Journal of Convex Analysis
Volume 5 (1998), No. 2, 397–403

Homogeneous Functions and Conjugacy

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Received January 9, 1997
Revised manuscript received December 24, 1997

(Positively) homogeneous functions play a special role in the Legendre-Fenchel duality. The Legendre-Fenchel conjugate of a p-homogeneous function is a q-homogeneous function with $1/p+1/q=1$. Further properties of the function, its conjugate, and of the maximizers are also derived. The same result also holds in the Max-Prod and Min-Max algebra for the analogue of the Legendre-Fenchel transform.

Keywords: Legendre-Fenchel duality, homogeneous functions

1. Introduction

The Legendre-Fenchel transform is a fundamental tool in Convex Analysis (the analogue in the MIN-PLUS algebra of the Laplace transform in the PLUS-PROD algebra (cf. e.g. [2])). For the interested reader, various properties of f^*, the Legendre-Fenchel transform of f, also called the conjugate of f, are presented in [1] and more recently in [9] (see also the use of an appropriate "scalar product" for the Min-Plus Analysis, in the recent work of Gondran [7]). For instance it is known that *quadratic* functions are transformed into quadratic functions. However, not only quadratic, but in fact (positively) homogeneous functions play a special role.

In this note, the class of (positively) homogeneous functions is shown to be globally invariant. More precisely, the p-homogeneous functions are Legendre-Fenchel dual to the q-homogeneous functions and vice-versa (with $1/p+1/q=1$). In other words, the exponent of the conjugate is the conjugate of the exponent.

Under additional assumptions, other properties are also derived for the function, its conjugate as well as for the "maximizers".

In addition, the same result also holds in other algebras such as the Max-Prod and the Min-Max algebra, for the analogue of the Legendre-Fenchel transform. This duality property of homogeneous functions is specific to the "sup" (or "inf") operator.

2. Legendre-Fenchel transform of homogeneous functions

With $0 \neq p \in R$, let $f : R^n \to \overline{R}$ be a (positively) homogeneous function of degree p (in short, p-homogeneous), i.e., for every $\lambda > 0$ and $x \in R^n$, $f(\lambda x) = \lambda^p f(x)$. One assumes that $f(x) \neq -\infty$ for all $x \in R^n$ and that f is strict (or proper, or non improper), i.e. $\emptyset \neq \mathrm{Dom}(f) := \{x \in R^n | f(x) < +\infty\}$.

ISSN 0944-6532 / \$ 2.50

2.1. The class of homogeneous functions is invariant

The Legendre-Fenchel transform $f^* : R^n \to \overline{R}$ of the function f is given by:

$$f^*(y) := \sup_x \,[\langle y, x\rangle - f(x)], \quad y \in R^n \tag{2.1}$$

Theorem 2.1. *Let f be strict and p-homogeneous. Then, the Legendre-Fenchel transform f^* of f, is q-homogeneous with $1/p + 1/q = 1$.*

Proof. Let $y \in \mathrm{Dom}(f^*)$ and let $1/p + 1/q = 1$.

$$\lambda^q f^*(y) = \sup_x[\langle \lambda y, \lambda^{q-1}x\rangle - \lambda^q f(x)] = \sup_x[\langle \lambda y, \lambda^{q-1}x\rangle - f(\lambda^{q-1}x)],$$

since $f(\lambda^{q-1}x) = \lambda^{p(q-1)}f(x) = \lambda^q f(x)$. Therefore, with the change of variable $z := \lambda^{q-1}x$ we finally get

$$\lambda^q f^*(y) \;=\; \sup_z[\langle \lambda y, z\rangle - f(z)] \;=\; f^*(\lambda y),$$

the desired result. □

Note that p and q need not be nonnegative. Theorem 2.1 could also be proved using standard calculus rules for the conjugates of $f(\lambda.)$ and $\lambda f(.)$, as found in e.g. [9] and [10].

