wenbin results

Let us compare the results of the previous section to those of Rubio and Wenbin in [14]. Following these authors, we define the set

$$W = \{\xi \in V \mid \exists \xi_n \in \overline{\bigcup_{\lambda>0} \lambda(K - z_n)},\ \xi_n \to \xi \\ \text{and } \limsup_{n\to+\infty}\ n \langle \alpha T(y_n, v_n) + \Lambda(z_n - y_n), \xi_n \rangle \le 0\}.$$

Since

$$n \langle \alpha T(y_n, v_n) + \Lambda(z_n - y_n), \xi - y_n \rangle = \\ n \langle \alpha T(y_n, v_n) + \Lambda(z_n - y_n), \xi - z_n \rangle + \langle \alpha T(y_n, v_n) + \Lambda(z_n - y_n), p_n \rangle$$

and

$$\lim_{n\to\infty} \langle \alpha T(y_n, v_n) + \Lambda(z_n - y_n), p_n \rangle = 0,$$

it is easy to see that the passage to the limit in relation (4.13) gives

$$\forall \xi \in W \quad \left\langle J'_y(y^*, v^*) - T'^*_y(y^*, v^*)\ q^*, \xi \right\rangle \ge 0. \tag{5.1}$$

Proposition 5.1. *We have*

$$\overline{\bigcup_{\lambda>0} \lambda[(K - z_n) \cap (z_n - K)]} \subset W.$$

Moreover, if the map $y \mapsto (K - y) \cap (y - K)$ is lower semi-continuous at y^*,

$$\overline{\bigcup_{\lambda>0} \lambda[(K - y^*) \cap (y^* - K)]} \subset W.$$

Proof. Let ξ be in $\overline{\bigcup_{\lambda>0}\lambda[(K-z_n)\cap(z_n-K)]}$ and $\xi_n\in\lambda_n[(K-z_n)\cap(z_n-K)]$ such that $\xi_n\to\xi$. As $\xi_n\in\overline{\bigcup_{\lambda>0}\lambda(K-z_n)}$ it remains to show that

$$\limsup_{n\to+\infty} n\,\langle\alpha T(y_n,v_n)+\Lambda(z_n-y_n),\xi_n\rangle\le 0. \tag{5.2}$$

As $z_n=P_K(y_n-\alpha\Lambda^{-1}T(y_n,v_n))$, the characterization of the projection on K yields

$$\forall z\in K\quad \langle\alpha T(y_n,v_n)+\Lambda(z_n-y_n),z_n-z\rangle\le 0.$$

As $\xi_n\in\lambda_n(z_n-K)$ we may choose $z=z_n-\frac{\xi_n}{\lambda_n}\in K$ in the previous relation to get

$$\langle\alpha T(y_n,v_n)+\Lambda(z_n-y_n),\xi_n\rangle\le 0.$$

Then it is clear that (5.2) is satisfied and $\xi\in W$.

The end of the proposition is obvious since $z_n\to y^*$. □

This proposition means that our results include those of Rubio and Wenbin [14]. Therefore, they can be applied in all the cases given as examples in the cited paper.

5.2. Lower-semicontinuity criteria for $\mathcal{A}$

We are going to precise a little more the set-valued application $\mathcal{A}$ and give some cases where one has the desired lower semi-continuity. To simplify the presentation we define the functional $\Phi: V\times U\to V$ as

$$\Phi(y,v)=\frac{1}{\alpha}\left([P_K(y-\alpha\Lambda^{-1}T(y,v))]-[y-\alpha\Lambda^{-1}T(y,v)]\right),$$

so that

$$\mathcal{A}(y,v)=\{z\in K|\,(\Phi(y,v),z-y)_V\le 0\},$$

if $(y,v)\in K\times U_{ad}$. Note that Φ is continuous and $\Phi(y^*,v^*)=\Lambda^{-1}T(y^*,v^*)$.