Example 2.2. Let $f : R \to \overline{R}$ with $f(x) := 1/x$ for $x > 0$, $+\infty$ otherwise; one may easily check that f is (-1)-homogeneous (on $(-\infty, 0]$ we also have $f(\lambda x) = +\infty = \lambda^{-1} \times +\infty$ since $\lambda > 0$).

It is easily shown that $f^*(y) = +\infty$ on $(0, +\infty)$ and $f^*(y) = -2\sqrt{-y}$ on $(-\infty, 0]$, i.e. f^* is $(1/2)$-homogeneous, and $-1 + 2 = 1$.

Another example of homogeneous functions is $f(x) := \inf_z\{g(z) | Az = x,\ z \geq 0\}$ where $A : R^n \to R^m$ is a linear mapping and g is p-homogeneous.

Then f is p-homogeneous since $f(\lambda x) = \inf_{z/\lambda}\{\lambda^p g(z/\lambda) | Az/\lambda = x,\ z \geq 0\} = \lambda^p f(x)$. Therefore, from Theorem 2.1, f^* is q-homogeneous.

Remark 2.3. One is often interested in the transform of $x \mapsto f(x - x_0)$ (translation). If f is p-homogeneous the conjugate of $x \mapsto f(x - x_0)$ is $y \mapsto \langle y, x_0\rangle + f^*(y)$ with f^* q-homogeneous.

2.2. Further properties

We now investigate some properties of f, its conjugate f^*, and the maximizers in (2.1) when f is p-homogeneous. Let $x(y)$ denote a maximizer in (2.1) when it exists.

Lemma 2.4. *Let f be p-homogeneous and for $y \in \mathrm{Dom}(f^*)$ (resp. $x \in \mathrm{Dom}(f^{**})$), let $x(y)$ (resp. $y(x)$) be a maximizer (when it exists) in the definition of $f^*(y)$ (resp. $f^{**}(x)$). Then:*

(a) $f^*(y) = (p-1)f(x(y))$; $f^{**}(x) = (q-1)f^*(y(x))$.

(b) *If $x(y)$ is a maximizer for $y \in \mathrm{Dom}(f^*)$, then $x(\lambda y) := \lambda^{q-1}x(y)$ is a maximizer for λy. Similarly, if $y(x)$ is a maximizer for $x \in \mathrm{int}(\mathrm{Dom}(f^{**}))$, then $y(\lambda x) := \lambda^{p-1}y(x)$ is a maximizer for λx, i.e. the maximizers are $(q-1)$-homogeneous and $(p-1)$-homogeneous respectively.*

Proof. (a) Let $y \in \mathrm{Dom}(f^*)$ and assume that $x(y)$ is a maximizer in (2.1). Let $df(x;u)$ denote the directional derivative of f at x in the direction u, when it exists. As f is p-homogeneous, for $u := x(y)$ we immediately have

$$df(x(y);x(y)) = \lim_{\lambda \downarrow 0} \left[\lambda^{-1} f(x(y)(1+\lambda)) - f(x(y))\right] = pf(x(y)). \tag{2.2}$$

This property is a weak version of the famous Euler's identity for continuously differentiable functions ($pf(x) = \langle \nabla f(x), x\rangle$).

Hence, as $x(y)$ is a maximizer, the directional derivatives of $\langle y, .\rangle - f(.)$ at $x(y)$ in the directions $u := x(y)$ and $u := -x(y)$ must be nonpositive, i.e.

$$\langle y, x(y)\rangle - pf(x(y)) \leq 0 \text{ and } \langle y, -x(y)\rangle + pf(x(y)) \leq 0,$$

i.e. $\langle y, x(y)\rangle = pf(x(y)$. But from $f^*(y) = \langle y, x(y)\rangle - f(x(y))$ we immediately deduce $f^*(y) = (p-1)f(x(y))$.

Using similar arguments with now f^* q-homogeneous, it follows that $f^{**}(x) = (q-1)f^*(y(x))$.