Let us call F and $G: V\times U\to V$ the following set-valued applications

$$F(y,v)=K-y,\quad\text{and}\quad G(y,v)=\{\ z\in V\ |\ (\Phi(y,v),z)_V\le 0\},$$

so that

$$\forall(y,v)\in K\times U_{ad},\quad \mathcal{A}(y,v)=[F(y,v)\cap G(y,v)]+y.$$

Lemma 5.2. *The set-valued applications F and G have convex and closed values. Moreover they are lower semi-continuous at any $(y,v)\in K\times U_{ad}$ such that $\Phi(y,v)\neq 0$.*

Proof. The first assertion is obvious. Let us show now the lower semi-continuity of F and G.

Let be (y,v) in $V\times U$ and define a sequence (y_n,v_n) of $V\times U$ converging to (y,v); choose $z\in K-y$. Then, it is clear that the element $z_n=z+y-y_n$ converges to z and belongs to $K-y_n$. Therefore F is lower semi-continuous at (y,v).

The lower semi-continuity of G is not so obvious. Let be $z\in G(y,v)$.

- If $(\Phi(y,v),z)_V < 0$ then the continuity of Φ implies $(\Phi(y_n,v_n),z)_V < 0$ for n large enough so that one can choose $z_n = z$.
- If $(\Phi(y,v),z)_V = 0$ then $\alpha_n = (\Phi(y_n,v_n),z)_V$ converges to 0. If $\Phi(y_n,v_n) \neq 0$ we set

$$z_n = z - |\alpha_n| \frac{\Phi(y_n,v_n)}{||\Phi(y_n,v_n)||^2}$$

and $z_n = z$ otherwise. So $(\Phi(y_n,v_n),z_n)_V$ is either equal to 0 or to $\alpha_n - |\alpha_n|$: it is less than 0 in any case; so $z_n \in G(y_n,v_n)$.

If $\Phi(y,v) \neq 0$ then $\Phi(y_n,v_n) \neq 0$ for n large enough and $\dfrac{\Phi(y_n,v_n)}{||\Phi(y_n,v_n)||^2}$ remains bounded. So z_n converges to z in V and the lower semi-continuity of G is proven in this case. □

Now we may conclude using a result of Penot [11, Proposition 2.3 and Corollary 3.3] on the persistence under intersection that we recall:

Proposition 5.3. *Let F and G be two convex-valued multifunctions from $V \times U$ into V which are l.s.c. at (y^*,v^*) and assume that F has a convex graph, is closed valued and that int $F(y^*,v^*)$ is non empty.*

Then, if $G(y^*,v^*) \cap \operatorname{int} F(y^*,v^*)$ is nonempty, $F \cap G$ is l.s.c. at (y^*,v^*).

Before we apply this result, we may notice that it is "symmetric" and one could replace F by G (and conversely) in the previous proposition. However, in our case

$$F(y^*,v^*) \cap \operatorname{int} G(y^*,v^*) = \{z \in K | \, \langle T(y^*,v^*), z - y^* \rangle < 0\},$$

is always empty. This justifies the choice of F and G.

We may now enounce the following

Theorem 5.4. *Suppose that assumptions of* Theorem 4.3 *are fulfilled and assume that K has a nonempty interior; let (y^*,v^*) be an optimal solution. Then, there exists $q^* \in V$ such that relations* (4.24) *and* (4.26) *are satisfied with*

- *either $T(y^*,v^*) = 0$,*
- *or*

$$\begin{aligned} &\forall z \in K \quad s.t. \;\; \langle T(y^*,v^*), z - y^* \rangle = 0 \\ &\left\langle J'_y(y^*,v^*) - T'^*_y(y^*,v^*)\, q^*, \;\; z - y^* \right\rangle \geq 0. \end{aligned} \tag{5.3}$$

Proof. If $T(y^*,v^*) \neq 0$, then $\Phi(y^*,v^*) \neq 0$ and Lemma 5.2 gives the lower semi-continuity of F and G at (y^*,v^*). Moreover F is closed-valued and its graph is convex since K and U_{ad} are convex. In addition if the interior of K is nonempty then the interior of $F(y^*,v^*) = K - y^*$ is nonempty as well. So Proposition 5.3 gives the lower semi-continuity of $F \cap G$ at (y^*,v^*). It is easy to conclude that $\mathcal{A}$ is lsc at (y^*,v^*) as well. □

Remark 5.5 (Finite Dimensional Case). As usual, hypothesis are simplified and much easy to ensure in the Finite Dimensional Case. As there is no difference between the strong and the weak convergence, the continuity of T yields the strong continuity.