Note that if f is proper, convex and lower semi-continuous, then $f^{**} \equiv f$ (see e.g. [9]). Hence, from (a)

$$f^{**}(x) = f(x) = (p-1)(q-1)f(x(y(x))) = f(x(y(x))),$$

since if $1/p + 1/q = 1$ then $(p-1)(q-1) = 1$.

To get (b) observe that if $x(y)$ is a maximizer, then

$$f^*(\lambda y) = \lambda^q f^*(y) = \langle \lambda y, \lambda^{q-1}x(y)\rangle - f(\lambda^{q-1}x(y)),$$

so that $\lambda^{q-1}x(y)$ is a maximizer for λy, or in other words, $x(\lambda y) := \lambda^{q-1}x(y)$ is a maximizer for λy. With a similar argument, $y(\lambda x) := \lambda^{p-1}y(x)$ is a maximizer for λx. $\square$

Finally, let $f_1 \overset{+}{\vee} f_2$ denote the inf-convolution of f_1 and f_2, i.e. $f_1 \overset{+}{\vee} f_2(x) := \inf_{x_1,x_2} \{f_1(x_1) + f_2(x_2) |\ x_1 + x_2 = x\}$ (see for instance [9]). An interesting property of the Legendre-Fenchel transform is to replace an inf-convolution by a sum. We also get

Corollary 2.5. *Let $f_1, f_2 : R^n \to \overline{R}$ (not identically $+\infty$) be p-homogeneous and satisfy $\mathrm{Dom}(f_1^*) \cap \mathrm{Dom}(f_2^*) \neq \emptyset$. Then, $f_1 \overset{+}{\vee} f_2$ is p-homogeneous. In addition, $(f_1 \overset{+}{\vee} f_2)^* = f_1^* + f_2^*$ is q-homogeneous, with $1/p + 1/q = 1$.*

Proof. We have

$$\begin{aligned} f_1 \overset{+}{\vee} f_2(\lambda x) &= \inf_{x_1/\lambda,\ x_2/\lambda} \{\lambda^p (f_1(x_1/\lambda) + f(x_2/\lambda)) |\ (x_1 + x_2)/\lambda = x\} \\ &= \lambda^p (f_1 \overset{+}{\vee} f_2)(x), \end{aligned}$$

i.e. $f_1 \overset{+}{\vee} f_2$ is p-homogeneous. That $(f_1 \overset{+}{\vee} f_2)^* = f_1^* + f_2^*$ is immediate from Corollary 2.1.3 in e.g. [9], and the result follows from Theorem 2.1. $\square$

Now, as in [7], [9], consider the "scalar product" of two functions $f, g : R^n \to \overline{R}$, given by

$$(f, g) := \inf_x \{f(x) + g(x)\} \tag{2.3}$$

and consider the case where $f(x) := \tilde{f}(x - a)$, $g(x) := \tilde{g}(x - b)$ and both $\tilde{f}$ and $\tilde{g}$ are (positively) homogeneous.

Lemma 2.6. *Let $\tilde{f}$ (resp. $\tilde{g}$) be p-homogeneous (resp. q-homogeneous) convex lower-semicontinuous. Let $f(x) := \tilde{f}(x - a)$ and $g(x) := \tilde{g}(x - b)$ be such that:*

(a) $\mathrm{Dom}(f) \cap \mathrm{Dom}(g) \neq \emptyset$ *and* $0 \in \mathrm{ri}(\mathrm{Dom}(f) - \mathrm{Dom}(g))$
(b) (f, g) *is finite.*

Then:

$$-(f, g) = \min_y \{f^*(y) + g^*(-y)\} = f^*(\bar{y}) + g^*(-\bar{y}) \tag{2.4}$$

and

$$f^*(\bar{y}) + g^*(-\bar{y}) + (p - 1)\tilde{f}^*(\bar{y}) + (q - 1)\tilde{g}^*(-\bar{y}) = 0 \tag{2.5}$$

or equivalently,

$$\langle \bar{y}, a\rangle + p\tilde{f}^*(\bar{y}) = -(\langle -\bar{y}, b\rangle + q\tilde{g}^*(-\bar{y})) \tag{2.6}$$

for every minimizer $\bar{y}$ where both f^* and g^* have directional derivatives.

In addition, if $p = q$ then

$$(f, g) = (p - 1)(\tilde{f}^*(\bar{y}) + \tilde{g}^*(-\bar{y})) \tag{2.7}$$

Proof. The first equality in (2.4) is the Fenchel's duality Theorem (cf. [9] p. 63). Now, let $\bar{y}$ be a minimizer and assume that both f^* and g^* have directional derivatives at $\bar{y}$ and $-\bar{y}$, denoted by $df^*(\bar{y}; u)$ (resp. $dg^*(-\bar{y}; u)$).

From the necessary conditions of optimality we must have

$$df^*(\bar{y}; \bar{y}) + dg^*(-\bar{y}; -\bar{y}) \geq 0 \ \text{ and } df^*(\bar{y}; -\bar{y}) + dg^*(-\bar{y}; +\bar{y}) \geq 0. \tag{2.8}$$

In addition, $f^*(y) = \langle y, a\rangle + \tilde{f}^*(y)\ \forall y$ and $g^*(y) = \langle y, b\rangle + \tilde{g}^*(y)\ \forall y$. As $\tilde{f}$ is p-homogeneous, from (2.2) we get

$$\langle \bar{y}, a - b\rangle + p\tilde{f}^*(\bar{y}) + q\tilde{g}^*(-\bar{y}) \geq 0$$

and

$$\langle \bar{y}, -a + b\rangle - p\tilde{f}^*(\bar{y}) - q\tilde{g}^*(-\bar{y}) \geq 0,$$

which yields $\langle \bar{y}, a - b\rangle + p\tilde{f}^*(\bar{y}) + q\tilde{g}^*(-\bar{y}) = 0$, i.e. (2.5).

From (2.5) and $-(f, g) = f^*(\bar{y}) + g^*(-\bar{y})$, if $p = q$ we immediately get (2.7). □

Thus, computing (f, g) reduces to solving the equation (2.5) or (2.6).

3. Extension to other algebras

It has been known for some time that many problems that are *non-linear* when formulated in the standard $(\overline{R}, +, \times)$ (or PLUS-PROD) algebra, become *linear* when formulated in some appropriate algebra. In some optimization problems, the semi-ring algebraic structures $(\overline{R}, \min, +)$, $(\overline{R}, \min, \times)$, or $(\overline{R}, \min, \max)$, are particularly useful and the interested reader is referred to e.g. [2], [5], [6], [11] and [7] for recent results along these lines.

For instance, the change of algebra (PLUS-PROD) $\rightarrow$ (MIN-PLUS) permits to establish an interesting correspondence between some results in Probability and Optimal control as shown in [2] and [5], [6]. The Legendre-Fenchel transform is the analogue in the MIN-PLUS algebra of the Laplace Transform in the PLUS-PROD algebra. If we formally replace $\int_R$ by $\inf_x$, i.e. using $\min(a, b)$ instead of $a + b$, one recognizes the exponential of the Legendre-Fenchel transform of $\log f$. Note in passing that as Legendre-Fenchel transform changes the quadratics into quadratics, Laplace transform changes exponentials of a quadratic into exponentials of a quadratic. However, the Laplace conjugate of a p-homogeneous function (of a real variable) is $-(p+1)$-homogeneous since

$$\mathcal{L}f(\lambda y) = \int e^{-y(\lambda x)}\lambda^{-p} f(\lambda x)\lambda^{-1} d(\lambda x) = \lambda^{-(p+1)}\mathcal{L}(f)(y).$$

Thus, although the class of homogeneous functions is globally invariant under the Laplace transform, q-homogeneous functions are not Laplace-dual of the p-homogeneous functions when $1/p + 1/q = 1$.