On the other hand, one can use the previous results easily since convex sets with nonempty interiors are more easy to describe in finite dimensional spaces. This in particular the case for the obstacle problem where K is a convex-cone of the type

$$K = \{ y \in \mathbb{R}^n \mid y \geq \psi \} .$$

This set has a nonempty interior in $\mathbb{R}^n$ (but unfortunately the interior of such a set is empty in the infinite dimensional space $L^2(\Omega)$ for example).

5.3. The Linear Obstacle Problem

Let us present the linear obstacle problem case as given in Mignot-Puel [10] as an example.

Let Ω be an open, bounded subset of $\mathbb{R}^n$ ($n \leq 3$) with a smooth boundary $\partial\Omega$. We set $V = H^1_o(\Omega)$, $U = L^2(\Omega)$ and $T(y,v) = Ay - v - f$ where A is the continuous linear operator from $H^1_o(\Omega)$ to $H^{-1}(\Omega)$ defined by

$$\begin{gathered} Ay = -\sum_{i,j=1}^{n} \partial_{x_i}(a_{ij}(x)\partial_{x_j} y) + a_0(x) y \quad \text{with} \\ a_{ij}, a_0 \in \mathcal{C}^2(\bar{\Omega}) \;\; \text{for} \;\; i,j = 1, \dots, n, \;\; \inf\{a_0(x) \mid x \in \bar{\Omega}\} > 0 \\ \sum_{ij=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \delta \sum_{i=1}^{n} \xi_i^2 \;, \forall x \in \bar{\Omega}, \forall \xi \in \mathbb{R}^n, \delta > 0, \end{gathered} \tag{5.4}$$

and $f \in L^2(\Omega)$. The compactness of the injection of $H^1_o(\Omega)$ in $L^2(\Omega)$ implies that (2.2) is satisfied. The linearity of A gives (2.1). Moreover is we choose

$$J(y,v) = \frac{1}{2}\left[\|y - y_d\|^2_{L^2(\Omega)} + \rho \|v - v_d\|^2_{L^2(\Omega)} \right] ,$$

where $y_d \in L^2(\Omega)$, $u_d \in L^2(\Omega)$ and $\rho > 0$, assumption (2.3) is fulfilled as well. So $(\mathcal{H})$ is satisfied. Moreover, the $H^1(\Omega)$-ellipticity of A yields (4.2).

Let us consider

$$K = \{ y \in H^1_o(\Omega) \mid y \geq \; \varphi \geq 0 \text{ a.e. in } \Omega \} , \tag{5.5}$$

where $\varphi \in H^1_o(\Omega)$. This set is a non empty, closed, convex subset of $H^1_o(\Omega)$. Unfortunately, the $H^1(\Omega)$-interior of K is empty except for $n = 1$. So we would like to choose another suitable state-space, using the regularity properties of A which is an isomorphism from $H^2(\Omega) \cap H^1_o(\Omega)$ to $L^2(\Omega)$.

So, we set from now $V = H^2(\Omega) \cap H^1_o(\Omega)$ which is an Hilbert-space continuously embedded in $\mathcal{C}^o(\Omega)$, since $n \leq 3$. As the interior of K is non empty for the L^∞- norm, it is nonempty as well for the V-norm. Moreover, assumption $(\mathcal{H})$ is still fulfilled.