Perhaps less known is the Bellman-Karush, or *Maximum transform* (cf. [3], [4]). It is particularly useful in dynamic optimization problems with multiplicative (positive) costs. For a function $f : R \rightarrow R$, let

$$f \mapsto f^*(y) := \sup_x [e^{-yx} f(x)] \tag{3.1}$$

be its Bellman-Karush conjugate. Again, formally, this transform can be seen as the analogue of the Laplace Transform, but now in the MAX-PROD algebra (i.e. the "+" is now replaced by the "max" so that $\int_R$ becomes $\sup_x$). As for the Laplace transform, the Bellman-Karush transform replaces an inf-convolution by a product, i.e.

$$h(x) := \sup_z \{f(z).g(x-z)\} \;\mapsto\; h^*(y) = f^*(y).g^*(y).$$

Another algebra of interest is the MIN-MAX algebra (e.g., for worst-case performance analysis). The reader is referred to [8] for various results concerning the Min-Max analysis of s.c.i and quasiconvex s.c.i. functions. In particular, the *inf-max-affine* transform is introduced in [8].

3.1. The MAX-PROD algebra

From the definition (3.1) of the Bellman-Karush transform of f, it suffices to make the change $f' = \log f$ $(f > 0)$, and since the exponential preserves order, one applies Theorem 2.1 to f'.

Thus, if $\log f(z)$ is p-homogeneous, then $\log f^*(y)$ is also q-homogeneous (provided everything is well-defined).

3.2. The MIN-MAX algebra

Formally, the equivalent of the Legendre-Fenchel transform is:

$$f \mapsto \tilde{f}(y) := \inf_x \max\{\langle y, x\rangle, f(x)\} = \inf_x\{\langle y, x\rangle \vee f(x)\} \tag{3.2}$$

Lemma 3.1. *Let $f : R^n \to \overline{R}$ and $\tilde{f}(y) := \inf_x\{\langle y, x\rangle \vee f(x)\}$ be its Min-Max conjugate. If f is p-homogeneous then $\tilde{f}$ is q-homogeneous with $1/p + 1/q = 1$.*

Proof. The proof is similar to the proof of Theorem 2.1. For $y \in \mathrm{Dom}(\tilde{f})$, we have (with $\lambda > 0$)

$$\lambda^q \tilde{f}(y) = \inf_x\{\langle \lambda y, \lambda^{q-1}x\rangle \vee f(\lambda^{q-1}x)\} = \inf_z\{\langle \lambda y, z\rangle \vee f(z)\} = \tilde{f}(\lambda y),$$

which is the desired result. □

Example 3.2. On the real line, consider the function $f(x) = 1/x^2$, $\forall x \in (0, \infty)$ and $f(x) = +\infty$ otherwise, i.e. f is (-2)-homogeneous. $\tilde{f}(z) = z^{2/3}$ on $[0, \infty)$ and 0 otherwise, i.e. $\tilde{f}$ is q-homogeneous with $q = 2/3$ and one may check that $1/p+1/q = -1/2+3/2 = 1$.

A similar conclusion hold with $f(x) = x^{-1}$, $x \neq 0$ and $f(0) := +\infty$. $\tilde{f}(z) = -z^{1/2}$ on $[0, \infty)$ and 0 otherwise. Hence, $\tilde{f}$ is 1/2-homogeneous and $1/p + 1/q = -1 + 2 = 1$.

Consider now the *inf-max-affine* transform introduced in [8], i.e.

$$\tilde{f}(y, z) := \inf_x [f(x) \vee (\langle y, x\rangle + z)].$$

Then, if f is p-homogeneous, one has

$$\tilde{f}(\lambda y, \lambda^q z) = \lambda^q \tilde{f}(y, z).$$

4. Conclusion

We have shown that homogeneous functions play a special role in Legendre-Fenchel-like transforms. Indeed, the p-homogeneous and q-homogeneous functions are in duality when $1/p + 1/q = 1$, i.e. when their exponents are also in duality. This strong property holds for transforms involving a "sup" (or "inf") operator (e.g. the Legendre-Fenchel, Bellman-Karush and MIN-MAX ..).

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