Nevertheless it remains a difficulty: A is (generally) no longer V-elliptic. We may conclude however. We precisely need the results of Theorems 4.1 and 4.2. As relation (4.2) seems to be impossible to ensure, let us focus on (4.3). The compactness of the injection of $L^2(\Omega)$ in $H^{-1}(\Omega)$ (and the continuity of the injection of $H^{-1}(\Omega)$ in $(H^2(\Omega) \cap H^1_o(\Omega))'$) yields this property immediately. So results of Theorem 4.1 are still valid. In Theorem 4.2 we have seen that p_n could be bounded. We may prove this also if $U_{ad} = L^2(\Omega)$:

Lemma 5.6. *If* $U_{ad} = L^2(\Omega)$, *then the result of* Theorem 4.2 *is still valid.*

Proof. If $U_{ad} = L^2(\Omega)$, relation (4.14) with $u = v_n + v$ gives

$$\forall v \in L^2(\Omega) \quad (\rho(v_n - v_d) + v_n - v^*, v)_{L^2(\Omega)} - \alpha \langle p_n, v\rangle_{V,V'} + \varepsilon_n \|v\|_{L^2(\Omega)} \geq \quad 0.$$

As the n-quantities (except p_n) are bounded, this implies that p_n is bounded in $L^2(\Omega)$ independently of n and therefore weakly converges to p^* in $L^2(\Omega)$. $\square$

We can conclude that Corollary 4.4 may be applied. Let us precise the notations: we set

$$\xi^* = Ay^* - f - v^* \ \text{ and } \ K_{y^*} = \{\ z \in K - y^* \ \mid \ (\xi^*, z)_{L^2(\Omega)} = 0\ \}\ .$$

We may note that $(\xi^*, y^*) = 0$ (this is a genuine property of the obstacle problem) so that we get $K_{y^*} = K \cap (y^*)^\perp$. This gives

Theorem 5.7. *Assume* $U_{ad} = L^2(\Omega)$ *and let* (y^*, v^*) *be an optimal solution of* $(\mathcal{P})$*. Then, there exists* $q^* \in L^2(\Omega)$ *such that*

$$q^* + \rho(v^* - v_d) = 0\ , \tag{5.6}$$

Either

$$Ay^* - v^* - f = 0\ , \tag{5.7}$$

or

$$(\ \xi^*,\ q^*)_{L^2(\Omega)} = 0\ , \tag{5.8}$$

$$\forall z \in K_{y^*} \qquad (y^* - y_d - A^*\ q^*, z)_{L^2(\Omega)} \geq\ 0. \tag{5.9}$$

Let us compare this result with the one of Mignot and Puel [10]. Let us set

$$S_{y^*} = \{\ z \in L^2(\Omega)\ \mid\ (\xi^*, z)_{L^2(\Omega)} = 0 \ \text{ and } z \geq 0 \text{ on } \{\ y^* = 0\ \}\}\ .$$

In [10], these authors obtain the following

Theorem 5.8. *Assume* $U_{ad} = L^2(\Omega)$ *and let* (y^*, v^*) *be an optimal solution of* $(\mathcal{P})$*. Then, there exists* $q^* \in L^2(\Omega)$ *such that* (5.6) *is satisfied and*

$$(\ \xi^*,\ q^*)_{L^2(\Omega)} = 0\ ,\ \textit{and } q^* \geq 0 \textit{ on } \{\ y^* = 0\ \}\ , \tag{5.10}$$

$$\forall z \in S_{y^*} \qquad (y^* - y_d - A^*\ q^*, z)_{L^2(\Omega)} \geq\ 0. \tag{5.11}$$

As $K_{y^*} \subset S_{y^*}$ we see that we have lost informations between (5.9) and (5.11). We cannot avoid it because of the (general) method we have used. From the very beginning (in the penalized system) we have been considering elements of K; so it is hopeless to get an optimality system dealing with elements which would not belong to K. Another lack of information concerns q^*. We have lost the property that $q^* \geq 0$ on $\{\ y^* = 0\ \}$. Once again, this comes from the fact that we have used a quite general method for quite general problems and we have not been taking into account all theparticular properties of the *linear* obstacle problem.

Remark 5.9. Though the exact computation of the h_α function is not useful, since it does not appear in the final optimality system, we may precise it in the linear case, for $T(y,v) = -\Delta y - f - v$ for instance. Of course, it depends on the norm of the space V. If $V = H^1_o(\Omega)$ is endowed with the norm $\|y\|_{1,o} = \|\nabla y\|_{L^2(\Omega)}$ we see that $\Lambda = -\Delta = T'_y$. So, using Theorem 2.3(ii), we get $\Lambda^{-1}T(y,v) = y - (-\Delta^{-1})(f+v)$ and

$$h_\alpha(y,v) = \frac{1}{2}\left[\alpha^2 \|y - (-\Delta^{-1})(f+v))\|^2_{1,o} - d^2_K((1-\alpha)y + \alpha(-\Delta^{-1})(f+v))\ \right].$$

Let us set $y_\alpha = (1-\alpha)y + \alpha(-\Delta^{-1})(f+v))$ and compute $d^2_K(y_\alpha) = \|y_\alpha - y_K\|^2_{1,o}$ where y_K is the $H^1_o(\Omega)$-projected element of y_α on K. A classical calculus shows that y_K is the solution of the following obstacle problem

$$\begin{aligned} &\int_\Omega \nabla y_K \nabla(z - y_K)\ dx \geq \int_\Omega f_\alpha\ (z - y_K) dx, \quad \forall z \in K \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad y_K \in K, \end{aligned} \tag{5.12}$$

where $f_\alpha = -\Delta y_\alpha = -\Delta y - \alpha(-\Delta y - f - v)$. Finally

$$h_\alpha(y,v) = \frac{1}{2}\left[\|y_\alpha - y\|^2_{1,o} - \|y_\alpha - y_K\|^2_{1,o}\ \right] = \int_\Omega [\nabla(y_\alpha - y)]^2 - [\nabla(y_\alpha - y_K)]^2\ dx.$$

References

[1] J.-P. Aubin, H. Frankowska: Set-valued Analysis, Birkhäuser, Basel, 1990.

[2] V. Barbu: Optimal Control of Variational Inequalities, Research Notes in Mathematics 100, Pitman, Boston, 1984.

[3] V. Barbu: Analysis and Control of Non Linear Infinite Dimensional Systems, Math. in Science and Engineering 190, Academic Press, 1993.

[4] C. Berge: Espaces Topologiques et Fonctions Multivoques, Dunod, Paris, 1959.

[5] M. Bergounioux: Optimal control of an obstacle problem, Applied Mathematics and Optimization 36 (1997) 147–172.

[6] M. Bergounioux: Optimal control of problems governed by abstract elliptic variational inequalities with state constraints, Siam Journal on Control and Optimization 36(1) (1998) 273–289.

[7] F. Clarke: Optimization and Nonsmooth Analysis, Classics in Applied Mathematics 5, SIAM, 1990.

[8] I. Ekeland, R. Temam: Analyse Convexe et Problèmes Variationnels, Dunod-Gauthier-Villars, Paris, 1974.

[9] F. Mignot: Contrôle dans les inéquations variationnelles elliptiques, Journal of Functional Analysis 22(2) (1976) 130–185.

[10] F. Mignot, J. P. Puel: Optimal control in some variational inequalities, SIAM Journal on Control and Optimization 22(3) (1984) 466–476.

[11] J. P. Penot: Preservation of persistence and stability under intersections and operations, Part 1: Persistence, Journal of Optimization Theory and Applications 79(3) (1993) 525–550.

[12] S. Shi: Optimal control of strongly monotone variational inequalities, SIAM Journal on Control and Optimization 25 (1988) 274–290.

[13] S. Shi: Erratum: Optimal control of strongly monotone variational inequalities, SIAM Journal on Control and Optimization 28 (1990) 243–249.

[14] L. Wenbin, J. E. Rubio: Optimality conditions for strongly monotone variational inequalities, Applied Mathematics and Optimization 27(3) (1993) 291–312.

[15] E. Zeidler: Nonlinear Functional Analysis and Its Applications, Vol. IIB: Nonlinear Monotone Operators, Springer, Berlin, 1992.

[16] J. Zowe, S. Kurcyusz: Regularity and stability for the mathematical programming problem in Banach spaces, Applied Mathematics and Optimization 5 (1979) 49–62.